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THE NUMBER OF BUCKLED STATES OF CIRCULAR PLATES

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Summary. This paper deals with the exact number of solutions of von Kármán equations for a rotationally symmetric buckling of a thin elastic plate. The plate of constant thickness is in static equilibrium under a uniform compressive thrust applied along its edge in the plane of the plate. The theory of M. G. Crandall, P. H. Rabinowitz [4], is used and the theory of M. S. Berger [1], [3] and M. S. Berger and P. C. Fife [2] is adapted. This work is a part of [6].

Keywords: Fredholm map of index zero, proper, singular point, bifurcation.

AMS Classification: 34B15, 58F14, 73C50.

1. INTRODUCTION

We deal with the von Kármán equations of the plate theory. We shall prove that for the parameter values $\lambda > \lambda_n$ the problem of rotationally symmetric buckling of a circular plate has exactly n pairs of rotationally symmetric solutions lying on smooth branches, and the nodal structure of solutions is preserved along the bifurcation branches. In this paper we treat only the case when the edge of the plate is clamped. All considerations remain valid for the case of a simply supported plate provided minor changes are made in formulations and considerations which follow.

The rotationally symmetric buckling of a circular plate was previously investigated by many authors. The review of these results can be found in the paper by J. H. Wolkowisky [9]. Using the Schauder fixed point theorem he proved that for the parameter values $\lambda > \lambda_n$ there exist at least n pairs of nontrivial solutions $(\lambda, \pm u_j)$, $j = 1, 2, \ldots, n$ where u_j has exactly $j - 1$ internal simple zero points. His paper, however, does not tell anything about the exact number of solutions and the continuity of the bifurcation branches he found.

2. FORMULATION OF THE BOUNDARY VALUE PROBLEM

We consider a thin elastic plate having a radius c, a thickness h, Young's modulus E, and the Poisson ratio μ. The plate is subjected to a uniform edge thrust p applied in the midplane of the plate. The case $p < 0$ is a tension, the case $p > 0$ is a compression of the plate. We denote by $W(r)$ the normal displacement of the unstrained midplane and by $F(r)$ the Airy stress function. Then the rotationally symmetric deformations of the plate are described by the von Kármán equations [8]

$$D \frac{\mathrm{d}}{\mathrm{d}r} (\Delta W) = \frac{h}{r} \frac{\mathrm{d}F}{\mathrm{d}r} \frac{\mathrm{d}W}{\mathrm{d}r} \quad r \in (0, c) , \tag{2.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}r} (\Delta F) = - \frac{E}{2r} \left(\frac{\mathrm{d}W}{\mathrm{d}r} \right)^2 \quad r \in (0, c) \tag{2.2}$$

with the boundary conditions

$$W|_{r=c} = 0 , \tag{2.3}$$

$$F|_{r=c} = 0 , \tag{2.4}$$

$$\left. \frac{\mathrm{d}W}{\mathrm{d}r} \right|_{r=c} = 0 , \tag{2.5}$$

$$\left. \frac{1}{r} \frac{\mathrm{d}F}{\mathrm{d}r} \right|_{r=c} = -p , \tag{2.6}$$

$$\left. \frac{\mathrm{d}W}{\mathrm{d}r} \right|_{r=0} = 0 , \tag{2.7}$$

$$\left. \frac{\mathrm{d}F}{\mathrm{d}r} \right|_{r=0} = 0 . \tag{2.8}$$

The equations (2.1), (2.2) with the boundary conditions (2.3)–(2.8) form the problem of the rotationally symmetric buckling of a circular plate clamped at the edge $r = c$. If the condition (2.5) is replaced by the condition

$$\left. \left(\frac{\mathrm{d}^2 W}{\mathrm{d}r^2} + \frac{\mu}{r} \frac{\mathrm{d}W}{\mathrm{d}r} \right) \right|_{r=c} = 0 \tag{2.9}$$

then we consider the simply supported plate problem. Here

$$D = \frac{Eh}{12(1 - \mu^2)}$$

is the flexural rigidity of the plate. We introduce the quantities

$$r = xc , \tag{2.10}$$

$$u(x) = \left(\frac{Eh}{2D} \right)^{1/2} \frac{\mathrm{d}w(xc)}{\mathrm{d}x} ,$$

$$v(x) = \frac{h}{D} \frac{\mathrm{d}f(xc)}{\mathrm{d}x} ,$$

$$\lambda = \frac{c^2 ph}{D},$$

$$g(x) = v(x) + \lambda x ,$$

$$u(x) = x\, w(x) , \quad g(x) = x f(x) .$$

The equations (2.1), (2.2) can be reduced to the form

$$(x^3 w'(x))' + \lambda x^3 w(x) = x^3 w(x) f(x) \quad x \in (0, 1) , \tag{2.11}$$

$$(x^3 f'(x))' = -x^3 w^2(x) \quad x \in (0, 1) \tag{2.12}$$

where $\lambda \in \mathbb{R}$, with the following boundary conditions for the clamped plate case:

$$\lim_{x \to 0^+} x\, w(x) = 0 , \quad w(1) = 0 , \tag{2.13}$$

$$\lim_{x \to 0^+} x f(x) = 0 , \quad f(1) = 0 , \tag{2.14}$$

the second condition in (2.13) being replaced by the condition

$$w'(1) + (\mu + 1)\, w(1) = 0 , \tag{2.15}$$

the boundary conditions for the simply supported plate case. From now on we denote the problem of solving the equations (2.11), (2.12) with the boundary conditions (2.13), (2.14) as the (CP) problem, and we will make all considerations only for the (CP) problem.

3. CLASSICAL AND GENERALIZED SOLUTIONS AND OPERATOR FORMULATION OF (CP) PROBLEM

Definition 3.1. *The classical solution of the problem (CP) is a pair of functions $w(x)$, $f(x)$ with the following properties:*

a) $w(x), f(x) \in C^2((0, 1)) \cap C((0, 1])$;

b) $w(x), f(x)$ *satisfy* (2.11), (2.12) *pointwise for a real parameter* λ;

c) $w(x)$ *satisfies* (2.13), $f(x)$ *satisfies* (2.14).

Let $W^{1,2}((0, 1), x^3)$ be the real Sobolev space with the weight x^3, the inner product

$$(u, v)_{1,2,3} = \int_0^1 x^3 u'(x)\, v'(x)\, \mathrm{d}x + \int_0^1 x^3 u(x)\, v(x)\, \mathrm{d}x$$

and the corresponding norm

$$\|u\|_{1,2,3} = [(u, u)_{1,2,3}]^{1/2} . \tag{3.1}$$

We denote

$$M = \{u \in C^\infty([0, 1]), u(1) = 0\} ,$$

and introduce a real Hilbert space V defined as the closure of the set M in the norm

(3.1). Then V consists of functions from $W^{1,2}((0, 1), x^3)$ which vanish at $x = 1$. A more convenient norm and inner product for the space V can be obtained in the following way. The bilinear form

$$(u, v) = \int_0^1 x^3 u'(x) v'(x) \,\mathrm{d}x \tag{3.2}$$

can be used as the inner product on V, and throughout this paper the corresponding norm will be denoted by $\|\cdot\|$. The fact that $\|\cdot\|$ is a norm on V follows from the inequality

$$\|u\| \leqq \|u\|_{1,2,3} \leqq K_1 \|u\| . \tag{3.3}$$

The first inequality in (3.3) is obvious, the second is based on Hardy's inequality [5].

