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SHAPE OPTIMIZATION OF ELASTIC AXISYMMETRIC BODIES

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Summary. The shape of the meridian curve of an elastic body is optimized within a class of Lipschitz functions. Only axisymmetric mixed boundary value problems are considered. Four different cost functionals are used and approximate piecewise linear solutions defined on the basis of a finite element technique. Some convergence and existence results are derived by means of the theory of the appropriate weighted Sobolev spaces.

Keywords: shape optimization, axisymmetric elliptic problems, finite elements, elasticity.

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INTRODUCTION

If both the domain occupied by an elastic body and the data (prescribed forces and displacements) are axially symmetric, the use of cylindrical coordinates reduces the problem to a two-dimensional domain — meridian section. Let the meridian curve be optimized so that a cost functional attains its minimum. The weak solution of the (quasistatic) state problem is defined in a weighted Sobolev space of displacement vector-functions with finite energy.

The present paper is a continuation of the previous paper [2], where the state problem was defined by a single elliptic equation with mixed boundary conditions.

In Section 1 we introduce the appropriate weighted Sobolev space and derive some auxiliary results. Section 2 contains the state problem formulated in displacements and the proof of continuous dependence of its solution on the domain in a certain sense. We formulate four Shape Optimization Problems in Section 3 and prove the existence of a solution to each of them. Approximations by finite elements are introduced in Section 4. Here we prove that having a sequence of approximate solutions with the mesh-size tending to zero, one can choose a subsequence, which converges to a solution of the original problem.

1. DEFINITIONS AND AUXILIARY LEMMAS IN THE APPROPRIATE WEIGHTED SOBOLEV SPACE

Let a bounded elastic body occupy an axisymmetric domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary (see e.g. [4] — chapter 1). The displacement vector $\boldsymbol{u} = (u_1, u_2, u_3)$

belongs to the space $W(\Omega)$ of functions with finite energy if each component u_i in the Cartesian coordinate system $\mathbf{x} = (x_1, x_2, x_3)$ belongs to the Sobolev space $H^1(\Omega)$ ([4] – chapter 6), $W(\Omega) = [H^1(\Omega)]^3$.

Let us denote the strain component by

$$\varepsilon_{ij}(\mathbf{u}) = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2 .$$

Henceforth $\|\cdot\|_{0,\Omega}$ denotes the norm in $[L^2(\Omega)]^3$ and $\|\cdot\|_0$ the norm in $L^2(0, 1)$.

Assume that the domain Ω is generated by the rotation of a two-dimensional domain D around the $z = x_3$-axis. Let us pass to the cylindrical coordinate system r, ϑ, z.

Let Z map each vector-function $\mathbf{u} \in W(\Omega)$, defined in Cartesian coordinates, onto the ordered triple

$$Z\mathbf{u} = \hat{\mathbf{u}} = (u_r, u_\vartheta, u_z)$$

of the physical components of the same vector at the corresponding point (r, ϑ, z). Then the space $W(\Omega)$ is transformed into $ZW(D \times [0, 2\pi))$. For brevity, let us denote

$$u_r = u , \quad u_\vartheta = v , \quad u_z = w .$$

Let $W_0(D)$ be the following subspace of axisymmetric displacements with finite energy

$$W_0(D) = \{\hat{\mathbf{u}} \in ZW(D \times [0, 2\pi)) \mid v = 0,\ \partial u/\partial\vartheta = 0,\ \partial w/\partial\vartheta = 0\} .$$

For $\hat{\mathbf{u}} \in W_0(D)$ we may write

$$(1.1) \qquad (2\pi)^{-1} \|\mathbf{u}\|^2_{W(\Omega)} = (2\pi)^{-1} \|\hat{\mathbf{u}}\|^2_{ZW} =$$
$$= \int_D [u^2 + (\partial u/\partial r)^2 + (\partial u/\partial z)^2 + u^2/r^2 + w^2 + (\partial w/\partial r)^2 +$$
$$+ (\partial w/\partial z)^2] \, r \, \mathrm{d}r \, \mathrm{d}z \equiv \|\hat{\mathbf{u}}\|^2_{\mathscr{H}(D)} .$$

On the basis of (1.1) the space $W_0(D)$ can be identified with the following space

$$\mathscr{H}(D) = \{\hat{\mathbf{u}} = (u, w) \in (W^{(1)}_{2,r}(D) \cap L_{2,1/r}(D)) \times W^{(1)}_{2,r}(D)\} .$$

Here $W^{(1)}_{2,r}(D)$ denotes the weighted Sobolev space with the norm

$$\|u\|_{1,r,D} = (\int_D [u^2 + (\partial u/\partial r)^2 + (\partial u/\partial z)^2] \, r \, \mathrm{d}r \, \mathrm{d}z)^{1/2} ,$$

$L_{2,1/r}(D)$ is the space of functions with the norm

$$\|u\|_{0,1/r,D} = (\int_D u^2/r \, \mathrm{d}r \, \mathrm{d}z)^{1/2} .$$

Let $L_{2,r}(D)$ be the space of functions with the following norm

$$\|u\|_{0,r,D} = (\int_D u^2 r \, \mathrm{d}r \, \mathrm{d}z)^{1/2} ,$$

$L_{2,r}(\Gamma)$ the space of functions defined on $\Gamma \subset \partial D$ with the norm

$$\|u\|_{0,r,\Gamma} = (\int_\Gamma u^2 r \, \mathrm{d}s)^{1/2} .$$

Lemma 1.1. *The embedding of the space $\mathscr{H}(D)$ into $[L_{2,r}(D)]^2$ is compact.*
For the proof – see [1] – Lemma 1.

Remark 1.1. Let Γ be a part of the boundary $\partial D \dot{-} \mathcal{O}$, where $\mathcal{O}$ denotes the z-axis and let Γ have a positive length. In $\mathscr{H}(D)$ we can define the *trace operator*. In fact, since each component u or w of $\hat{\boldsymbol{u}} \in \mathscr{H}(D)$ belongs to the space $W_{2,r}^{(1)}(D)$, we can use the linear continuous mapping

$$\gamma\colon W_{2,r}^{(1)}(D) \to L_{2,r}(\Gamma)$$

(see e.g. [2] – Section 1).

We shall consider a specific class of domains $D(\alpha)$, where

$$D(\alpha) = \{(r, z) \mid 0 < r < \alpha(z),\ 0 < z < 1\}$$

and the function α belongs to the following set

$$\mathscr{U}_{\mathrm{ad}} = \{\alpha \in C^{(0),1}([0, 1]) \quad \text{(i.e., Lipschitz function)},$$
$$\alpha_{\min} \leqq \alpha(z) \leqq \alpha_{\max},\ |\mathrm{d}\alpha/\mathrm{d}z| \leqq C_1,\ \textstyle\int_0^1 \alpha^2(z)\,\mathrm{d}z = C_2\},$$

where $\alpha_{\min}, \alpha_{\max}$ and C_1, C_2 are given positive constants.

Let $\Gamma(\alpha)$ denote the graph of the function α, $\Gamma_1(\alpha) = \partial D(\alpha) \cap \{z = 0\}$, $\Gamma_2(\alpha) = \partial D(\alpha) \cap \{z = 1\}$ (see Fig. 1).

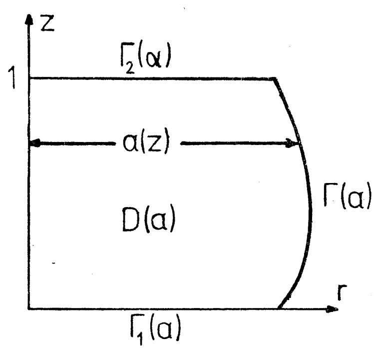

Fig. 1

Lemma 1.2. *There exists a constant C independent of α and such that*

$$\|\gamma u\|_{0,r,\Gamma_1(\alpha)\cup\Gamma(\alpha)} \leqq C\|u\|_{1,r,D(\alpha)}$$

holds for any $u \in W_{2,r}^{(1)}(D(\alpha))$, $\alpha \in \mathscr{U}_{\mathrm{ad}}$.

For the proof – see [2] – Lemma 1.

Lemma 1.3. *Let $\alpha \in \mathscr{U}_{\mathrm{ad}}$. Then the set*

$$M(D(\alpha)) = \{\hat{\boldsymbol{u}} = (u, w) \in [C^\infty(\mathrm{Cl}\, D(\alpha))]^2,$$

$$\operatorname{supp} u \cap (\mathcal{O} \cup \Gamma_2(\alpha)) = \emptyset,\ \operatorname{supp} w \cap \Gamma_2(\alpha) = \emptyset\}$$

is dense in the following subspace

$$V(D(\alpha)) = \{\hat{\boldsymbol{u}} \in \mathscr{H}(D(\alpha)) \mid \gamma u = 0,\ \gamma w = 0 \text{ on } \Gamma_2(\alpha)\}.$$

Proof. 1° Let us denote $D(\alpha) = D$, $\Gamma_2(\alpha) = \Gamma_2$ and let

$$H_2^1(D, r, r^{-1}) = W_{2,r}^{(1)}(D) \cap L_{2,1/r}(D)$$

with the norm

$$\|u\|_{H_2^1(D,r,r^{-1})} = (\|u\|_{1,r,D}^2 + \|u\|_{0,1/r,D}^2)^{1/2}.$$

Let $\hat{\boldsymbol{u}} = (u, w) \in V(D)$ be given. There exists a sequence of functions $u_n \in H_2^1(D, r, r^{-1})$ such that

$$\operatorname{supp} u_n \cap \mathcal{O} = \emptyset, \quad \gamma u_n = 0 \quad \text{on} \quad \Gamma_2, \tag{1.2}$$

$$u_n \to u \quad \text{in} \quad H_2^1(D, r, r^{-1})$$

(see the proof of Theorem 3.2.4/1 in the book [5]).

