Aplikace matematiky

Vladimír Drápalík; Vladimír Janovský

Numerical treatment of 3-dimensional potential problem

Aplikace matematiky, Vol. 33 (1988), No. 6, 456–469

Persistent URL: `http://dml.cz/dmlcz/104324`

33 (1988) APLIKACE MATEMATIKY No. 6, 456—469

NUMERICAL TREATMENT OF 3-DIMENSIONAL POTENTIAL PROBLEM

VLADIMÍR DRÁPALÍK, VLADIMÍR JANOVSKÝ

(Received March 13, 1986)

Summary. Assuming an incident wave to be a field source, we calculate the field potential in a neighborhood of an inhomogeneous body. This problem which has been formulated in $\mathbb{R}^3$ can be reduced to a bounded domain. Namely, a boundary condition for the potential is formulated on a sphere. Then the potential satisfies a well posed boundary value problem in a ball containing the body.

A numerical approximation is suggested and its convergence is analysed.

Keywords: diffraction, nonlocal boundary condition, finite elements.

AMS Subject classification: 31B10, 65N30, 35J15, 35J67, 78A20, 78A45.

1. INTRODUCTION

The present paper is an extension of [1] to the 3-dimensional case. Let $f = f(x)$ be the density of an electric charge in $\mathbb{R}^3$. Let $w = w(x)$ be the potential of the relevant electric field in vacuum. Suppose that an inhomogeneous body Ω is placed in the field. If $u = u(x)$ is the potential of the resulting field on $\mathbb{R}^3$ (due to scattering, $u \neq w$) then our aim is to find u on Ω.

We say that u is a smooth solution if $u = u(x)$ is continuous in $\mathbb{R}^3$, $\lim_{|x|\to\infty} u(x) = 0$, and

$$Au \equiv -\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\right) = f(x) \tag{1.1}$$

in $\mathbb{R}^3$ in the sense of distributions. We assume

(i) $a_{ij} \in L_\infty(\mathbb{R}^3)$, $a_{ij}(x) \equiv \delta_{ij}$ outside Ω, where Ω is a bounded domain in $\mathbb{R}^3$;

(ii) there exists a positive constant c such that

$$\sum_{i,j=1}^{3} a_{ij}\xi_i\xi_j \geqq c\sum_{i=1}^{3}\xi_i^2 \quad \text{for each} \quad \xi \in \mathbb{R}^3 \quad \text{a.e. on } \Omega\,;$$

(iii) $f \in L_2(\mathbb{R}^3)$, $\operatorname{supp} f$ is compact in $\mathbb{R}^3$, $\bar{\Omega} \cap \operatorname{supp} f = \emptyset$.

The function w (the so called *incident wave*) is assumed to be continuous in $\mathbb{R}^3$, such that $\lim_{|x|\to\infty} w(x) = 0$ and

$$-\Delta w = f \quad \text{in} \quad \mathbb{R}^3 \tag{1.2}$$

in the sense of distributions.

We note that both u and w are harmonic in a neighborhood of ∞. Due to the assumptions

$$\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} w(x) = 0\,,$$

they behave asymptotically as follows:

$$u(x) = 0(|x|^{-1})\,, \qquad w(x) = 0(|x|^{-1})\,,$$
$$\operatorname{grad} u = 0(|x|^{-2})\,, \quad \operatorname{grad} w = 0(|x|^{-2})$$

as $|x| \to \infty$.

Using some standard arguments of potential theory, we can prove

Theorem 1.1. *Let the boundary $\partial\Omega$ of Ω be sufficiently smooth. If u is a smooth solution of* (1.1) *then*

$$\frac{1}{2}u(x) + \frac{1}{4\pi}\int_{\partial\Omega}\left\{u(y)\frac{\partial}{\partial\mu(y)}\left(\frac{1}{|x-y|}\right) + \frac{\partial u}{\partial\nu(y)}\frac{1}{|x-y|}\right\}d\sigma(y) = w(x) \tag{1.3}$$

for each $x \in \partial\Omega$ with the following notation:

(a) *$\mu = \mu(y)$ is the outward normal vector at $y \in \partial\Omega$ with respect to the complement Ω^c of $\bar{\Omega}$ in* $\mathbb{R}^3$;

(b) *$\partial/\partial\mu(y)$ is the derivative at $y \in \partial\Omega$ along the direction $\mu(y)$ with respect to Ω^c, i.e.*

$$\frac{\partial u}{\partial\mu(y)} \overset{\text{def}}{=} \sum_{i=1}^{3} \mu_i(y) \lim_{\substack{z\to y\\ z\in\Omega^c}} \frac{\partial u}{\partial x_i}(z)\,;$$