Let φ, ψ be smooth functions in V. Then from (2.11), (2.12), integrating by parts over (0, 1), one obtains

$$\int_0^1 x^3 w'(x) \varphi'(x) \,\mathrm{d}x - \lambda \int_0^1 x^3 w(x) \varphi(x) \,\mathrm{d}x + \int_0^1 x^3 w(x) f(x) \varphi(x) \,\mathrm{d}x = 0 , \tag{3.4}$$

$$\int_0^1 x^3 f'(x) \psi'(x) \,\mathrm{d}x = \int_0^1 x^3 w^2(x) \psi(x) \,\mathrm{d}x . \tag{3.5}$$

Definition 3.2. *The generalized solution of the problem* (CP) *is a pair of functions* w, f *in* V *satisfying* (3.4) *and* (3.5) *for all functions* φ, ψ *in* V.

Theorem 3.1. *Any classical solution of the problem* (*CP*) *is a generalized solution. Conversely, any generalized solution of the problem* (*CP*) *is a classical solution.*

Proof. The first assertion is obvious. We prove the second assertion. Let w, f from V be the generalized solution of the problem (CP). From (3.4) and (3.5) we obtain

$$\int_0^1 \left[x^3 w'(x) + \lambda \int_0^x t^3 w(t) \,\mathrm{d}t - \int_0^x t^3 w(t) f(t) \,\mathrm{d}t\right] \varphi'(x) \,\mathrm{d}x = 0 , \tag{3.6}$$

$$\int_0^1 \left[x^3 f'(x) + \int_0^x t^3 w^2(t) \,\mathrm{d}t\right] \psi'(x) \,\mathrm{d}x = 0 . \tag{3.7}$$

We denote

$$p_1(x) = x^3 w'(x) + \lambda \int_0^x t^3 w(t) \,\mathrm{d}t - \int_0^x t^3 w(t) f(t) \,\mathrm{d}t ,$$
$$p_2(x) = x^3 f'(x) + \int_0^x t^3 w^2(t) \,\mathrm{d}t$$

and set

$$\varphi(x) = \int_0^x (p_1(t) - c_1) \,\mathrm{d}t , \quad c_1 = \int_0^1 p_1(t) \,\mathrm{d}t ,$$
$$\psi(x) = \int_0^x (p_2(t) - c_2) \,\mathrm{d}t , \quad c_2 = \int_0^1 p_2(t) \,\mathrm{d}t .$$

Then (3.6) and (3.7) imply

$$x^3 w'(x) = -\lambda \int_0^x t^3 w(t) \,\mathrm{d}t + \int_0^x t^3 w(t) f(t) \,\mathrm{d}t + c_1 ,$$
$$x^3 f'(x) = -\int_0^x t^3 w^2(t) \,\mathrm{d}t + c_2$$

almost everywhere in $[0, 1]$. Then $h(x) = x^3 w'(x)$ and $g(x) = x^3 f'(x)$ are continuous functions on $[0, 1]$. We assert: $c_i = 0$, $i = 1, 2$. If this were not true, then

$$h(0) = \lim_{x\to 0^+} h(x) = \lim_{x\to 0^+} x^3 w'(x) = c_1 \neq 0 . \tag{3.8}$$

Without loss of generality we can suppose $c_1 > 0$. In virtue of (3.8) we have: for every $\varepsilon > 0$ there exists $\delta > 0$ such that all x, $0 < x < \delta$ satisfy

$$0 < \frac{c_1 - \varepsilon}{x^3} < w'(x) < \frac{c_1 + \varepsilon}{x^3} .$$

As $w \in V$, we have

$$+\infty > \|w\|^2 = \int_0^1 x^3 w'^2(x)\,dx \geqq \int_0^\delta x^3 w'^2(x)dx > (c_1 - \varepsilon)^2 \int_0^\delta \frac{1}{x^3}\,dx ,$$

a contradiction. The same holds for c_2. We have

$$x^3 w'(x) = -\lambda \int_0^x t^3 w(t)\,dt + \int_0^x t^3 w(t) f(t)\,dt \tag{3.9}$$

$$x^3 f'(x) = -\int_0^x t^3 w^2(t)\,dt \tag{3.10}$$

The functions $h(x) = x^3 w'(x)$ and $g(x) = x^3 f'(x)$ are continuous on $[0, 1]$, so

$$w(x) = \int_1^x w'(t)\,dt , \quad f(x) = \int_1^x f'(t)\,dt$$

are continuous functions on $(0, 1]$. We assert:

$$\lim_{x\to 0^+} x\,w(x) = 0 , \quad \lim_{x\to 0^+} x f(x) = 0 . \tag{3.11}$$

In virtue of (3.10) the function $f(x)$ is a nonincreasing continuous function on $(0, 1]$ and we have $f(x) \geqq 0$ in $(0, 1]$. Then (3.10) implies

$$f'(x) = -\frac{1}{x^3}\int_0^x t^3 w^2(t)\,dt \quad \text{in} \quad (0, 1] .$$

We can estimate

$$0 \leqq f(x) = \int_1^x f'(t)\,dt = -\int_1^x \frac{1}{t^3}\left(\int_0^t s^3 w^2(s)\,ds\right) dt \leqq$$

$$\leqq \int_x^1 \frac{1}{t}\left(\int_0^t s\,w^2(s)\,ds\right) dt \leqq \int_x^1 \frac{1}{t}\left(\int_0^1 s\,w^2(s)\,ds\right) dt \leqq -\ln x \|w\|^2$$

and we have

$$0 \leqq x f(x) \leqq -x \ln x \,\|w\|^2 . \tag{3.12}$$

From (3.12) we obtain

$$\lim_{x\to 0^+} x f(x) = 0 .$$

Similarly, (3.9) implies

$$w'(x) = -\frac{\lambda}{x^3}\int_0^x t^3\, w(t)\, dt + \frac{1}{x^3}\int_0^x t^3\, w(t) f(t)\, dt$$

and we have

$$w(x) = \lambda \int_x^1 \frac{1}{t^3}\left(\int_0^t s^3\, w(s)\, ds\right) dt - \int_x^1 \frac{1}{t^3}\left(\int_0^t s^3\, w(s) f(s)\, ds\right) dt\,,$$

which yields an estimate

$$|w(x)| \leqq \frac{|\lambda|}{2}\, \|w\|\,(1 - x) - \|w\|\, \|f\| \ln x\,.$$

Thus, we obtain the first part of the assertion (3.11). Using the above estimates and the relations (3.9), (3.10) we obtain that the functions $w(x)$, $f(x)$ are bounded. Hence, differentiating (3.9) and (3.10) we obtain the desired result. Q.E.D.