Let us choose the domain

$$D_k = \{(r, z) \in D \mid r > k\},$$

such that $\operatorname{supp} u_n \subset D_k$.

There exists a sequence $\{u_{nj}\}$, $j = 1, 2, \ldots$, such that

$$u_{nj} \in C^\infty(\operatorname{Cl} D_k), \quad \operatorname{supp} u_{nj} \cap (\Gamma_2 \cup O_k) = \emptyset,$$

(where $O_k = \{(r, z) \mid r = k\}$),

$$u_{nj} \to u_n \quad \text{in} \quad H^1(D_k) \quad \text{for} \quad j \to \infty.$$

Since the norms in $H^1(D_k)$ and $H_2^1(D_k, r, r^{-1})$ are equivalent, we have also

$$u_{nj} \to u_n \quad \text{in} \quad H_2^1(D_k, r, r^{-1}).$$

If we extend u_{nj} by zero to $D \dot{-} D_k$, we obtain $u_{nj} \in C^\infty(\operatorname{Cl} D)$,

$$u_{nj} \to u_n \quad \text{in} \quad H_2^1(D, r, r^{-1}). \tag{1.3}$$

Combining (1.2) and (1.3), we arrive at the following result

$$u_{nj} \to u \quad \text{in} \quad H_2^1(D, r, r^{-1}) \tag{1.4}$$

for $n \to \infty$, $j \to \infty$, $j > j(n)$.

2° There exists a sequence of $w_n \in C^\infty(\operatorname{Cl} D)$ such that

$$\operatorname{supp} w_n \cap \Gamma_2 = \emptyset, \quad w_n \to w \quad \text{in} \quad W_{2,r}^{(1)}(D) \tag{1.5}$$

(see [2] – Lemma 2). Now Lemma 1.3 follows from (1.4) and (1.5), since

$$\|\hat{\boldsymbol{u}}\|_{\mathscr{H}(D)}^2 = \|u\|_{H_2^1(D,r,r^{-1})}^2 + \|w\|_{1,r,D}^2.$$

Q.E.D.

2. THE STATE PROBLEM AND THE CONTINUOUS DEPENDENCE OF ITS SOLUTION ON THE DESIGN VARIABLE

For simplicity we shall restrict ourselves to isotropic elastic bodies. We shall formulate the state problem via the principle of virtual displacements (see e.g. [4], § 10.3). For physical components of the stress tensor τ and of the strain tensor ε the following relations hold:

$$\text{(2.1)} \qquad \tau_{rr} = \lambda e + 2\mu\varepsilon_{rr}\,, \quad \tau_{zz} = \lambda e + 2\mu\varepsilon_{zz}\,,$$
$$\tau_{\vartheta\vartheta} = \lambda e + 2\mu\varepsilon_{\vartheta\vartheta}\,, \quad \tau_{rz} = 2\mu\varepsilon_{rz}\,,$$

where

$$e = \varepsilon_{rr} + \varepsilon_{zz} + \varepsilon_{\vartheta\vartheta}\,,$$

$\lambda = \lambda(r, z)$ and $\mu = \mu(r, z)$ are Lamé's coefficients.

Moreover, we have the strain-displacement relations

$$\text{(2.2)} \qquad \varepsilon_{rr} = \partial u/\partial r\,, \quad \varepsilon_{zz} = \partial w/\partial z\,, \quad \varepsilon_{\vartheta\vartheta} = u/r\,, \quad \varepsilon_{rz} = (\partial u/\partial z + \partial w/\partial r)/2\,.$$

Let us define the following bilinear form

$$\text{(2.3)} \qquad a(\alpha; \hat{u}, \hat{u}^*) = \int_{D(\alpha)} \left[2\mu \left(\frac{\partial u}{\partial r}\frac{\partial u^*}{\partial r} + \frac{u}{r}\frac{u^*}{r} + \frac{\partial w}{\partial z}\frac{\partial w^*}{\partial z}\right) + \right.$$
$$+ \lambda \left(\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}\right)\left(\frac{\partial u^*}{\partial r} + \frac{u^*}{r} + \frac{\partial w^*}{\partial z}\right) +$$
$$\left. + \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\left(\frac{\partial u^*}{\partial z} + \frac{\partial w^*}{\partial r}\right)\right] r \,\mathrm{d}r \,\mathrm{d}z$$

for all $\hat{u} = (u, w)$ and $\hat{u}^* = (u^*, w^*)$.

Note that

$$a(\alpha; \hat{u}, \hat{u}^*) = \int_{D(\alpha)} [\tau_{rr}(\hat{u})\,\varepsilon_{rr}(\hat{u}^*) + \ldots + 2\,\tau_{rz}(\hat{u})\,\varepsilon_{rz}(\hat{u}^*)]\, r \,\mathrm{d}r \,\mathrm{d}z\,.$$

Denote by $S_i(\alpha)$ the disc generated by the rotation of $\Gamma_i(\alpha)$ around the z-axis. Let $\hat{\Omega}$ be a cylindrical domain generated by the rotation of the rectangle

$$\hat{D} = (0, \delta) \times (0, 1)\,, \quad \delta > \alpha_{\max}\,.$$

Assume that axisymmetric body forces $F \in [L^2(\hat{\Omega})]^3$ are given and the surface tractions are defined by an axisymmetric function

$$G = \begin{cases} 0 & \text{on} \quad S(\alpha) \\ G^1 & \text{on} \quad S_1(\alpha) \end{cases},$$

where G^1 is determined as the restriction to $S_1(\alpha)$ of an axisymmetric function $G^1 \in [L^2(\hat{S}_1)]^3$, where

$$\hat{S}_1 = \partial\hat{\Omega} \cap \{x_3 = 0\}\,;$$

$S(\alpha)$ denotes the surface generated by the rotation of $\Gamma(\alpha)$ around the z-axis.

Assume that the functions λ and μ are given in $L^\infty(\hat{D})$ and $\lambda \geqq 0$, $\mu \geqq \mu_0 > 0$ holds a.e. in $\hat{D}$, where μ_0 is a constant.

Passing to the cylindrical coordinate system, we transform the work of external forces

$$\sum_{i=1}^{3} \left(\int_{\Omega(\alpha)} F_i u_i \,\mathrm{d}x + \int_{S_1(\alpha)} G_i \gamma u_i \,\mathrm{d}s \right)$$

into the following integral

$$L(\alpha; u) = \int_{D(\alpha)} (f_r u + f_z w) \, r \,\mathrm{d}r \,\mathrm{d}z + \int_{\Gamma_1(\alpha)} (g_r \gamma u + g_z \gamma w) \, r \,\mathrm{d}r \,,$$

where the functions $f_r, f_z \in L_{2,r}(\hat{D})$ and $g_r, g_z \in L_{2,r}(\hat{\Gamma}_1)$, $(\hat{\Gamma}_1 = \partial \hat{D} \cap \{z = 0\})$ are given.

The principle of virtual displacements yields the following variational formulation of the State Problem:

find $\boldsymbol{u} \in V(D(\alpha))$ such that

$$a(\alpha; \boldsymbol{u}, \boldsymbol{v}) = L(\alpha; \boldsymbol{v}) \quad \forall \boldsymbol{v} \in V(D(\alpha)) \,. \tag{2.4}$$

(See Lemma 1.3 for the definition of $V(D(\alpha))$.)

Lemma 2.1. (Uniform Korn's inequality). *There exists a positive constant C, independent of $\alpha \in \mathscr{U}_{\mathrm{ad}}$, such that*

$$\int_{D(\alpha)} [\varepsilon_{rr}^2(\boldsymbol{u}) + \varepsilon_{\vartheta\vartheta}^2(\boldsymbol{u}) + \varepsilon_{zz}^2(\boldsymbol{u}) + 2\varepsilon_{rz}^2(\boldsymbol{u})] \, r \,\mathrm{d}r \,\mathrm{d}z \geqq C \|\boldsymbol{u}\|_{\mathscr{H}(D(\alpha))}^2$$

holds for all $\boldsymbol{u} \in V(D(\alpha))$, $\alpha \in \mathscr{U}_{\mathrm{ad}}$.

For the Proof – see [1] – Theorem 1 and Example 1.

Lemma 2.2. *There exist positive constants C_3, C_4, independent of α and such that the inequalities*

$$a(\alpha; \boldsymbol{u}, \boldsymbol{u}) \geqq C_3 \|\boldsymbol{u}\|_{\mathscr{H}(D(\alpha))}^2 \quad \forall \boldsymbol{u} \in V(D(\alpha)) \,, \tag{2.5}$$

$$a(\alpha; \boldsymbol{u}, \boldsymbol{v}) \leqq C_4 \|\boldsymbol{u}\|_{\mathscr{H}(D(\alpha))} \|\boldsymbol{v}\|_{\mathscr{H}(D(\alpha))} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathscr{H}(D(\alpha)) \tag{2.6}$$

hold for all $\alpha \in \mathscr{U}_{\mathrm{ad}}$.

Proof. Since $\lambda \geqq 0$ and $\mu_0 \leqq \mu$, we have

$$a(\alpha; \boldsymbol{u}, \boldsymbol{u}) \geqq 2\mu_0 \int_{D(\alpha)} [\varepsilon_{rr}^2(\boldsymbol{u}) + \ldots + 2\varepsilon_{rz}^2(\boldsymbol{u})] \, r \,\mathrm{d}r \,\mathrm{d}z \,.$$

Then (2.5) follows from Lemma 2.1. Making use of the boundedness of λ and μ, we obtain the estimate (2.6).