(c) *$\partial/\partial\nu(y)$ is the derivative at $y \in \partial\Omega$ along the co-normal $\nu(y)$ with respect to Ω, i.e.*

$$\frac{\partial u}{\partial\nu(y)} \overset{\text{def}}{=} \sum_{i,j=1}^{3} a_{i,j}(y)\,(-\mu_i(y)) \lim_{\substack{z\to y\\ z\in\Omega}} \frac{\partial u}{\partial x_j}(z)\,.$$

Proof. We omit the proof which would follow almost word-by-word the proof of Theorem 2.1 in [1]. We note only that the fundamental solution $-1/(2\pi)\log|x|$ of the Laplace operator in $\mathbb{R}^2$ should be replaced by $1/(4\pi|x|)$ which is the fundamental solution of the same operator in $\mathbb{R}^3$. Moreover, the asymptotic behavior of u and w is different in $\mathbb{R}^3$ (see above).

2. BOUNDARY CONDITION ON A SPHERE

We assume Ω to be a ball $\{x \in \mathbb{R}^3: |x| < R\}$ with radius R. Let us rewrite (1.3) making use of the spherical coordinates (r, α, ϑ): $x_1 = r \sin\alpha \cos\vartheta$, $x_2 = r \sin\alpha \sin\vartheta$, $x_3 = r \cos\alpha$.

If $x \in \partial\Omega$, $y \in \partial\Omega$, $x \neq y$ then, if (R, α, ϑ) and (R, α', ϑ') are the spherical coordinates of x and y, respectively, we have

$$\frac{1}{|x-y|} = 4\pi\, \mathscr{K}(\alpha, \vartheta; \alpha', \vartheta'),$$

$$\frac{\partial}{\partial\mu(y)} \frac{1}{|x-y|} = \frac{2\pi}{R} \mathscr{K}(\alpha, \vartheta; \alpha', \vartheta')$$

where

$$(2.1)\quad \mathscr{K}(\alpha, \vartheta; \alpha', \vartheta') = \frac{1}{4\pi R\sqrt{2}} (1 - \sin\alpha \sin\alpha' \cos(\vartheta - \vartheta') - \cos\alpha\cos\alpha')^{-1/2}.$$

Substituting into (1.3), we obtain the boundary conditions in the form

$$(2.2)\quad \frac{1}{2} u(R, \alpha, \vartheta) + \frac{R}{2} \int_0^\pi \int_0^{2\pi} u(R, \alpha', \vartheta')\, \mathscr{K}(\alpha, \vartheta; \alpha', \vartheta') \sin\alpha' \, d\alpha' \, d\vartheta' + R^2 \int_0^\pi \int_0^{2\pi} \frac{\partial u}{\partial \nu(y)}\bigg|_{y=(R,\alpha',\vartheta')} \mathscr{K}(\alpha, \vartheta; \alpha', \vartheta') \sin\alpha' \, d\alpha' \, d\vartheta' = w(R, \alpha, \vartheta).$$

In order to simplify notation, we define an operator $\mathbf{K}$,

$$(2.3)\quad \mathbf{K}v = (\mathbf{K}v)(\alpha, \vartheta) \overset{\text{def}}{=} \int_0^\pi \int_0^{2\pi} \mathscr{K}(\alpha, \vartheta; \alpha', \vartheta')\, v(\alpha', \vartheta')\, R^2 \sin\alpha' \, d\alpha' \, d\vartheta',$$

which acts on sufficiently smooth functions on $\partial\Omega$.

Thus, the traces u and $\partial u/\partial\nu$ of a smooth solution u should satisfy

$$(2.4)\quad \frac{1}{2} u + \frac{1}{2R} \mathbf{K}u + \mathbf{K} \frac{\partial u}{\partial \nu} = w \quad \text{on} \quad \partial\Omega.$$

In the next step, we find the spectrum of $\mathbf{K}$. To this end we make an observation: If v is a harmonic function on $\bar{\Omega}$ then the classical Green's formula yields

$$v(x) = \frac{1}{2\pi} \int_{\partial\Omega} \left\{ \frac{\partial v}{\partial s}(y) \frac{1}{|x-y|} - v(y) \frac{\partial}{\partial s} \frac{1}{|x-y|} \right\} d\sigma(y)$$

at each $x \in \partial\Omega$. The vector $s = (y_1/R, y_2/R, y_3/R)^{\mathsf{T}}$ is the outward normal vector at $y \in \partial\Omega$ with respect to Ω. In terms of the operator $\mathbf{K}$, the above identity can be written as follows:

$$(2.5)\quad v = 2\mathbf{K} \frac{\partial v}{\partial s} + \frac{1}{R} \mathbf{K}v = 2\mathbf{K}\left(\frac{\partial v}{\partial s} + \frac{1}{2R} v\right) \quad \text{on } \partial\Omega.$$