We consider the Hilbert space V with the norm $\|\cdot\|$. Let u and v in V be fixed. For arbitrary $\varphi \in V$ we define the functionals

$$c(u, \cdot): V \to R\,, \quad c(u, \varphi) = \textstyle\int_0^1 x^3\, u(x)\, \varphi(x)\, dx\,;$$

$$b(u, v, \cdot): V \to R\,, \quad b(u, v, \varphi) = \textstyle\int_0^1 x^3\, u(x)\, v(x)\, dx\,.$$

The functionals b and c are continuous linear functionals on V. Here we use the fact that the space $W^{1,2}((0, 1), x^3)$ is continuously imbedded in $L_2((0, 1), x^{1+\varkappa})$, in $L_2((0, 1), x^{5-\varkappa})$ and in $L_4((0, 1), x^{5-\varkappa})$ for $\varkappa \in (0, 1)$, where $L_p((0, 1), x^\alpha)$ is the Lebesgue space L_p with the weight x^α. The Riesz representation theorem implies that there exists a unique $Lu \in V$ and a unique $M(u, v) \in V$ such that

$$(Lu, \varphi) = c(u, \varphi)\,, \tag{3.13}$$

$$(M(u, v), \varphi) = b(u, v, \varphi)\,. \tag{3.14}$$

Using the fact that $W^{1,2}((0, 1), x^3)$ is compactly embedded in $L_2((0, 1), x^{1+\varkappa})$ we obtain the following theorem:

Theorem 3.2. *The generalized solutions of the problem (CP) are identical with the solutions of the operator equations of the form*

$$w - \lambda L w + N(w) = 0\,, \tag{3.15}$$

$$f = M(w, w) \tag{3.16}$$

defined on the Hilbert space V, M is a bounded bilinear symmetric compact operator defined on $V \times V$ with range in V, L is a bounded linear selfadjoint compact operator mapping V into itself, N a bounded compact nonlinear operator mapping V into itself. Here

$$N(w) = D(w, w, w) \tag{3.17}$$

where

$$(3.18) \qquad D(u, v, z) = \tfrac{1}{3}[M(u, M(v, z)) + M(v, M(z, u)) + M(z, M(u, v))]$$

is a bounded trilinear symmetric compact operator defined on $V \times V \times V$ with range in V.

Now we show some properties of the operators M and N.

Lemma 3.1. *Let $u, v, z \in V$, then the form $(M(u, v), z)$ is symmetric in u, v, z.*

Proof is obvious from (3.14) and the definition of the functional b.

Lemma 3.2. *The equality*

$$M(v, v) = 0$$

in the Hilbert space V holds if and only if $v = 0$ in V.

Proof is straightforward.

Lemma 3.3. *The nonlinear operator N defined by (3.17) has the following properties:*

(i) *N is a continuous cubic operator, satisfying for every $\alpha \in \mathbb{R}$ and $v \in V$*

$$N(\alpha v) = \alpha^3 N(v) ,$$

$$\|N(v)\| \leqq \|M\|^2 \|v\|^3 .$$

(ii) If *we denote*

$$(3.19) \qquad j(v) = \tfrac{1}{4}(N(v), v) = \tfrac{1}{4}\|M(v, v)\|^2$$

then j is a functional defined on V. For $0 \neq v \in V$ we have

$$(3.20) \qquad j(v) > 0 , \quad j(\alpha v) = \alpha^4 j(v) , \quad j(0) = 0$$

and the functional j is infinitely Frèchet differentiable on V, the operator N is the gradient of j with respect to the inner product (3.2), i.e., for $v \in V$ and for every $h \in V$ we have

$$(3.21) \qquad j'(v)(h) = (N(v), h)$$

where $j'(v)(h)$ is the Frèchet derivative at the point v in the direction h.

Proof. The continuity of N follows from the continuity of M, the homogeneity of the degree three is obvious. Using Lemmas 3.1 and 3.2 we obtain all assertions of the lemma. Q.E.D.

We denote the left hand side of (3.15) by F_λ, i.e., F_λ is the mapping defined on V with the range in V,

$$(3.22) \qquad F_\lambda : V \to V ,$$

$$F_\lambda(u) = u - \lambda L u + N(u) .$$

Theorem 3.3. *Let $\lambda \in \mathbb{R}$ be fixed. The mapping F_λ is a nonlinear Fredholm operator of index zero.*

Proof. The mapping F_λ is infinitely Frèchet differentiable and we have

$$F'_\lambda(u)(v_1) = v_1 - \lambda L v_1 + 3D(u, u, v_1), \tag{3.23}$$

$$F''_\lambda(u)(v_1, v_2) = 6D(u, v_1, v_2), \tag{3.24}$$

$$F'''_\lambda(u)(v_1, v_2, v_3) = 6D(v_1, v_2, v_3), \tag{3.25}$$

$$F^{(n)}_\lambda(u)(v_1, v_2, \ldots, v_n) = 0 \quad \text{for} \quad n \geqq 4 \tag{3.26}$$

where $F^{(n)}_\lambda(u)(v_1, \ldots, v_n)$ is the n-th Frèchet derivative of F_λ at u in the directions $v_1, \ldots, v_n$. It is easy to see that $F'_\lambda(u)(\cdot)$ is a selfadjoint operator on V, it is a compact perturbation of the identity. Hence $F'_\lambda(u)(\cdot)$ is a linear Fredholm operator of index zero, so F_λ is a nonlinear Fredholm operator of index zero. Q.E.D.

Theorem 3.4. *Let $u, v \in V$ be arbitrary, then*

$$(M(u, u), M(v, v)) \geqq 0$$

and the equality holds only if $u = 0$ or $v = 0$ in V.

Proof. Let $u, v \in V$, then there exist unique $g, h \in V$ such that

$$g = M(u, u), \quad h = M(v, v).$$

Arguing similarly as in the proof of Theorem 3.1 we obtain

$$x^3 g'(x) = -\int_0^x t^3 u^2(t)\, dt, \quad x^3 h'(x) = -\int_0^x t^3 v^2(t)\, dt$$

which means

$$g'(x) \leqq 0 \quad \text{and} \quad h'(x) \leqq 0 \quad \text{for} \quad x \in (0, 1),$$

and the equality holds only if $u = 0$ or $v = 0$ in V. Hence

$$(M(u, u), M(v, v)) = (g, h) = \int_0^1 x^3 g'(x) h'(x)\, dx \geqq 0.$$

Q.E.D.

Theorem 3.5. *Let $u, v \in V$ be arbitrary, then*

$$(N'(u)(v), v) \geqq 0$$

and the equality holds only if $u = 0$ or $v = 0$ in V.

Proof. We obtain

$$(N'(u)(v), v) = 3(D(u, u, v), v) = 2(M(u, M(u, v)), v) + \\ + (M(v, M(u, u)), v) = 2\|M(u, v)\|^2 + (M(u, u), M(v, v))$$

and the assertion follows immediately from Theorem 3.4. Q.E.D.

We reformulate our problem (CP). We define a mapping

$$(4.1) \qquad F: V \times \mathbb{R} \to V,$$

$$F(u, \lambda) = u - \lambda Lu + N(u) = F_\lambda(u)\,.$$

Lemma 4.1. *The nontrivial solutions of the problem (CP) appear in pairs, that is, if $u \in V$ is a nontrivial solution then $-u \in V$ is a nontrivial solution for some $\lambda \in \mathbb{R}$. The trivial solution $u = 0$ is a solution for each $\lambda \in \mathbb{R}$.*

Proof. From the properties of the operators L, N it is easy to see that the assertions of the lemma hold. Q.E.D.