Lemma 2.3. *There exists a positive constant C_5, independent of α and such that*

$$L(\alpha; \boldsymbol{u}) \leqq C_5 \|\boldsymbol{u}\|_{\mathscr{H}(D(\alpha))} \quad \forall \boldsymbol{u} \in \mathscr{H}(D(\alpha))$$

holds for all $\alpha \in \mathscr{U}_{\mathrm{ad}}$.

Proof. Since both components u and w of $\mathbf{u}$ belong to $W_{2,r}^{(1)}(D(\alpha))$, we can apply Lemma 1.2 to them.

Lemma 2.4. *The State Problem* (2.4) *has a unique solution* $\mathbf{u} = \mathbf{u}(\alpha)$ *for any* $\alpha \in \mathcal{U}_{ad}$.

Proof – follows from the Riesz-Frechet Theorem, since the space $V(D(\alpha))$ can be equipped with the inner product $a(\alpha; \mathbf{u}, \mathbf{v})$ on the basis of Lemma 2.2. Moreover, we employ Lemma 2.3 to verify the continuity of the right-hand side in (2.4).

Proposition 2.1. *Assume that a sequence* $\{\alpha_n\}$, $n = 1, 2, \ldots$, $\alpha_n \in \mathcal{U}_{ad}$, *converges to a function* α *in* $C([0, 1])$. *Let us define the domains*

$$G_m = \{(r, z) \mid 0 < r < \alpha(z) - 1/m,\ 0 < z < 1\}\,, \quad m = 2, 3, \ldots .$$

Let $\mathbf{u}(\alpha_n)$ *be the solution of the State Problem* (2.4) *on* $D(\alpha_n)$. *Then*

$$\mathbf{u}(\alpha_n) \longrightarrow \mathbf{u}(\alpha) \quad (\textit{weakly}) \ \textit{in} \ \mathcal{H}(G_m) \quad \forall m\,,$$

where $\mathbf{u}(\alpha)$ *is the solution of* (2.4) *on* $D(\alpha)$.

Proof. Let us denote $D(\alpha) = D$, $D(\alpha_n) = D_n$, $\mathbf{u}(\alpha_n) = \mathbf{u}_n$. Inserting $\mathbf{v} = \mathbf{u}_n$ in (2.4) and using Lemmas 2.2, 2.3, we obtain

$$C_3 \|\mathbf{u}_n\|^2_{\mathcal{H}(D_n)} \leqq a(\alpha_n; \mathbf{u}_n, \mathbf{u}_n) = L(\alpha_n; \mathbf{u}_n) \leqq C_5 \|\mathbf{u}_n\|_{\mathcal{H}(D_n)}$$

so that

$$(2.7) \qquad \|\mathbf{u}_n\|_{\mathcal{H}(D_n)} \leqq C_5/C_3 \quad \forall n\,.$$

Let m be fixed for a time being. Since $G_m \subset D_n$ for $n > n_0(m)$, we have

$$(2.8) \qquad \|\mathbf{u}_n\|_{\mathcal{H}(G_m)} \leqq C_5/C_3 \equiv C_6 \quad \forall n > n_0(m)\,.$$

The space $\mathcal{H}(G_m)$ is a Banach reflexive space (see e.g. [5]) and therefore it is weakly compact. There exists $\mathbf{u}^{(m)} \in \mathcal{H}(G_m)$ and a subsequence $\{\mathbf{u}_{n_1}\} \subset \{\mathbf{u}_n\}$ such that

$$(2.9) \qquad \mathbf{u}_{n_1} \longrightarrow \mathbf{u}^{(m)} \quad (\text{weakly}) \ \text{in} \ \mathcal{H}(G_m) \quad \text{for} \quad n_1 \to \infty\,.$$

Passing to G_{m+1}, we may argue in the same way, choosing a subsequence $\{\mathbf{u}_{n_2}\} \subset \subset \{\mathbf{u}_{n_1}\}$ such that

$$\mathbf{u}_{n_2} \longrightarrow \mathbf{u}^{(m+1)} \quad (\text{weakly}) \ \text{in} \ \mathcal{H}(G_{m+1})\,.$$

Let us consider the diagonal subsequence $\{\mathbf{u}_{n_D}\}$ of all subsequences $\{\mathbf{u}_{n_1}\}, \{\mathbf{u}_{n_2}\}, \ldots$. We can prove that a function $\mathbf{u} \in \mathcal{H}(D)$ exists such that

$$(2.10) \qquad \mathbf{u}_{n_D} \longrightarrow \mathbf{u}|_{G_m} \quad (\text{weakly}) \ \text{in} \ \mathcal{H}(G_m)$$

holds for any m, if $n_D \to \infty$.

First we show that

$$u^{(m+k)}|_{G_m} = u^{(m)} \quad \text{a.e. in} \quad G_m \tag{2.11}$$

holds for any positive integer k. In fact, let us denote

$$u^{(m+k)}|_{G_m} - u^{(m)} = \psi$$

and let $\tilde{\psi}$ be an extension of ψ by zero to $G_{m+k} - G_m$. Consider the equation

$$\int_{G_m} \psi \, . \, u_{n_D} r \, dr \, dz = \int_{G_{m+k}} \tilde{\psi} \, . \, u_{n_D} r \, dr \, dz \, ,$$

(where $\psi \, . \, \varphi = \psi_r \varphi_r + \psi_z \varphi_z$) and pass to the limit with $n_D \to \infty$ on both sides. Then (2.9) implies

$$\int_{G_m} \psi \, . \, u^{(m)} r \, dr \, dz = \int_{G_{m+k}} \tilde{\psi} \, . \, u^{(m+k)} r \, dr \, dz = \int_{G_m} \psi \, . \, u^{(m+k)}|_{G_m} r \, dr \, dz \, ,$$

so that

$$\|\psi\|^2_{0,r,G_m} = \int_{G_m} \psi \, . \, (u^{(m+k)}|_{G_m} - u^{(m)}) r \, dr \, dz = 0$$

and (2.11) follows.

Consequently, we may define

$$u|_{G_m} = u^{(m)} \quad \forall m \, . \tag{2.12}$$

Since any closed convex set in $\mathscr{H}(G_m)$ is weakly closed, (2.8) and (2.9) imply

$$\|u^{(m)}\|_{\mathscr{H}(G_m)} \leqq C_6 \quad \forall m \, ,$$

so that

$$\|u\|^2_{\mathscr{H}(D)} = \lim_{m \to \infty} \|u^{(m)}\|^2_{\mathscr{H}(G_m)} \leqq C_6 \, .$$

Hence u defined by (2.12) belongs to $\mathscr{H}(D)$ and (2.10) holds.

2° Let us show that $u = u(\alpha)$, i.e., u is a solution of the State Problem (2.4) on $D(\alpha)$. Let any $v \in V(D)$ be given. By virtue of Lemma 1.3 there exists a sequence $\{\omega_k\}$, $k \to \infty$, such that $\omega_k \in M(D)$ and

$$\omega_k \to v \quad \text{in} \quad \mathscr{H}(D) \, . \tag{2.13}$$

Let $\varrho_k \in H(\hat{D})$ denote any extension of ω_k to the rectangular domain $\hat{D}$, which saves the homogeneous boundary conditions on the line $z = 1$. For instance, we can define

$$\varrho_k(r, z) = \omega_k(2\alpha(z) - r, z)$$

for $(r, z) \in \hat{D} - D(\alpha)$ (provided $\alpha_{\max} < \delta \leqq 2\alpha_{\min}$).

Then we have

$$\varrho_k|_{D_n} \in V(D_n)$$

and therefore

$$a(\alpha_{n_D}; u_{n_D}, \varrho_k) = L(\alpha_{n_D}; \varrho_k) \quad \forall n_D \, .$$

Let k be fixed for the time being. We shall write n instead of n_D and denote $\alpha(z) - 1/m$ by $\alpha^m(z)$. We have

$$|a(\alpha_n; \mathbf{u}_n, \varrho_k) - a(\alpha^m; \mathbf{u}, \varrho_k)| \leqq$$
$$\leqq |a(\alpha^m; \mathbf{u}_n - \mathbf{u}, \varrho_k)| + |\tilde{a}(\alpha_n - \alpha^m; \mathbf{u}_n, \varrho_k)| = I_1 + I_2 ,$$

where $\tilde{a}(\alpha_n - \alpha^m; \cdot, \cdot)$ denotes the bilinear form a with the integration over the domain $D_n \dot{-} G_m$ only. From the weak convergence (2.10) $I_1 \to 0$ follows for $n \to \infty$. By an analogue of (2.6) and using (2.7), we obtain

$$I_2 \leqq C\|\varrho_k\|_{\mathscr{H}(D \dot{-} G_m)} .$$

(Here we always assume that n is large enough so that $n > n_0(m)$ implies that $G_m \subset D_n$.) Consequently, we may write

$$|a(\alpha_n; \mathbf{u}_n, \varrho_k) - a(\alpha; \mathbf{u}, \varrho_k)| \leqq |a(\alpha_n; \mathbf{u}_n, \varrho_k) - a(\alpha^m; \mathbf{u}, \varrho_k)| +$$
$$+ |\tilde{a}(\alpha - \alpha^m; \mathbf{u}, \varrho_k)| \leqq I_1 + C\|\varrho_k\|_{\mathscr{H}(D_n - G_m)} + \tilde{C}\|\varrho_k\|_{\mathscr{H}(D \dot{-} G_m)} .$$

Since

$$\text{meas } (D_n - G_m) < 1/m + \|\alpha_n(z) - \alpha(z)\|_{C([0,1])} ,$$

we conclude that

$$\lim_{n\to\infty} a(\alpha_n; \mathbf{u}_n, \varrho_k) = a(\alpha; \mathbf{u}, \varrho_k) .$$

It is easy to see that

$$\lim_{n\to\infty} L(\alpha_n; \varrho_k) = L(\alpha; \varrho_k) ,$$

so that we arrive at the following result:

$$a(\alpha; \mathbf{u}, \varrho_k) = L(\alpha; \varrho_k) \quad \forall k .$$

Let us pass to the limit with $k \to \infty$. On the basis of Lemma 2.2, Lemma 2.3 and (2.13), we may write

$$|a(\alpha; \mathbf{u}, \varrho_k) - a(\alpha; \mathbf{u}, \mathbf{v})| \leqq C\|\varrho_k - \mathbf{v}\|_{\mathscr{H}(D)} \to 0 ,$$
$$|L(\alpha; \varrho_k) - L(\alpha; \mathbf{v})| \leqq C\|\omega_k - \mathbf{v}\|_{\mathscr{H}(D)} \to 0 .$$

Consequently, $\mathbf{u}$ satisfies the condition (2.4) for any $\mathbf{v} \in V(D)$.

The subspace $V(G_m)$ is weakly closed in $\mathscr{H}(G_m)$, being convex and closed (as follows from Lemma 1.2 applied to G_m). We have

$$\mathbf{u}_n|_{G_m} \in V(G_m) \quad \forall n > n_0(m) .$$

Then $\mathbf{u}|_{G_m} \in V(G_m)$ follows from (2.10) and since m was arbitrary, $\mathbf{u} \in V(D)$ holds.

By means of Lemma 2.4 we conclude that (i) $\mathbf{u} = \mathbf{u}(\alpha)$ is the solution of (2.4) and (ii) the whole sequence $\{\mathbf{u}_n\}$ is weakly convergent to the solution $\mathbf{u}(\alpha)$ in $\mathscr{H}(G_m)$ for any m. Q.E.D.

3. OPTIMIZATION PROBLEMS. EXISTENCE OF AN OPTIMAL DOMAIN

It this Section we shall choose four different cost functionals and present definitions of four corresponding shape optimization problems. Finally we shall prove the existence of a solution to any of these problems.

Let us consider the following cost functionals:

$$j_1(\alpha; \boldsymbol{u}) = \int_{D(\alpha)} (u^2 + w^2)\, r \,\mathrm{d}r\, \mathrm{d}z\,;$$

$$j_2(\alpha; \boldsymbol{u}) = \int_0^1 [(u(\alpha(z), z) - u_g)^2 + (w(\alpha(z), z) - w_g)^2]\, \mathrm{d}z\,,$$

where $u(\alpha(z), z)$ and $w(\alpha(z), z)$ denote the traces of u and w on the curve $\Gamma(\alpha)$, respectively and u_g, w_g are given functions from $L^2(0, 1)$;

$$j_3(\alpha; \boldsymbol{u}) = L(\alpha; \boldsymbol{u})\,;$$

$$j_4(\alpha; \boldsymbol{u}) = \int_{D(\alpha)} 4\mu^2(\varepsilon_{rr}^2(\boldsymbol{u}) + \varepsilon_{\vartheta\vartheta}^2(\boldsymbol{u}) + \varepsilon_{zz}^2(\boldsymbol{u}) + 2\varepsilon_{rz}^2(\boldsymbol{u}) - \tfrac{1}{3}e^2(\boldsymbol{u}))\, r\,\mathrm{d}r\,\mathrm{d}z\,.$$

Note that the integrand of j_4 is proportional to the squared yield function of Mises (see e.g. [4] – chapter 3). Another form of the latter cost functional is

$$j_4(\alpha; \boldsymbol{u}) = \int_{D(\alpha)} (\tau_{rr}^2 + \tau_{\vartheta\vartheta}^2 + \tau_{zz}^2 + 2\tau_{rz}^2 - \tfrac{1}{3}(\tau_{rr} + \tau_{\vartheta\vartheta} + \tau_{zz})^2)\, r\,\mathrm{d}r\,\mathrm{d}z\,,$$

where the stress components are determined by (2.1), (2.2) as functions of the displacement $\boldsymbol{u} = (u, w)$.

We define the Shape Optimization Problems:

find $\alpha^0 \in \mathscr{U}_{\mathrm{ad}}$ *such that*

$$(3.1)_i \qquad j_i(\alpha^0; \boldsymbol{u}(\alpha^0)) \leqq j_i(\alpha; \boldsymbol{u}(\alpha)) \quad \forall \alpha \in \mathscr{U}_{\mathrm{ad}}\,, \quad i \in \{1, 2, 3, 4\}\,,$$

where $\boldsymbol{u}(\alpha)$ *denotes the solution of the State Problem* (2.4).

For the proof of the existence of an optimal solution α^0 we shall need the following

Proposition 3.1. *Let the assumptions of Proposition* 2.1 *be satisfied. Then*

$$(3.2) \qquad \lim_{n\to\infty} j_i(\alpha_n; \boldsymbol{u}(\alpha_n)) = j_i(\alpha; \boldsymbol{u}(\alpha))\,, \quad i = 1, 2, 3\,,$$

$$\liminf_{n\to\infty} j_4(\alpha_n; \boldsymbol{u}(\alpha_n)) \geqq j_4(\alpha; \boldsymbol{u}(\alpha))\,.$$

Proof. *Case* $i = 1$. Let us denote again $\alpha(z) - 1/m = \alpha^m(z)$, $\boldsymbol{u}(\alpha_n) = \boldsymbol{u}_n$, $D(\alpha_n) = D_n$, $\boldsymbol{u}(\alpha) = \boldsymbol{u}$, $D(\alpha) = D$. We have (for n large enough)

$$(3.3) \qquad |j_1(\alpha_n; \boldsymbol{u}_n) - j_1(\alpha; \boldsymbol{u})| = |\, \|\boldsymbol{u}_n\|_{0,r,G_m}^2 - \|\boldsymbol{u}\|_{0,r,G_m}^2 + \|\boldsymbol{u}_n\|_{0,r,D_n \dot- G_m}^2 - \|\boldsymbol{u}\|_{0,r,D \dot- G_m}^2| \leqq$$

$$\leqq |\, \|\boldsymbol{u}_n\|_{0,r,G_m}^2 - \|\boldsymbol{u}\|_{0,r,G_m}^2| + \|\boldsymbol{u}_n\|_{0,r,D_n \dot- G_m}^2 + \|\boldsymbol{u}\|_{0,r,D \dot- G_m}^2\,.$$

Using Lemma 1.1 and Proposition 2.1, we obtain

$$\boldsymbol{u}_n \to \boldsymbol{u} \quad \text{in} \quad [L_{2,r}(G_m)]^2$$

so that

$$\lim_{n\to\infty} \|\boldsymbol{u}_n\|^2_{0,r,G_m} = \|\boldsymbol{u}\|^2_{0,r,G_m} \quad \forall m\,. \tag{3.4}$$

We can derive the estimate

$$\|\boldsymbol{u}_n\|^2_{0,r,D_n \dot{-} G_m} \leqq C\alpha_{\max} \|\alpha_n - \alpha^m\|_{C([0,1])} \tag{3.5}$$

for $n > n_0(m)$, $m > m_0$, with some constant C independent of n, m. Indeed, (3.5) follows from [6] – (Appendix), if we use the equivalence of norms in $W^{(1)}_{2,r}(D_0)$ and $H^1(D_0)$, where

$$D_0 = \{(r, z) \mid \alpha_{\min}/2 < r < \alpha_{\max}\}\,.$$

Combining (3.4) and (3.5), we deduce

$$\lim_{n\to\infty} j_1(\alpha_n; \boldsymbol{u}_n) = j_1(\alpha; \boldsymbol{u})$$

on the basis of (3.3).

Case $i = 2$. Let us denote the graph of α_n, α by Γ_n, Γ, respectively, graph of $\alpha^m = \alpha - 1/m$ by γ_m. We may write

$$j_2(\alpha_n; \boldsymbol{u}_n) - j_2(\alpha; \boldsymbol{u}) = K_1 + K_2 + K_3\,,$$

where (for $\boldsymbol{u}_n = (u_n, w_n)$)

$$K_1 = \int_0^1 [(u_n|_{\Gamma_n} - u_g)^2 + (w_n|_{\Gamma_n} - w_g)^2]\,\mathrm{d}z - \int_0^1 [(u_n|_{\gamma_m} - u_g)^2 + (w_n|_{\gamma_m} - w_g)^2]\,\mathrm{d}z\,,$$

$$K_2 = \int_0^1 [(u_n|_{\gamma_m} - u_g)^2 + (w_n|_{\gamma_m} - w_g)^2]\,\mathrm{d}z - \int_0^1 [(u|_{\gamma_m} - u_g)^2 + (w|_{\gamma_m} - w_g)^2]\,\mathrm{d}z\,,$$

$$K_3 = \int_0^1 [(u|_{\gamma_m} - u_g)^2 + (w|_{\gamma_m} - w_g)^2]\,\mathrm{d}z - \int_0^1 [(u|_{\Gamma} - u_g)^2 + (w|_{\Gamma} - w_g)^2]\,\mathrm{d}z\,.$$

We have a splitting

$$K_1 = K_{1u} + K_{1w}\,,$$

and the estimates

$$|K_{1u}| \leqq \int_0^1 |u_n|_{\Gamma_n} - u_n|_{\gamma_m}| \cdot |u_n|_{\Gamma_n} + u_n|_{\gamma_m} - 2u_g|\,\mathrm{d}z \leqq \leqq \|u_n|_{\Gamma_n} - u_n|_{\gamma_m}\|_0 \cdot \|u_n|_{\Gamma_n} + u_n|_{\gamma_m} - 2u_g\|_0\,,$$

$$\|u_n|_{\Gamma_n} - u_n|_{\gamma_m}\|_0^2 = \int_0^1 (u_n|_{\Gamma_n} - u_n|_{\gamma_m})^2\,\mathrm{d}z = \int_0^1 \mathrm{d}z\Big(\int_{\alpha^m}^{\alpha_n} \partial u_n/\partial r\,\mathrm{d}r\Big)^2 \leqq \leqq (1/m + \beta_n)\int_0^1 \mathrm{d}z \int_{\alpha^m}^{\alpha_n} (\partial u_n/\partial r)^2\,\mathrm{d}r \leqq \alpha_{\min}^{-1}(1/m + \beta_n)\,\|\boldsymbol{u}_n\|^2_{\mathscr{H}(D_n)}\,,$$

where

$$\beta_n = \|\alpha - \alpha_n\|_{C([0,1])}\,.$$

By virtue of (2.7), the latter expression is bounded by $\tilde{C}(1/m + \beta_n)$.