We shall study the behaviour of the partial Frèchet derivative of the mapping F at the point $(0, \lambda)$ for $\lambda \in \mathbb{R}$. We have

$$(4.2) \qquad F'_{|u}(0, \lambda)(v) = v - \lambda Lv + 3D(0, 0, v) = v - \lambda Lv$$

so we must analyze the problem

$$(4.3) \qquad v - \lambda Lv = 0 \quad \text{in the space } V\,.$$

By virtue of Theorem 3.2, L is a linear selfadjoint continuous positive operator, therefore the classical theory of linear compact selfadjoint operators yields

Lemma 4.2. *The problem (4.3) has nontrivial solutions for a countable infinite number of real numbers*

$$0 < \lambda_1 \leqq \lambda_2 \leqq \dots \leqq \lambda_n \leqq \dots,$$

whose only limit point is at infinity. Every eigenvalue λ_n has finite multiplicity and the orthogonal system of eigenfunctions is complete in the Hilbert space V.

Lemma 4.3. *Any eigenvalue of the eigenvalue problem (4.3) is simple.*

Proof. By Theorem 3.1, any generalized solution of the problem (4.3) is a classical solution of the problem

$$(4.4) \qquad (x^3 v'(x))' + \lambda x^3 v(x) = 0 \quad \text{in} \quad (0, 1)\,,$$

$$\lim_{x \to 0^+} x\, v(x) = 0\,, \quad v(1) = 0\,.$$

The problem (4.4) is equivalent to the eigenvalue problem

$$(4.5) \qquad w''(x) + 1/x\, w'(x) + (\lambda - 1/x)\, w(x) = 0 \quad x \in (0, 1)\,,$$

$$w(0) = w(1) = 0$$

with the solutions

$$w_n(x) = c_n J_1(\sqrt{\lambda_n} x)$$

where c_n is a constant and the eigenvalue λ_n is the root of the equation

$$J_1(\sqrt{\lambda_n}) = 0$$

where J_1 is the Bessel function of the first order. It means, that any eigenvalue of the problem (4.3) is simple. Then $v_n(x) = (c_n/x)\, J_1(\sqrt{\lambda_n} x)$ are eigenfunctions of the problem (4.4). Q.E.D.

5. EXISTENCE OF SOLUTIONS AND THE LOCAL BIFURCATION THEORY OF PROBLEM (CP)

We define the functional

$$A : V \to R\,, \tag{5.1}$$

$$A(u) = \|u\|^2 - \lambda(Lu, u) + 2\,j(u)$$

where the functional j is defined by (3.19).

Theorem 5.1. *The equation $u - \lambda Lu + N(u) = 0$ has at least one solution in the Hilbert space V, i.e., there exists at least one element $u_0 \in V$ such that*

$$A(u_0) \leqq A(u)$$

for any $u \in V$. If the parameter value $\lambda > \lambda_1$, the equation has at least one non-trivial solution in the space V. λ_1 is the first eigenvalue of the linearized problem (4.3).

Proof. We shall show that A is sequentially weakly lower semicontinuous and coercive. We easily see that A is weakly lower semicontinuous. A is also coercive. If it were not true, there would exist a sequence $\{u_n\}$ with the following properties: $\lim_{n \to \infty} \|u_n\| = \infty$ and the sequence $\{A(u_n)\}$ is bounded, i.e. $A(u_n) \leqq K$ where K is a positive constant. Hence

$$\|u_n\|^2 - \lambda(Lu_n, u_n) + 2\,j(u_n) \leqq K\,, \tag{5.2}$$

$n = 1, 2, \ldots$. We can suppose $u_n \neq 0$ for all n. We set

$$w_n = \frac{u_n}{\|u_n\|}\,.$$

From (5.2) we obtain

$$1 - \lambda(Lw_n, w_n) + 2\|u_n\|^2\, j(w_n) \leqq \frac{K}{\|u_n\|^2}\,. \tag{5.3}$$

We can choose from $\{w_n\}$ a weakly convergent subsequence, which we denote again by $\{w_n\}$ for simplicity, such that

$$w_n \rightharpoonup w \quad \text{weakly in} \quad V\,.$$

If n in (5.3) tends to infinity, then for the limit element w we have

$$j(w) = 0 \quad \text{in} \quad V.$$

So $w = 0$ by Lemma 3.2. As $j(w_n) \geqq 0$, (5.3) implies

$$1 - \lambda(Lw_n, w_n) \leqq \frac{K}{\|u_n\|^2}. \tag{5.4}$$

Then letting n tend to infinity we obtain a contradiction, which means the functional A is coercive. To prove the second assertion we express the second variation of the functional A at the point 0 in the direction v_n, where v_n is the n-th eigenvalue of the problem (4.3). We have

$$\delta^2 A(0)(v_n) = \frac{d^2}{dt^2} A(tv_n)|_{t=0} = 2\{\|v_n\|^2 - \lambda(Lv_n, v_n)\} =$$

$$= 2\left(1 - \frac{\lambda}{\lambda_n}\right)\|v_n\|^2 < 0 \quad \text{for} \quad \lambda > \lambda_n. \qquad \text{Q.E.D.}$$

Theorem 5.2. *Let $f(x, \lambda)$ be a C^p mapping, $p \geqq 0$, of a neighbourhood of the point $(0, \lambda_0)$ from $X \times \mathbb{R}$ to Y (X, Y are Banach spaces) and let $f(0, \lambda_0) = 0$. We suppose:*

(i) $f'_{|\lambda}(0, \lambda_0) = 0$ (*$f'_{|\lambda}$ is the partial Frèchet derivative with respect to λ*);

(ii) $\operatorname{Ker} f'_{|x}(0, \lambda_0)$ *is the one-dimensional subspace generated by x_0*;

(iii) $R f'_{|x}(0, \lambda_0) = Y_1$ *has codimension* 1 ($Rf'_{|x}(0, \lambda_0)$ is *the range of* $f'_{|x}(0, \lambda_0)$);

(iv) $f''_{|\lambda\lambda}(0, \lambda_0) \in Y_1$ *and* $f''_{|x\lambda}(0, \lambda_0)\, x_0 \notin Y_1$.

Then the point $(0, \lambda_0)$ is a bifurcation point of the mapping f. Moreover, the set of solutions of the equation $f(x, \lambda) = 0$ in a small neighbourhood of the bifurcation point consists of two C^{p-2} curves Γ_1 and Γ_2 which intersect only at the point $(0, \lambda_0)$, and we can parametrize the former by

$$\Gamma_1\ (x(\lambda), \lambda) \quad |\lambda - \lambda_0| \leqq \varepsilon,$$

while the curve Γ_2 can be parametrized by a variable s, $|s| \leqq \varepsilon$, as

$$\Gamma_2\colon (sx_0 + x_2(s), \lambda(s)), \quad \text{where} \quad x_2(0) = x_2'(0) = 0,$$

$$\lambda(0) = \lambda_0, \quad x_2 \in Y_1.$$

Supplement. If the mapping f fulfils the condition $f(0, \lambda) \equiv 0$ for all $\lambda \in \mathbb{R}$, the curve Γ_1 coincides with the axis λ.