Next we have

$$(3.6)\qquad \left\|u_n|_{\Gamma_n} + u_n|_{\gamma_m} - 2u_g\right\|_0^2 \leqq$$
$$\leqq 3\left(\left\|u_n|_{\Gamma_n}\right\|_0^2 + \left\|u_n|_{\gamma_m}\right\|_0^2 + 4\|u_g\|_0^2\right) \leqq C\,,$$

where C is independent of all sufficiently great $n, m, n > n_0(m)$.

Indeed, making use of Lemma 1.2 and (2.7), we obtain

$$\left\|u_n|_{\Gamma_n}\right\|_0^2 \leqq \alpha_{\min}^{-1} \int_{\Gamma_n} (\gamma u_n)^2\, r \,\mathrm{d}s \leqq \alpha_{\min}^{-1} C \|u_n\|_{1,r,D_n}^2 \leqq C_7\,.$$

Second, we may write

$$\left\|u_n|_{\gamma_m}\right\|_0^2 \leqq 2\left\|u_n|_{\Gamma_n}\right\|_0^2 + 2\left\|u_n|_{\gamma_m} - u_n|_{\Gamma_n}\right\|_0^2 \leqq$$
$$\leqq 2C_7 + 2\tilde{C}(1/m + \beta_n) \leqq C_8$$

and thus we arrive at (3.6). Altogether, we have

$$|K_{1u}| \leqq C(1/m + \beta_n)^{1/2}\,.$$

The same estimate can be derived for K_{1w}. Consequently, we obtain

$$(3.7)\qquad |K_1| \leqq C(1/m + \beta_n)^{1/2} \quad \forall n, m\,, \quad n > n_0(m)\,.$$

For K_2 we may obviously write

$$K_2 = K_{2u} + K_{2w}$$

with a selfexplanatory splitting. We can prove that for $m > m_1$

$$(3.8)\qquad \lim_{n\to\infty} \left\|u_n|_{\gamma_m} - u|_{\gamma_m}\right\|_0^2 = 0\,.$$

In fact, let us define $G_m^0 = G_m \cap D_0$. We easily realize that the weak convergence of $\boldsymbol{u}_n$ in $\mathscr{H}(G_m)$ to $\boldsymbol{u}$ implies the weak convergence of the u_n-components in $H^1(G_m^0)$ to u. The trace mapping of $H^1(G_m^0)$ into $L^2(\partial G_m^0)$ is compact (see [7] Chapt. 2, § 6.2), so that

$$\lim_{n\to\infty} \left\|u_n|_{\gamma_m} - u|_{\gamma_m}\right\|_{L^2(\gamma_m)} = 0\,.$$

Consequently, we have

$$\int_0^1 \left(u_n|_{\gamma_m} - u|_{\gamma_m}\right)^2 \mathrm{d}z \leqq \int_{\gamma_m} \left(u_n|_{\gamma_m} - u|_{\gamma_m}\right)^2 \mathrm{d}s \to 0\,, \quad n \to \infty\,,$$

which proves (3.8).

Then

$$(3.9)\qquad K_{2u} = \int_0^1 \left[(u_n|_{\gamma_m})^2 - (u|_{\gamma_m})^2 + 2u_g(u|_{\gamma_m} - u_n|_{\gamma_m})\right] \mathrm{d}z \to 0$$

by virtue of (3.8). By an analogous argument, we arrive at the similar assertion for K_{2w}, so that

$$(3.10)\qquad \lim_{n\to\infty} K_2 = 0 \quad \forall m > m_1\,.$$

For K_3 we have the following splitting and estimate

$$K_3 = K_{3u} + K_{3w},$$

$$|K_{3u}| = \left|\int_0^1 (u|_{\gamma_m} - u|_\Gamma)(u|_{\gamma_m} + u|_\Gamma - 2u_g)\,dz\right| \leqq$$
$$\leqq C\|u|_{\gamma_m} - u|_\Gamma\|_0 \to 0\,, \quad m \to \infty\,,$$

which follows by an argument similar to that for K_{1u}. Consequently, we obtain

$$\lim_{n\to\infty} K_3 = 0\,. \tag{3.11}$$

Combining (3.7), (3.10) and (3.11), we deduce that

$$\lim_{n\to\infty} (K_1 + K_2 + K_3) = 0$$

and (3.2) follows for $i = 2$.

Case $i = 3$. We have (denoting $\Gamma_1(\alpha_n) = \Gamma_{1n}$ and $\Gamma_1(\alpha) = \Gamma_1$)

$$\begin{aligned}|j_3(\alpha_n; \boldsymbol{u}_n) - j_3(\alpha; \boldsymbol{u})| &= |L(\alpha_n; \boldsymbol{u}_n) - L(\alpha; \boldsymbol{u})| = \\ &= \left|\int_{D_n} (f_r u_n + f_z w_n)\, r\,dr\,dz + \int_{\Gamma_{1n}} (g_r \gamma u_n + g_z \gamma w_n)\, r\,dr - \right.\\ &\quad \left.- \int_D (f_r u + f_z w)\, r\,dr\,dz - \int_{\Gamma_1} (g_r \gamma u + g_z \gamma w)\, r\,dr\right|.\end{aligned} \tag{3.12}$$

In particular, we may write

$$\left|\int_{D_n} f_r u_n r\,dr\,dz - \int_D f_r u r\,dr\,dz\right| \leqq \left|\int_{G_m} f_r(u_n - u)\, r\,dr\,dz\right| +$$
$$+ \left|\int_{D_n \dot{-} G_m} f_r u_n r\,dr\,dz\right| + \left|\int_{D \dot{-} G_m} f_r u r\,dr\,dz\right|.$$

The first integral on the right-hand side tends to zero for any m on the basis of the weak convergence of u_n – see Proposition 2.1. Using (2.7), for the second integral we obtain the following estimate

$$\left|\int_{D_n \dot{-} G_m} f_r u_n r\,dr\,dz\right| \leqq C\|f_r\|_{0,r,D_n \dot{-} G_m} \to 0\,,$$

if $n \to \infty$, $m \to \infty$, $n > n_0(m)$. Since meas $(D \dot{-} G_m)$ tends to zero for m growing to infinity, the last integral converges to zero for $m \to \infty$. Altogether, we deduce

$$\lim_{n\to\infty} \int_{D_n} f_r u_n r\,dr\,dz = \int_D f_r u r\,dr\,dz\,. \tag{3.13}$$

Denoting $\Gamma_1(\alpha) \cap \partial G_m = \Gamma_{1m}$, we may write

$$\left|\int_{\Gamma_{1n}} g_r \gamma u_n r\,dr - \int_{\Gamma_1} g_r \gamma u r\,dr\right| \leqq \left|\int_{\Gamma_{1m}} g_r(\gamma u_n - \gamma u)\, r\,dr\right| +$$
$$+ \left|\int_{\Gamma_{1n} - \Gamma_{1m}} g_r \gamma u_n r\,dr\right| + \left|\int_{\Gamma_1 - \Gamma_{1m}} g_r \gamma u r\,dr\right|.$$

The first integral tends to zero for any m on the basis of Remark 1.1 and Proposition 2.1 – weak convergence of $\boldsymbol{u}_n$. The second integral has the following upper bound

$$C\|g_r\|_{0,r,\Gamma_{1n}-\Gamma_{1m}}\|u_n\|_{1,r,D_n}$$

with the constant C independent of n (cf. Lemma 1.2). Using (2.7), we conclude that the latter bound tends to zero with $n \to \infty$, $m \to \infty$, $n > n_0(m)$. Since

$$\text{meas}\,(\Gamma_1 \dot{-} \Gamma_{1m}) = 1/m\,,$$

the third integral tends to zero with m growing. Altogether, we obtain

$$\lim_{n\to\infty} \int_{\Gamma_{1n}} g_r \gamma u_n r \,\mathrm{d}r = \int_{\Gamma_1} g_r \gamma u r \,\mathrm{d}r\,. \tag{3.14}$$

Since analogous results can be derived for the integrals involving w_n instead of u_n, we are led to (3.2).

Case $i = 4$. Obviously, we may write

$$j_4(\alpha_n; \mathbf{u}_n) = 4 \int_{D_n} \mu^2(\varepsilon_{rr}^2 + \varepsilon_{\vartheta\vartheta}^2 + \varepsilon_{zz}^2 + 2\varepsilon_{rz}^2 - \tfrac{1}{3}e^2)\, r \,\mathrm{d}r\,\mathrm{d}z \geqq$$

$$\geqq 4 \int_{G_m} \mu^2 \left(\left(\frac{\partial u_n}{\partial r}\right)^2 + \left(\frac{u_n}{r}\right)^2 + \left(\frac{\partial w_n}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial u_n}{\partial z} + \frac{\partial w_n}{\partial r}\right)^2 - \right.$$

$$\left. - \frac{1}{3}\left(\frac{\partial u_n}{\partial r} + \frac{u_n}{r} + \frac{\partial w_n}{\partial z}\right)^2\right) r \,\mathrm{d}r\,\mathrm{d}z$$

for all $n, m, n > n_0(m)$, since the integrand is non-negative everywhere. The functional on the right-hand side is weakly lower semi-continuous in $\mathscr{H}(G_m)$ (being convex and Gateaux-differentiable). Making use of the weak convergence of $\mathbf{u}_n$ in $\mathscr{H}(G_m)$, we obtain

$$\liminf_{n\to\infty} j_4(\alpha_n; \mathbf{u}_n) \geqq$$

$$\geqq 4 \int_{G_m} \mu^2 \left(\left(\frac{\partial u}{\partial r}\right)^2 + \ldots - \frac{1}{3}\left(\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}\right)^2\right) r \,\mathrm{d}r\,\mathrm{d}z$$

for any m. Passing to the limit with $m \to \infty$, we arrive at the inequality

$$\liminf_{n\to\infty} j_4(\alpha_n; \mathbf{u}_n) \geqq j_4(\alpha; \mathbf{u})\,. \qquad \text{Q.E.D.}$$

Theorem 3.1. *There exists at least one solution of the Shape Optimization Problem* $(3.1)_i$, $i \in \{1, 2, 3, 4\}$.

Proof. Let $\{\alpha_n\}$, $n \to \infty$, $\alpha_n \in \mathscr{U}_{\text{ad}}$, be a minimizing sequence of $j_i(\alpha; \mathbf{u}(\alpha))$, i.e.,

$$\lim_{n\to\infty} j_i(\alpha_n; \mathbf{u}(\alpha_n)) = \inf_{\alpha\in\mathscr{U}_{\text{ad}}} j_i(\alpha, \mathbf{u}(\alpha))\,. \tag{3.15}$$

By means of the Arzelà-Ascoli Theorem we can show that the set $\mathscr{U}_{\text{ad}}$ is compact in $C([0, 1])$. Hence there exist a subsequence $\{\alpha_k\}$ and $\alpha^0 \in \mathscr{U}_{\text{ad}}$ such that

$$\alpha_k \to \alpha^0 \quad \text{in} \quad C([0, 1])\,.$$

Proposition 3.1 and (3.15) imply

$$\inf_{\alpha \in \mathcal{U}_{ad}} j_i(\alpha, \mathbf{u}(\alpha)) = \lim_{k \to \infty} j_i(\alpha_k, \mathbf{u}(\alpha_k)) \geqq j_i(\alpha^0, \mathbf{u}(\alpha^0)) .$$

Consequently, a minimum is attained at α^0.

4. APPROXIMATIONS BY FINITE ELEMENTS

In the present Section we propose an approximate solution of the Shape Optimization Problems, making use of piecewise linear design variables and linear triangular finite elements for solving the State Problem.

Let N be a positive integer and $h = 1/N$. We denote by Δ_j, $j = 1, 2, \ldots, N$, the subintervals $[(j-1)h, jh]$ and introduce the set

$$\mathcal{U}^h_{ad} = \{\alpha_h \in \mathcal{U}_{ad} \mid \alpha_h|_{\Delta j} \in P_1(\Delta_j) \ \forall j\} ,$$

where $P_1(\Delta_j)$ is the set of linear functions defined on Δ_j. Let D_h denote the domain $D(\alpha_h)$ bounded by the graph $\Gamma_h = \Gamma(\alpha_h)$ of the function $\alpha_h \in \mathcal{U}^h_{ad}$. The polygonal domain D_h will be partitioned into triangles by the following way. We choose $\alpha_0 \in (0, \alpha_{min})$ and introduce a uniform triangulation of the rectangle $\mathcal{R} = [0, \alpha_0] \times \times [0, 1]$, independent of α_h, if h is fixed.

In the remaining part $D_h \dot{-} \mathcal{R}$ let the nodal points divide the segments $[\alpha_0, \alpha_h(jh)]$, $j = 0, 1, 2, \ldots, N$, into M equal segments, where $M = 1 + [(\alpha_{max} - \alpha_0) N]$ and the square brackets denote the integer part of the number inside.

One can verify that the segments parallel with the r-axis are not longer than h and shorter than $h(\alpha_{min} - \alpha_0)/(\alpha_{max} - \alpha_0)$. One also deduces the following estimate for the interior angles ω of the triangulation

$$\operatorname{tg} \omega \geqq \frac{\alpha_{min} - \alpha_0}{\alpha_{max} - \alpha_0} (1 + C_1 + C_1^2)^{-1} .$$

Consequently, one obtains a strongly regular family $\{\mathcal{T}_h(\alpha_h)\}$, $h \to 0$, $\alpha_h \in \mathcal{U}^h_{ad}$, of triangulations. Note that for any $\alpha_h \in \mathcal{U}^h_{ad}$ we construct a unique triangulation $\mathcal{T}_h(\alpha_h)$.

Let us consider the standard space V_h of linear finite elements

$$V_h(D_h) = \{\mathbf{v}_h \in [C(\text{Cl } D_h)]^2 \cap V(D_h) \, |\mathbf{v}_h|_T \in [P_1(T)]^2 \quad \forall T \in \mathcal{T}_h(\alpha_h)\} .$$

Note that $u_h = 0$ for $r = 0$ follows from $\mathbf{u}_h = (u_h, w_h) \in V_h(D_h)$.

We define the *Approximate State Problem*:

find $\mathbf{u}_h = \mathbf{u}_h(\alpha_h) \in V_h(D_h)$ *such that*

$$a(\alpha_h; \mathbf{u}_h, \mathbf{v}_h) = L_h(\alpha_h; \mathbf{v}_h) \quad \forall \mathbf{v}_h \in V_h(D_h) . \tag{4.1}$$

Here $L_h(\alpha_h; \mathbf{v}_h)$ denotes a suitable approximation of the functional $L(\alpha_h; \mathbf{v}_h)$, which satisfies the following conditions: there exist positive constants C_7, C_8 and λ, independent of α_h and such that

$$|L_h(\alpha_h; \mathbf{v}_h) - L(\alpha_h; \mathbf{v}_h)| \leqq C_7 h^\lambda \|\mathbf{v}_h\|_{\mathscr{H}(D_h)}, \tag{4.2}$$

$$|L_h(\alpha_h; \mathbf{v}_h)| \leqq C_8 \|\mathbf{v}_h\|_{\mathscr{H}(D_h)} \tag{4.3}$$

holds for any $\alpha_h \in \mathscr{U}^h_{\text{ad}}$ and any $\mathbf{v}_h \in V_h(D_h)$.

For example, let us define

$$L_h(\alpha_h; \mathbf{u}_h) = \sum_{T \in \mathscr{T}_h(\alpha_h)} [f_r u_h r + f_z r w_h]_{G(T)} \text{ meas}(T) + \\ + \sum_{I \in \mathscr{T}_h(\alpha_h) \cap \Gamma_1(\alpha_h)} [g_r r u_h + g_z r w_h]_{G(I)} \text{ meas}(I), \tag{4.4}$$

where $G(T)$ denotes the centre of gravity of the triangle T and $G(I)$ is the midpoint of the interval $I = T \cap \Gamma_1(\alpha_h)$.

Lemma 4.1. *Let $L_h(\alpha_h; \mathbf{u}_h)$ be defined by the formula (4.4). Assume that $f_r, f_z \in H^1(\hat{D}) \cap C(\text{Cl}\,\hat{D})$, $r^2 D^\alpha f_r$, $r^2 D^\alpha f_z \in L^2(\hat{D})$ for $|\alpha| = 2$ and g_r, g_z are piecewise from C^2.*

Then (4.2) and (4.3) hold, with $\lambda = 1$.

The proof follows immediately from Lemma 8 in [2], since both components u_h and w_h belong to the space $W^{(1)}_{2,r}(D_h)$.

Remark 4.1. One can weaken the assumptions somewhat, employing the fact that u_h vanishes on the z-axis.

Lemma 4.1. *The Approximate State Problem (4.1) has a unique solution $\mathbf{u}_h(\alpha_h)$ for any $\alpha_h \in \mathscr{U}^h_{\text{ad}}$ and any $h = 1/N$.*

Proof. Lemma 2.2 and Lemma 4.1 enable us to apply to Riesz-Theorem in the Hilbert space $V_h(D_h)$ with the inner product $(\mathbf{u}, \mathbf{v}) \equiv a(\alpha_h; \mathbf{u}, \mathbf{v})$.

Proposition 4.1. *Let the assumptions (4.2), (4.3) be satisfied. Let $\{\alpha_h\}$, $h \to 0$, be a sequence of $\alpha_h \in \mathscr{U}^h_{\text{ad}}$, converging to α in $C([0, 1])$.*

Then

$$\mathbf{u}_h(\alpha_h)|_{G_m} \longrightarrow \mathbf{u}(\alpha)|_{G_m} \quad (\textit{weakly}) \quad \textit{in} \quad \mathscr{H}(G_m) \quad \forall m, \tag{4.5}$$

where $\mathbf{u}(\alpha)$ is the solution of the State Problem (2.4) on the domain $D(\alpha)$.

Proof. Denote $D(\alpha_h) = D_h$, $D(\alpha) = D$.