Proof can be found in [7].
Setting $f = F$ defined by (4.1) in Theorem 5.2 we obtain

Theorem 5.3. *Let λ_n be an eigenvalue of the problem (4.3). Then for any $n = 1, 2, \ldots$, the point $(0, \lambda_n)$ is a bifurcation point of the problem (CP) and the set* *of its solutions in a small neighbourhood of the point* $(0, \lambda_n)$ *consists of two analytic curves* Γ_1 *and* Γ_2 *which intersect only at the point* $(0, \lambda_n)$. *The curve* Γ_1 *coincides with the* λ*-axis, it is the trivial solution, while the curve* Γ_2 *can be parametrized by a real variable* $|s| \leqq \varepsilon$:

$$\Gamma_2 : (sv_n + v_{2n}(s), \lambda(s)),$$

where v_n *is the n-th eigenvalue of the problem* (4.3) *and* $v_{2n}(0) = v'_{2n}(0) = 0$, $\lambda(0) = \lambda_n$ *and* $v_{2n} \in R(I - \lambda_n L)$.

Proof. We can easily verify that all assumptions of Theorem 5.2 are fulfilled by $f = F$.

Now we study the behaviour of the curves Γ_1, Γ_2 in a neighbourhood of the bifurcation point $(0, \lambda_n)$. The curve Γ_1 coincides with the λ-axis, as it is the trivial solution

$$\Gamma_1 : (0, \lambda) \quad |\lambda - \lambda_n| \leqq \varepsilon .$$

We know from Lemma 4.1 that the trivial solution exists for every $\lambda \in \mathbb{R}$. The curve Γ_2 is an analytical curve parametrized by $|s| \leqq \varepsilon$:

$$\Gamma_2 : (u(s), \lambda(s)) = (sv_n + v_{2n}(s), \lambda(s)) .$$

We set

$$\lambda(s) - \lambda_n = \mu(s) .$$

From the equation

$$(I - \lambda_n L)\, u(s) = \mu(s)\, Lu(s) - N(u(s))$$

using the facts that $v_n \in \operatorname{Ker}(I - \lambda_n L)$, $v_{2n} \in R(I - \lambda_n L)$ we obtain by easy computation

$$\mu(0) = \mu'(0) = 0 , \quad \mu''(0) > 0 . \tag{5.5}$$

Lemma 5.1. *There exists* $\delta > 0$ *such that*

(i) *if* $\lambda \in (\lambda_n - \delta, \lambda_n)$ *there exists only the trivial solution of the problem* (*CP*);

(ii) *if* $\lambda \in (\lambda_n, \lambda_n + \delta)$ *there are exactly three solutions of the problem* (*CP*), *one trivial, two nontrivial* u, $-u$ *lying on the curve* Γ_2 *which differ only by the sign.*

Proof follows immediately from (5.5).

6. THE EXACT NUMBER OF GLOBAL SOLUTIONS

Definition 6.1. *Let* $F: X \to Y$ *be a mapping from the Banach space* X *into the Banach space* Y. *If* $K \subset Y$ *is compact implies* $F^{-1}(K) \subset X$ *is compact, then* F *is called a proper mapping. Here* $F^{-1}(K) = \{x \in X \mid F(x) \in K\}$.

Theorem 6.1. *Let* $\lambda \in \mathbb{R}$ *be fixed. A mapping* $F_\lambda: V \to V$, $F_\lambda(u) = u - \lambda Lu + N(u)$ *is a proper mapping of the Hilbert space* V *into itself.*

Proof. Let $K \subset V$ be compact. Let $\{u_n\} \subset F_\lambda^{-1}(K)$ be such that

$$F_\lambda(u_n) = g_n \in K\,. \tag{6.1}$$

Compactness of K implies boundedness of the sequence $\{g_n\}$. Suppose the sequence $\{u_n\}$ is not bounded, i.e.

$$\lim_{n\to\infty} \|u_n\| = \infty\,.$$

Rewriting (6.1) we have

$$u_n - \lambda L u_n + N(u_n) = g_n\,. \tag{6.2}$$

Taking the inner product of (6.2) with the element u_n we obtain

$$\|u_n\|^2 - \lambda(Lu_n, u_n) + (N(u_n), u_n) = (g_n, u_n) \leqq \|g_n\|\,\|u_n\|\,.$$

The proof of boundedness of the sequence $\{u_n\}\}$ is the same, but for small changes, as the proof of coercivity of the functional A in the proof of Theorem 5.1. So, the sequence $\{u_n\}$ is bounded and we can choose a subsequence (which we denote u_n again) such that

$$u_n \rightharpoonup u \quad \text{weakly} \quad \text{in} \quad V.$$

Then

$$Lu_n \to Lu\,, \quad N(u_n) \to N(u) \quad \text{strongly in } V.$$

Since $\{g_n\} \subset K$, we can suppose $g_n \to g$ in K. Then $u_n = \lambda L u_n - N(u_n) + g_n \to \to \lambda Lu - N(u) + g$ strongly. It means $F_\lambda^{-1}(K)$ is relatively compact. The mapping F_λ is continuous, so $F_\lambda^{-1}(K)$ is closed, hence $F_\lambda^{-1}(K)$ is compact. Q.E.D.

Now we consider the extended mapping

$$\mathscr{F}: V \times \mathbb{R} \to V \times \mathbb{R}\,, \tag{6.3}$$

$$\mathscr{F}(u, \lambda) = (F_\lambda(u), \lambda)\,.$$

The mapping F_λ is a C^∞-Fredholm operator of index zero and consequently, $\mathscr{F}(u, \lambda)$ is a C^∞-Fredholm operator of index zero.

Theorem 6.2. *The mapping $\mathscr{F}$ defined above is a proper mapping of $V \times \mathbb{R}$ into itself.*

Proof is the same as the proof of Theorem 6.1, with only small modifications.

Definition 6.2. *Let X, Y be real Banach spaces and $\Omega \subset X$ an open set. Let $f\colon \Omega \to Y$ be a C^1-mapping.* A point *$x_0 \in \Omega$ is called a singular point of the mapping f if the Frèchet derivative $f'(x_0)$ is not an isomorphism of X onto Y. The image of the singular point x_0 in Y, i.e., $f(x_0) \in Y$, is called the singular value.*

We denote

$$I_n = (\lambda_1, \lambda_2) \cup (\lambda_2, \lambda_3) \cup \ldots \cup (\lambda_{n-1}, \lambda_n)$$

where $n \geqq 2$ and λ_k is an eigenvalue of the problem (4.3).