1° Let us define $\mathbf{u}^*_h \in V_h(D_h)$ to be the solution of the problem

$$a(\alpha_h; \mathbf{u}^*_h, \mathbf{v}_h) = L(\alpha_h; \mathbf{v}_h) \quad \forall \mathbf{v}_h \in V(D_h). \tag{4.6}$$

Subtracting (4.1), we obtain

$$a(\alpha_h; \boldsymbol{u}_h^* - \boldsymbol{u}_h, \boldsymbol{v}_h) = L(\alpha_h; \boldsymbol{v}_h) - L_h(\alpha_h, \boldsymbol{v}_h)$$

and inserting $\boldsymbol{v}_h := \boldsymbol{u}_h^* - \boldsymbol{u}_h$, we arrive at the inequalities

$$(4.7) \qquad C_3 \|\boldsymbol{u}_h^* - \boldsymbol{u}_h\|^2_{\mathscr{H}(D_h)} \leqq a(\alpha_h; \boldsymbol{u}_h^* - \boldsymbol{u}_h, \boldsymbol{u}_h^* - \boldsymbol{u}_h) =$$
$$= L(\alpha_h; \boldsymbol{u}_h^* - \boldsymbol{u}_h) - L_h(\alpha_h; \boldsymbol{u}_h^* - \boldsymbol{u}_h) \leqq C_7 h^\lambda \|\boldsymbol{u}_h^* - \boldsymbol{u}_h\|_{\mathscr{H}(D_h)},$$

using Lemma 2.2 and (4.2). From (4.6) we derive that

$$C_3 \|\boldsymbol{u}_h^*\|^2_{\mathscr{H}(D_h)} \leqq L(\alpha_h; \boldsymbol{u}_h^*) \leqq C_8 \|\boldsymbol{u}_h^*\|_{\mathscr{H}(D_h)}$$

holds by virtue of Lemma 2.2 and (4.3). Consequently, for $h < h_0(m)$ we have $G_m \subset D_h$ and

$$(4.8) \qquad \|\boldsymbol{u}_h^*\|_{\mathscr{H}(G_m)} \leqq \|\boldsymbol{u}_h^*\|_{\mathscr{H}(D_h)} \leqq C_8/C_3 \quad \forall m .$$

There exist a subsequence (we shall denote it by the same symbol) and $\boldsymbol{u}^{(m)} \in \mathscr{H}(G_m)$ such that

$$(4.9) \qquad \boldsymbol{u}_h^* \longrightarrow \boldsymbol{u}^{(m)} \quad \text{(weakly)} \quad \text{in} \quad \mathscr{H}(G_m) .$$

Arguing as in the proof of Proposition 2.1, we are led to a function $\boldsymbol{u} \in \mathscr{H}(D)$ such that

$$(4.10) \qquad \boldsymbol{u}_h^* \longrightarrow \boldsymbol{u}|_{G_m} \quad \text{(weakly)} \quad \text{in} \quad \mathscr{H}(G_m)$$

holds for any m and a subsequence of $\{\boldsymbol{u}_h^*\}$. In what follows, we shall consider this subsequence.

2° Let us show that $\boldsymbol{u} = \boldsymbol{u}(\alpha)$, i.e. $\boldsymbol{u}$ is a solution of the State Problem (2.4). Let any $\boldsymbol{v} \in V(D)$ be given. By virtue of Lemma 1.3 there exists a sequence $\{\omega_k\}$, $k \to \infty$, $\omega_k \in M(D)$ such that

$$(4.11) \qquad \omega_k \to \boldsymbol{v} \quad \text{in} \quad \mathscr{H}(D) .$$

Let $\varrho_k \in \mathscr{H}(\hat{D})$ be any extension of ω_k to the rectangular domain $\hat{D}$, which fulfills the zero boundary condition on the line $z = 1$.

Consider the Lagrange linear interpolate $\pi_h \varrho_k$ of $\varrho_k|_{D_h}$ over the triangulation $\mathscr{T}_h(\alpha_h)$. Obviously, $\pi_h \varrho_k$ belongs to $V_h(D_h)$. Let k be fixed, for the time being. We can insert $\pi_h \varrho_k$ into (4.6) to obtain

$$(4.12) \qquad a(\alpha_h; \boldsymbol{u}_h^*, \pi_h \varrho_k) = L(\alpha_h; \pi_h \varrho_k) .$$

We shall pass to the limit $h \to 0$. Let us denote $\alpha^m = \alpha - 1/m$. We may write

$$(4.13) \qquad |a(\alpha_h; \boldsymbol{u}_h^*, \pi_h \varrho_k) - a(\alpha^m; \boldsymbol{u}, \varrho_k)| =$$
$$= |a(\alpha^m; \boldsymbol{u}_h^*, \varrho_k) + a(\alpha^m; \boldsymbol{u}_h^*, \pi_h \varrho_k - \varrho_k) +$$
$$+ \tilde{a}(\alpha_h - \alpha^m; \boldsymbol{u}_h^*, \pi_h \varrho_k) - a(\alpha^m; \boldsymbol{u}, \varrho_k)| \leqq$$
$$\leqq |a(\alpha^m; \boldsymbol{u}_h^* - \boldsymbol{u}, \varrho_k)| + |a(\alpha^m; \boldsymbol{u}_h^*, \pi_h \varrho_k - \varrho_k)| + |\tilde{a}(\alpha_h - \alpha^m; \boldsymbol{u}_h^*, \pi_h \varrho_k)| ,$$

where

$$\tilde{a}(\alpha_h - \alpha^m; \cdot, \cdot) = a(\alpha_h; \cdot, \cdot) - a(\alpha^m; \cdot, \cdot).$$

Let any positive ε be given. From (4.10) we conclude that the first term on the right-hand side of (4.13) is less than $\varepsilon/6$ if $h < h_1(\varepsilon, m)$.

To estimate the second term, we first employ for $\varrho_k = (w_k, y_k)$ and $\pi_h \varrho_k = (\pi_h w_k, \pi_h y_k)$ the well-known inequalities

$$\|\pi_h w_k - w_k\|_{1,r,D_h} \leqq \alpha_{\max}^{1/2} \|\pi_h w_k - w_k\|_{1,D_h} \leqq Ch\|w_k\|_{2,\tilde{D}},$$

$$\left(\int_{D_h} (\pi_h w_k - w_k)^2 / r \, dr \, dz\right)^{1/2} \leqq \left(r_{0k}^{-1} \int_{D_h} (\pi_h w_k - w_k)^2 \, dr \, dz\right)^{1/2} \leqq C_k h \|w_k\|_{2,\tilde{D}}$$

where $r_{0k} = \operatorname{dist}(\mathcal{O}, \operatorname{supp} w_k)$.

Consequently, combining these estimates for w_k and y_k, we obtain (cf. (1.1))

$$\|\pi_h \varrho_k - \varrho_k\|_{\mathscr{H}(D_h)} \leqq C_k h (\|w_k\|_{2,\tilde{D}}^2 + \|y_k\|_{2,\tilde{D}}^2)^{1/2}. \tag{4.14}$$

Using Lemma 2.2, (4.8) and (4.14), we arrive at

$$|a(\alpha^m; \mathbf{u}_h^*, \pi_h \varrho_k - \varrho_k)| \leqq C_4 \|\mathbf{u}_h^*\|_{\mathscr{H}(G_m)} \|\pi_h \varrho_k - \varrho_k\|_{\mathscr{H}(G_m)} \leqq$$
$$\leqq C(k)\, h \|\varrho_k\|_{2,\tilde{D}} < \varepsilon/6 \quad \text{for} \quad h < h_2(\varepsilon). \tag{4.15}$$

It remains to estimate the third term. To this end, we realize that

$$\|\pi_h w_k\|_{1,T} \leqq C\|w_k\|_{2,T} \quad \forall h$$

holds for all triangles $T \in \mathscr{T}_h(\alpha_h)$.

Let G_m^h be the smallest union U of triangles $T \in \mathscr{T}_h(\alpha_h)$ such that $D_h \dotminus G_m \subset U$. Obviously, we have

$$\operatorname{meas}(G_m^h) \leqq 1/m + 2h + \|\alpha_h - \alpha\|_{C([0,1])}. \tag{4.16}$$

Consequently, the following estimate holds

$$\|\pi_h w_k\|_{1,D_h \dotminus G_m}^2 \leqq \|\pi_h w_k\|_{1,G_m^h}^2 = \sum_{T \in G_m^h} \|\pi_h w_k\|_{1,T}^2 \leqq C\|w_k\|_{2,G_m^h}^2.$$

Similar estimates are true for $\pi_h y_k$. Using also (4.8), we may therefore write

$$|\tilde{a}(\alpha_h - \alpha^m; \mathbf{u}_h^*, \pi_h \varrho_k)| \leqq$$
$$\leqq C_4 \|\mathbf{u}_h^*\|_{\mathscr{H}(D_h)} \|\pi_h \varrho_k\|_{\mathscr{H}(D_h \dotminus G_m)} \leqq C\|\varrho_k\|_{2,G_m^h}, \tag{4.17}$$

since the norms in $[H^1(D_h \dotminus G_m)]^2$ and $\mathscr{H}(D_h \dotminus G_m)$ are equivalent (for m great enough).