Theorem 6.3. *We consider the mapping $F_\lambda: V \to V$, $F_\lambda(u) = g$, $F_\lambda(u) = u - \lambda Lu + N(u)$. The point $g = 0$ is not a singular value of the mapping F_λ for $\lambda \in I_n$. Moreover, the point $g = 0$ is a singular value of F_λ for $\lambda = \lambda_n$ with the single singular point $u = 0$.*

Proof. Suppose $g = 0$ is a singular value of F_λ for some $\lambda \in I_n$. Then there exists an element $0 \neq h \in V$ and a singular point $0 \neq u \in V$ such that

$$h + N'(u)\,h = \lambda Lh \quad \text{for} \quad u \in F_\lambda^{-1}(0)\,, \tag{6.4}$$

as $u = 0$ cannot be a singular point of F_λ for $\lambda \in I_n$. In virtue of (6.4) λ is an eigenvalue of the eigenvalue problem (6.4), i.e.,

$$\lambda = \lambda_k = \inf_{V_k} \max_{x \in V_k} \frac{\|x\|^2 + (N'(u)\,x, x)}{(Lx, x)} \tag{6.5}$$

where V_k denotes an arbitrary linear subspace of the space V with the dimension k. This value is attained at the point $x = h$. We can rewrite (6.5) in the form

$$\lambda = \lambda_k = \inf_{V_k} \max_{x \in V_k} \frac{\|x\|^2 + 3(D(u, u, x), x)}{(Lx, x)}\,. \tag{6.6}$$

This follows from (3.23). From (6.4) we know that $u \in F_\lambda^{-1}(0)$ is a nontrivial solution which does exist by Theorem 5.1, which means

$$u + N(u) = \lambda Lu$$

or

$$u + D(u, u, u) = \lambda Lu\,.$$

This equation can be rewritten as

$$v + D(u, u, v) = \lambda Lv \tag{6.7}$$

with $v = u$ a nontrivial solution. Then λ must be an eigenvalue of the problem (6.7):

$$\lambda = \lambda_l = \inf_{V_l} \max_{x \in V_l} \frac{\|x\|^2 + (D(u, u, x), x)}{(Lx, x)}\,, \tag{6.8}$$

and the value λ is attained at $x = u$. Here $l \geqq k$, as follows from Theorem 3.5:

$$0 \leqq (D(u, u, x)\,x) < 3(D(u, u, x), x)\,.$$

Setting $x = h$ in (6.8) we obtain

$$\lambda = \inf_{V_l} \max_{x \in V_l} \frac{\|x\|^2 + (D(u, u, x), x)}{(Lx, x)} \leqq \frac{\|h\|^2 + (D(u, u, h), h)}{(Lh, h)} =$$

$$= \frac{\|h\|^2 + 3(D(u, u, h), h) - 2(D(u, u, h), h)}{(Lh, h)} = \lambda - \frac{2(D(u, u, h), h)}{(Lh, h)} < \lambda\,,$$

a contradiction with the minimality of (6.8).

Suppose that for $\lambda = \lambda_n$ and the singular value $g = 0$ there exists a singular point $u \neq 0$. We obtain the same contradiction as above. Q.E.D.

The following theorem holds [3].

Theorem 6.4. *Let X, Y be Banach spaces and $f\colon U \to Y$ a C^1-proper Fredholm mapping of index zero. Here $U \subset X$ is an open set. Then the number of solutions of the equation $f(x) = y$ is finite and constant in each connected component of $Y - B(S)$, where $B(S)$ is the set of the singular values of the mapping f.*

Lemma 6.1. *The number of solutions of the equation $F_\lambda(u) = 0$ (problem (CP)) in space V is finite and constant for $\lambda \in (\lambda_{n-1}, \lambda_n)$, $n = 2, 3, \ldots$ independent of λ, where λ_n is an eigenvalue of (4.3).*

Proof. Theorem 6.1 implies that the set of solutions of the equation $F_\lambda(u) = 0$ is compact for each $\lambda \in (\lambda_{n-1}, \lambda_n)$. By virtue of Theorem 3.3. the set of solutions of $F_\lambda(u) = 0$ must be discrete, hence finite for each $\lambda \in (\lambda_{n-1}, \lambda_n)$. Theorem 6.3 shows that the point $(0, \lambda)$ is not a singular value of the mapping $\mathscr{F}(u, \lambda)$ for $\lambda \in \in (\lambda_{n-1}, \lambda_n)$, which means that the line segment $0 \times (\lambda_{n-1}, \lambda_n)$ lies in the same connected component of $V \times \mathbb{R} - \{\text{the singular value of } \mathscr{F}(u, \lambda)\}$. Then by Theorem 6.4 the number of solutions of

$$\mathscr{F}(u, \lambda) = (0, \lambda)$$

or equivalently

$$F_\lambda(u) = 0 \quad \text{for} \quad \lambda \in (\lambda_{n-1}, \lambda_n)$$

is finite and constant independently of λ. Q.E.D.

Theorem 6.5. *The equation*

$$u - \lambda Lu + N(u) = 0\,, \tag{6.9}$$

or equivalently the problem (CP), has a unique solution $u = 0$ for $\lambda \in (-\infty, \lambda_1]$ and exactly $2n + 1$ solutions in V for $\lambda \in (\lambda_n, \lambda_{n+1}]$, $n = 1, 2, \ldots$, where λ_n is an eigenvalue of (4.3). The solutions lie on smooth branches bifurcating from $(0, \lambda_n)$. There exist no secondary bifurcations from these branches.

Proof. (i) Suppose $\lambda \in (-\infty, \lambda_1]$ and let $0 \neq u \in V$ be a solution of (6.9). Then taking the inner product of (5.9) with u we have

$$(u, u) - \lambda(Lu, u) + (N(u), u) = 0\,.$$

By the variational characterization of λ_1 we know that

$$(u, u) - \lambda(Lu, u) \geqq 0 \quad \text{for} \quad \lambda \in (-\infty, \lambda_1]\,, \quad \text{and for each} \quad u \in V\,.$$

By Lemma 3.3. we have $(N(u), u) > 0$, so u must be 0.

(ii) Suppose $\lambda \in (\lambda_1, \lambda_2)$. The properness of $\mathscr{F}(u, \lambda)$ implies that the set of solutions of the equation

$$\mathscr{F}(u, \lambda) = (0, \lambda) \quad \text{for} \quad \lambda \in [\lambda_1, \lambda_2]$$

is compact and hence uniformly bounded independently of λ. Lemma 6.1 implies that the number of solutions of the equation $F_\lambda(u) = 0$ for $\lambda \in (\lambda_1, \lambda_2)$ is finite and constant. Let us have k solutions. As λ varies in (λ_1, λ_2), the Implicit Function Theorem and the fact that 0 is not a singular value of $F_\lambda(u)$ imply that these solutions lie on continuous mutually nonintersecting curves which we label

$$u_1(\lambda), \ldots, u_k(\lambda) . \tag{6.10}$$

We assert: For a fixed value $\lambda \in (\lambda_1, \lambda_2)$ the solutions (6.10) lie in a sphere $B((0, \lambda_1), R(\lambda))$ (the sphere with the center at $(0, \lambda_1)$ and the radius $R(\lambda)$), and $R(\lambda) \to 0$ as $\lambda \to \lambda_1$. If it were not true, there would be a weakly convergent sequence $\{u_{i_0}(\lambda_\beta)\}_{\lambda_\beta}$, where $i_0 \in \{1, \ldots, k\}$ is such that

$$\liminf_{\lambda_\beta \to \lambda_1+} \|u_{i_0}(\lambda_\beta)\| = \alpha > 0 . \tag{6.11}$$

Let $u_{i_0}(\lambda_\beta) \rightharpoonup u$, $\lambda_\beta \to \lambda_1^+$ (u is the weak limit), then

$$u_{i_0}(\lambda_\beta) = \lambda_\beta L u_{i_0}(\lambda_\beta) - N(u_{i_0}(\lambda_\beta)) \to \lambda_1 L u - N(u)$$

strongly, the limit element fulfils

$$u - \lambda_1 L u + N(u) = 0 , \tag{6.12}$$

and by (6.11)

$$\|u\| = \alpha > 0 .$$

However, this is a contradiction with part (i) of the proof. The assertion is proved. By Lemma 5.1 we know that in a sufficiently small right neighbourhood of λ_1 the equation (6.9) has exactly three solutions, namely, one trivial and two nontrivial ones which differ only by the sign. Now Lemma 6.1 implies that for $\lambda \in (\lambda_1, \lambda_2)$ the number of solutions of (6.9) is exactly three. One of them is trivial and the other two we will denote by $u_1(\lambda)$ and $-u_1(\lambda)$. We have

$$\|u_1(\lambda_2)\| = \beta > 0 .$$

Indeed, otherwise there would be a weakly convergent sequence $u_1(\lambda_\gamma)$ where $u_1(\lambda_\gamma) \neq 0$ such that $u_1(\lambda_\gamma) \rightharpoonup u_1(\lambda_2)$ weakly as $\lambda_\gamma \to \lambda_2^-$, $\liminf_{\lambda_\gamma \to \lambda_2-} \|u_1(\lambda_\gamma)\| = 0$, and each member $u_1(\lambda_\gamma)$ would satisfy

$$u_1(\lambda_\gamma) - \lambda_\gamma L u_1(\lambda_\gamma) + N(u_1(\lambda_\gamma)) = 0 . \tag{6.13}$$

By the compactness of L, N the limit $u_1(\lambda_2)$ is also the strong limit and we obtain

$$u_1(\lambda_2) - \lambda_2 L u_1(\lambda_2) + N(u_1(\lambda_2)) = 0 \quad \text{and} \quad \|u_1(\lambda_2)\| = 0 .$$

This means that in a sufficiently small left neighbourhood we find an element $u_1(\lambda_{\gamma_0}) \neq 0$ for which (6.13) holds. This is a contradiction with part (i) of Lemma 5.1. Hence, as $u_1(\lambda_2) \neq 0$, it is a regular point of $F_\lambda(u)$. By the Implicit Function Theorem $u_1(\lambda)$ can be continued smoothly to the interval (λ_2, λ_3) and no secondary bifurcation occurs in (λ_1, λ_2) and at the point λ_2. All the above assertions hold for $-u_1(\lambda)$.

(iii) Let $\lambda \in (\lambda_2, \lambda_3)$. By the same argument as in part (ii) the equation (6.9) has a finite number $m + 1$ of solutions labelled as

$$u_1(\lambda),\ -u_1(\lambda),\ u_2(\lambda), \dots, u_m(\lambda) \tag{6.14}$$

where $u_1(\lambda)$ and $-u_1(\lambda)$ continue smoothly from (λ_1, λ_2). We assert: For a fixed value $\lambda \in (\lambda_2, \lambda_3)$ the solutions (6.14) with the exception of $u_1(\lambda)$ and $-u_1(\lambda)$ lie in the sphere $B((0, \lambda_2), R_1(\lambda))$ where $R_1(\lambda) \to 0$ as $\lambda \to \lambda_2^+$. Indeed, otherwise there would be a weakly convergent sequence $\{u_{j_0}(\lambda_\delta)\}$, $j_0 \in \{2, 3, \dots, m\}$ such that

$$u_{j_0}(\lambda_\delta) \rightharpoonup u_{j_0}(\lambda_2) \quad \text{weakly as} \quad \lambda_\delta \to \lambda_2^+, \ \liminf_{\lambda_\delta \to \lambda_2^+} \|u_{j_0}(\lambda_\delta)\| = \delta > 0$$

and $u_{j_0}(\lambda_\delta)$ fulfils (6.9) with $\lambda = \lambda_\delta$.

However the weak limit is also the strong limit and we obtain

$$\|u_{j_0}(\lambda_2)\| = \delta > 0\,, \quad u_{j_0}(\lambda_2) - \lambda_2 L u_{j_0}(\lambda_2) + N(u_{j_0}(\lambda_2)) = 0\,.$$

It means $u_{j_0}(\lambda_2)$ is a regular point (Theorem 6.3) and by the Implicit Function Theorem in a sufficiently small neighbourhood of λ_2 there exists a smooth solution curve $u_{j_0}(\lambda)$, but part (ii) of the proof implies $u_{j_0}(\lambda) = u_1(\lambda)$ or $u_{j_0}(\lambda) = -u_1(\lambda)$, a contradiction. Then by Lemma 5.1 and part (ii) of this proof, in a small right neighbourhood of λ_2 the equation (6.9) has exactly five solutions. There are solutions smoothly continuing from (λ_1, λ_2): $u_1(\lambda)$, $-u_1(\lambda)$, nontrivial solutions which we label $u_2(\lambda)$, $-u_2(\lambda)$ such that $u_2(\lambda_2) = -u_2(\lambda_2) = 0$, and the trivial solution. Then for each $\lambda \in (\lambda_2, \lambda_3)$ the equation (6.9) has exactly five solutions lying on smooth mutually non-intersecting curves. By the same argument as in part (ii) of this proof we can prove that the solution curves do not intersect each other, no secondary bifurcation occurs, and they do not approach the trivial solution at λ_3.

(iv) Arguing as above we obtain by induction the statement of the theorem. Q,E,D,

7. NODAL PROPERTIES OF SOLUTIONS

Theorem 5.3 implies that each nontrivial solution in a sufficiently small neighbourhood of the bifurcation point $(0, \lambda_n)$ has the form

$$(s v_n + v_{2n}(s), \lambda(s))\,, \quad |s| \leqq \varepsilon \tag{7.1}$$

where v_n is the n-th eigenfunction corresponding to the n-th eigenvalue of the problem (4.3) and $v_{2n}(0) = v'_{2n}(0) = 0$, $\lambda(0) = \lambda_n$, $v_{2n}(s) \in R(I - \lambda_n L)$. We see that in such a neighbourhood the nontrivial solution (7.1) has the same nodal properties as the eigenfunction v_n. We show that the nodal structute of solutions is preserved along a smooth curve of solutions.

Theorem 7.1. *The following equality holds on the solution curve:*

$$\lim_{\lambda \to \lambda^*} \sup_{x \in (0,1)} |w_\lambda(x) - w_{\lambda^*}(x)| = 0$$

where $w_\lambda(x)$ and $w_{\lambda^*}(x)$ are solutions of the problem (CP) corresponding to the parameter values λ and λ^*, respectively.