Combining (4.13), (4.15) and (4.17), we derive the following inequality

$$|a(\alpha_h; \mathbf{u}_h^*, \pi_h \varrho_k) - a(\alpha; \mathbf{u}, \varrho_k)| \leqq$$
$$\leqq |a(\alpha_h; \mathbf{u}_h^*, \pi_h \varrho_k) - a(\alpha^m; \mathbf{u}, \varrho_k)| + |\tilde{a}(\alpha - \alpha^m; \mathbf{u}, \varrho_k) \leqq$$
$$\leqq \varepsilon/3 + C\|\varrho_k\|_{2,G_m^h} + C\|\mathbf{u}\|_{\mathscr{H}(D \dotminus G_m)} \|\varrho_k\|_{\mathscr{H}(D \dotminus G_m)}$$

for $h < h_3(\varepsilon, m)$. Making use of (4.16), we conclude that

$$\lim_{h \to 0} a(\alpha_h; \mathbf{u}_h^*, \pi_h \varrho_k) = a(\alpha; \mathbf{u}, \varrho_k) . \tag{4.18}$$

Next we may write, using Lemma 2.3 and (4.14)

$$|L(\alpha_h; \pi_h \varrho_k) - L(\alpha; \varrho_k)| \leqq$$
$$\leqq |L(\alpha_h; \pi_h \varrho_k - \varrho_k)| + |L(\alpha_h; \varrho_k) - L(\alpha; \varrho_k)| = \mathscr{L}_1 + \mathscr{L}_2 ,$$
$$|\mathscr{L}_1| \leqq C_5 \|\pi_h \varrho_k - \varrho_k\|_{\mathscr{H}(D_h)} \leqq Ch \|\varrho_k\|_{2,D} ,$$
$$|\mathscr{L}_2| \leqq \int_{\Delta(D_h, D)} |f_r w_k + f_z y_k| \, r \, dr \, dz + \int_{\Delta(\Gamma_{1h}, \Gamma_1)} |g_r w_k + g_z y_k| \, r \, dr ,$$

where $\Delta(A, B) = (A \dot{-} B) \cup (B \dot{-} A)$ denotes the symmetric difference,

$$\lim_{h \to 0} \text{meas } \Delta(D_h, D) = 0 , \quad \lim_{h \to 0} \text{meas } \Delta(\Gamma_{1h}, \Gamma_1) = 0 .$$

Thus we conclude that

$$\lim_{h \to 0} L(\alpha_h; \pi_h \varrho_k) = L(\alpha; \varrho_k) . \tag{4.19}$$

Passing to the limit with $h \to 0$ in (4.12) and using (4.18), (4.19), we obtain

$$a(\alpha; \mathbf{u}, \omega_k) = L(\alpha; \omega_k)$$

Passing to the limit with $k \to \infty$ and making use of Lemma 2.2, Lemma 2.3 and (4.11), we arrive at

$$a(\alpha; \mathbf{u}, \mathbf{v}) = L(\alpha, \mathbf{v}) .$$

The space $V(G_m)$ is weakly closed in $\mathscr{H}(G_m)$. In fact, $V(G_m)$ is convex and closed by virtue of the continuity of the trace mapping – see Remark 1.1. Since $\mathbf{u}_h^*|_{G_m} \in V(G_m)$, the weak limit $\mathbf{u}|_{G_m} \in V(G_m)$, as well. Passing to the limit with $m \to \infty$, we obtain $\mathbf{u} \in V(D)$. Consequently, $\mathbf{u}$ is the solution of the State Problem (2.4), $\mathbf{u} = \mathbf{u}(\alpha)$. Since $\mathbf{u}(\alpha)$ is unique (see Lemma 2.4), the whole sequence $\{\mathbf{u}_h^*\}$ tends weakly to $\mathbf{u}|_{G_m}$ in $\mathscr{H}(G_m)$.

The estimate

$$\|\mathbf{u}_h^* - \mathbf{u}_h\|_{\mathscr{H}(G_m)} \leqq \|\mathbf{u}_h^* - \mathbf{u}_h\|_{\mathscr{H}(D_h)} \leqq Ch^\lambda \tag{4.20}$$

follows from (4.7). Combining the weak convergence of $\mathbf{u}_h^*$ with (4.20), we arrive at the assertion (4.5). Q.E.D.

For any fixed parameter $h = 1/N$, we define the *Approximate Shape Optimization Problem*:

find $\alpha_h^0 \in \mathscr{U}_{\text{ad}}^h$ *such that*

$$j_i(\alpha_h^0, \mathbf{u}_h(\alpha_h^0)) \leqq j_i(\alpha_h, \mathbf{u}_h(\alpha_h)) \quad \forall \alpha_h \in \mathscr{U}_{\text{ad}}^h , \tag{4.21$_i$}$$

where $i \in \{1, 2, 3, 4\}$ *and* $\mathbf{u}_h(\alpha_h)$ *is the solution of the Approximate State Problem* (4.1).

Proposition 4.2. *The Approximate Shape Optimization Problems have at least one solution for any* $i \in \{1, 2, 3, 4\}$ *and any* $h = 1/N$, $N = 2, 3, \ldots$.

Proof. It is readily seen that

$$\alpha_h \in \mathcal{U}^h_{\text{ad}} \Leftrightarrow \boldsymbol{a} \in \mathcal{A}$$

if $\boldsymbol{a} \in \mathbb{R}^{N+1}$ denotes the vector of $\alpha_h(jh)$, $j = 0, 1, \ldots, N$ and $\mathcal{A}$ is a compact subset of $\mathbb{R}^{N+1}$. One can show that the nodal values of $\boldsymbol{u}_h(\alpha_h)$ depend continuously on $\boldsymbol{a}$. The same assertion can be then verified for $j_i(\alpha_h; \boldsymbol{u}_h(\alpha_h)) \equiv J_i(\boldsymbol{a})$. Consequently, the function $J_i(\boldsymbol{a})$ attains its minimum on the set $\mathcal{A}$.

Proposition 4.3. *Let the assumptions* (4.2), (4.3) *be satisfied. Let* $\{\alpha_h\}$, $h \to 0$, *be a sequence of* $\alpha_h \in \mathcal{U}^h_{\text{ad}}$, *converging to* α *in* $C([0, 1])$. *Then*

$$\lim_{h \to 0} j_i(\alpha_h, \boldsymbol{u}_h(\alpha_h)) = j_i(\alpha, \boldsymbol{u}(\alpha))$$

holds for $i \in \{1, 2, 3\}$, *where* $\boldsymbol{u}_h(\alpha_h)$ *and* $\boldsymbol{u}(\alpha)$ *is the solution of the problem* (4.1) *and* (2.4), *respectively.*

Proof is parallel to that of Proposition 3.1. We replace α_n by α_h, $\boldsymbol{u}_n$ by $\boldsymbol{u}_h$, D_n by D_h, Γ_n by Γ_h, instead of Proposition 2.1 and Lemma 2.3 we make use of Proposition 4.1 and (4.3), respectively. The boundedness of all $\boldsymbol{u}_h$ in $\mathcal{H}(D_h)$ is a consequence of (4.8) and (4.20).

Remark 4.2. The functional j_3 can be replaced by the approximation $L_h(\alpha_h; \boldsymbol{u}_h(\alpha_h))$. Then we employ also the estimate (4.2) and the boundedness of $\boldsymbol{u}_h$ in $\mathcal{H}(D_h)$ to verify the assertion of Proposition 4.3.

Theorem 4.1. *Let the assumptions* (4.2), (4.3) *be satisfied. Let* $\{\alpha_h\}$, $h \to 0$, *be a sequence of solutions of the Approximate Shape Optimization Problem* $(4.21)_i$, $i \in \{1, 2, 3\}$. *Then a subsequence* $\{\alpha_{\tilde{h}}\}$ *exists such that*

$$\alpha_{\tilde{h}} \to \alpha^0 \quad in \quad C([0, 1])\,, \tag{4.22}$$

$$\boldsymbol{u}_{\tilde{h}}(\alpha_{\tilde{h}}) \longrightarrow \boldsymbol{u}(\alpha^0) \quad (weakly) \quad in \quad \mathcal{H}(G_m) \tag{4.23}$$

for any m sufficiently great,
where α^0 *is a solution of the Shape Optimization Problem* $(3.1)_i$, $\boldsymbol{u}_{\tilde{h}}(\alpha_{\tilde{h}})$ *are solutions of the Approximation State Problem* (4.1) *and* $\boldsymbol{u}(\alpha^0)$ *is the solution of the State Problem* (2.4).

The limit of any uniformly convergent subsequence of $\{\alpha_h\}$ *represents a solution of* $(3.1)_i$ *and an analogue of* (4.23) *holds.*

Proof. Since $\mathcal{U}_{\text{ad}}$ is compact in $C([0, 1])$, a subsequence $\{\alpha_{\tilde{h}}\}$ exists such that (4.22) holds, with $\alpha^0 \in \mathcal{U}_{\text{ad}}$.

Let any $\alpha \in \mathcal{U}_{\text{ad}}$ be given. There exists a sequence $\{\beta_h\}$, $h \to 0$, $\beta_h \in \mathcal{U}^h_{\text{ad}}$, such that β^q tends to α in $C([0, 1])$. (This follows from Appendix in [2]). We have

$$j_i(\alpha_{\hat{h}}, \mathbf{u}_{\hat{h}}(\alpha_{\hat{h}})) \leqq j_i(\beta_{\hat{h}}, \mathbf{u}_{\hat{h}}(\beta_{\hat{h}})) \quad \forall \hat{h} ,$$

by definition. Passing to the limit with $\hat{h} \to 0$ and using Proposition 4.3 on both sides, we obtain

$$j_i(\alpha^0, \mathbf{u}(\alpha^0)) \leqq j_i(\alpha, \mathbf{u}(\alpha)) .$$

Consequently, α^0 is a solution of the problem $(3.1)_i$. The convergence (4.23) follows from Proposition 4.1. The rest of the theorem is obvious.

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Souhrn

OPTIMALIZACE TVARU OSOVĚ SYMETRICKÝCH PRUŽNÝCH TĚLES

IVAN HLAVÁČEK

Uvažuje se osově symetrická úloha teorie pružnosti s kombinovanými okrajovými podmínkami. Je třeba nalézt část hranice osového řezu tělesa tak, aby minimalizovala jeden ze čtyř typů účelového funkcionálu. Dokazuje se existence optimální hranice a konvergence přibližných, po částech lineárních řešení.

Резюме

ОПТИМИЗАЦИЯ ФОРМЫ УПРУГИХ ОСЕСИММЕТРИЧЕСКИХ ТЕЛ

IVAN HLAVÁČEK

Рассматривается осесимметрическая задача теории упругости со смешанными краевыми условиями. Требуется найти часть границы меридионального сечения области так, чтобы минимизировать один из четырех типов целевого функционала. Доказывается существование оптимальной границы и сходимость приближенных, кусочно-линейных решений.

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