Proof. The relations

$$w_\lambda(x) = \int_x^1 \frac{1}{t^3}\left(\int_0^t s^3\, w_\lambda(s)\,(\lambda - f_\lambda)(s)\,\mathrm{d}s\right)\mathrm{d}t\,,$$

$$f_\lambda(x) = \int_x^1 \frac{1}{t^3}\left(\int_0^t s^3\, w_\lambda^2(s)\,\mathrm{d}s\right)\mathrm{d}t$$

and the estimates

$$0 \leqq f(x) \leqq -C_1 \ln x\,,$$

$$|w(x)| \leqq -C_2 \ln x$$

follow from the proof of Theorem 3.1, C_1, C_2 being positive constants. From Theorem 6.5 we know that the following convergence takes place on the continuous solution curve: $w_\lambda \to w_{\lambda^*}$ strongly in V as $\lambda \to \lambda^*$. Hence we obtain

$$|f_\lambda(x) - f_{\lambda^*}(x)| = \left|\int_x^1 \frac{1}{t^3}\left[\int_0^t s^3(w_\lambda^2(s) - w_{\lambda^*}^2(s))\,\mathrm{d}s\right]\mathrm{d}t\right| \leqq$$

$$\leqq \int_x^1 \frac{1}{t^3}\left(\int_0^t s^3|w_\lambda(s)|\,|w_\lambda(s) - w_{\lambda^*}(s)|\,\mathrm{d}s\right)\mathrm{d}t +$$

$$+ \int_x^1 \frac{1}{t^3}\left(\int_0^t s^3|w_{\lambda^*}(s)|\,|w_\lambda(s) - w_{\lambda^*}(s)|\,\mathrm{d}s\right)\mathrm{d}t \leqq$$

$$\leqq \int_x^1 \frac{1}{t^3}\left(\int_0^t s^5 w_\lambda^2(s)\,\mathrm{d}s\right)^{1/2}\left(\int_0^t s(w_\lambda(s) - w_{\lambda^*}(s))^2\,\mathrm{d}s\right)^{1/2}\mathrm{d}t +$$

$$+ \int_x^1 \frac{1}{t^3}\left(\int_0^t s^5\, w_{\lambda^*}^2(s)\,\mathrm{d}s\right)^{1/2}\left(\int_0^t s(w_\lambda(s) - w_{\lambda^*}(s))^2\,\mathrm{d}s\right)^{1/2}\mathrm{d}t \leqq$$

$$\leqq \left(\int_0^1 s(w_\lambda(s) - w_{\lambda^*}(s))^2\,\mathrm{d}s\right)^{1/2}\left\{\int_x^1 \frac{1}{t^3}\left[\left(\int_0^t s^5\, w_\lambda^2(s)\,\mathrm{d}s\right)^{1/2} + \left(\int_0^t s^5 w_{\lambda^*}^2(s)\,\mathrm{d}s\right)^{1/2}\right]\mathrm{d}t\right\} \leqq$$

$$\leqq C\|w_\lambda - w_{\lambda^*}\|\left[\int_x^1 \frac{1}{t^3}\left(\int_0^t s^5 \ln^2 s\,\mathrm{d}s\right)^{1/2}\mathrm{d}t\right] \leqq C\|w_\lambda - w_{\lambda^*}\|\,K(x)$$

where $K(x)$ is bounded. Then

$$\sup_{x\in(0,1)} |f_\lambda(x) - f_{\lambda^*}(x)| \leqq C \sup_{x\in(0,1)} K(x)\,\|w_\lambda - w_{\lambda^*}\|\,,$$

which means

$$\lim_{\lambda\to\lambda^*}\ \sup_{x\in(01)} |f_\lambda(x) - f_{\lambda^*}(x)| = 0\,.$$

Similarly we obtain

$$\sup_{x\in(0,1)} |w_\lambda(x) - w_{\lambda^*}(x)| \leqq C_1 \sup_{x\in(0,1)} K_1(x)\, |\lambda - \lambda^*| + C_2 \sup_{x\in(0,1)} K_2(x)\, \|w_\lambda - w_{\lambda^*}\| +$$
$$+ C_3 \sup_{x\in(0,1)} K_3(x)\, \|f_\lambda - f_{\lambda^*}\|$$

where $K_1(x)$, $K_2(x)$, $K_3(x)$ are bounded. Then

$$\lim_{\lambda\to\lambda^*} \sup_{x\in(0,1)} |w_\lambda(x) - w_{\lambda^*}(x)| = 0\,. \qquad \text{Q.E.D.}$$

Definition 7.1. *A point $s \in (0, 1)$ is a simple zero point of the function $u(x) \in \in C^2((0, 1))$ if $u(s) = 0$ and $u'(s) \neq 0$.*

Lemma 7.1. *If (u, f) is a classical solution of the problem (CP) and $u(x)$ has a double zero point in $(0, 1)$ (i.e., there is $s \in (0, 1)$ such that $u(s) = u'(s) = 0$) then $u(x) = 0$, $f(x) = 0$.*

Proof. By the proof of Theorem 3.1 we have

$$f(x) = \int_x^1 \frac{1}{t^3}\left(\int_0^t s^3\, w^2(s)\, ds\right) dt\,.$$

It means $f(x) \geqq 0$. Let $s \in (0, 1)$ be a double zero point. Then the first equation of the problem (CP) can be written in the form

$$v'(x) + \lambda x^3\, u(x) = x^3\, u(x) f(x)\,.$$
$$u'(x) = \frac{1}{x^3}\, v(x)\,,$$

which yields

$$\tfrac{1}{2}(v^2(x))' + \lambda x^3\, u(x)\, v(x) = x^3\, u(x) f(x)\, v(x)\,,$$
$$\tfrac{1}{2}(u^2(x))' = \frac{1}{x^3}\, u(x)\, v(x)\,.$$

Consequently,

$$\tfrac{1}{2}(u^2(x) + v^2(x))' = -\lambda x^3\, u(x)\, v(x) + \frac{1}{x^3}\, u(x)\, v(x) + x^3\, u(x)\, v(x) f(x)$$

and hence in a neighbourhood of the point s we have

$$(u^2(x) + v^2(x)' \leqq c(u^2(x) + v^2(x))$$

which implies

$$u^2(x) + v^2(x) \leqq (u^2(s) + v^2(s)) \exp\left[c(x - s)\right] \equiv 0\,.$$

This means $u(x) = 0$ in the whole neighbourhood of s, and by continuation $u(x) = 0$ in $(0, 1)$, hence $f(x) = 0$. Q.E.D.

Theorem 7.2. *A solution of the problem* (*CP*) *lying on the continuous curve of nontrivial solutions bifurcating from the point* $(0, \lambda_k)$ *has exactly* $k - 1$ *simple internal zero points.* λ_k *is the k-th eigenvalue of the problem* (4.3).

Proof. In a small neighbourhood of the bifurcation point $(0, \lambda_k)$, say for $\lambda \in \in (\lambda_k, \lambda^*)$, the solution of the problem (CP) has exactly $k - 1$ simple internal zero points. Suppose that for the parameter value $\lambda - \lambda^*$ the solution w_{λ^*} has k internal zero points. Then due to Theorem 7.1 one of them must be a double zero point. Then Lemma 7.1 implies $w_{\lambda^*}(x) = 0$ and $f_{\lambda^*}(x) = 0$ in $(0, 1)$, which leads to contradistion. Q.E.D.

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Súhrn

POČET DEFORMOVANÝCH STAVOV KRUHOVÝCH DOSÁK

ĽUBOMÍR MARKO

Práca sa zaoberá určením presného počtu riešení von Kármánových rovníc pre rotačne symetrický ohyb tenkej elastickej kruhovej dosky v závislosti od parametra napätia.

Резюме

ЧИСЛО ДЕФОРМИРУЕМЫХ СОСТОЯНИЙ КРУГЛЫХ ПЛАСТИНОК

ĽUBOMÍR MARKO

Работа занимается определением точного числа решений уравнений фон Кармана для осесимметрического изгиба круглой пластинки.

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