It is well known that there exist homogeneous polynomials in $\mathbb{R}^3$ which are harmonic. In spherical coordinates they equal $r^n Y_n(\alpha, \vartheta)$, where n is an integer and Y_n is a spherical function of degree n. Setting $v = r^n Y_n(\alpha, \vartheta)$, we observe that

$$\frac{\partial v}{\partial s} = \frac{\partial v}{\partial r} = nR^{n-1} Y_n(\alpha, \vartheta) = \frac{n}{R} v \quad \text{on} \quad \partial\Omega .$$

Thus, substituting into (2.5), $v = (2n + 1)/R\, \boldsymbol{K} v$ on $\partial\Omega$. Since $\boldsymbol{K}$ acts on the trace of v, we conclude that

$$Y_n = \frac{2n+1}{R} \boldsymbol{K} Y_n \quad \text{on} \quad \partial\Omega . \tag{2.6}$$

We recall the definition and basic properties of spherical functions: If $n \geqq 0$ is an integer then

$$P_n(t) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left[(t^2 - 1)^n\right]$$

is the Legendre polynomial of the n-th degree. For each integer k, $0 \leqq k \leqq n$, we define the conjugate Legendre functions

$$P_n^{(k)}(t) = (1 - t^2)^{k/2} \frac{\mathrm{d}^k P_n(t)}{\mathrm{d}t^k} .$$

The space of spherical functions $Y_n = Y_n(\alpha, \vartheta)$ of degree n is spanned by the basis $\{Y_n^{(k)}\}_{k=-n}^n$, where

$$\begin{aligned}
Y_n^{(0)}(\alpha, \vartheta) &= P_n(\cos\alpha) , \\
Y_n^{(k)}(\alpha, \vartheta) &= P_n^{(k)}(\cos\alpha) \sin k\vartheta , \qquad k = 1, \dots, n , \\
Y_n^{(k)}(\alpha, \vartheta) &= P_n^{(-k)}(\cos\alpha) \cos k\vartheta , \quad k = -1, \dots, -n .
\end{aligned}$$

The above basis is orthogonal in $L_2(\partial\Omega)$. The functions $Y_n^{(k)}$ can be normalized in $L_2(\partial\Omega)$. Namely, setting

$$N_n^{(k)} = \left(\frac{2\pi R^2}{2n+1} c_k \frac{(n + |k|)!}{(n - |k|)!}\right)^{-1/2} Y_n^{(k)}$$

for $k = -n, \dots, n$, where $c_0 = 2$, $c_k = 1$ for $k \neq 0$, we obtain the relevant orthonormal basis $\{N_n^{(k)}\}_{k=-n}^n$ of spherical functions Y_n. Taking into account (2.6), we can write

$$\boldsymbol{K} N_n^{(k)} = \frac{R}{2n+1} N_n^{(k)} \quad \text{on} \quad \partial\Omega \tag{2.7}$$

for integers $n = 0, 1, \dots$ and $k = -n, \dots, n$.

The spherical functions are eigenfunctions of the Laplace-Beltrami operator $\Delta_{\partial\Omega}$,

$$\Delta_{\partial\Omega} Y \stackrel{\text{def}}{=} -\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left((\sin\vartheta\frac{\partial Y}{\partial\vartheta}\right) - \frac{1}{\sin^2\vartheta}\frac{\partial^2 Y}{\partial\alpha^2}$$

for each sufficiently smooth Y defined on $\partial\Omega$. Namely

$$-\Delta_{\partial\Omega} N_n^{(k)} = \lambda_n N_n^{(k)} \quad \text{on} \quad \partial\Omega \tag{2.8}$$

for integers $n = 0, 1, \ldots$ and $k = -n, \ldots, n$, where

$$\lambda_n = n(n+1)\,. \tag{2.9}$$

The set $\{\{N_n^{(k)}\}_{k=-n}^{n}\}_{n=0}^{\infty}$ is known to be an orthonormal basis of $L_2(\partial\Omega)$.

Let $H^r(\partial\Omega)$, r real, be the usual Sobolev space with the norm $\|\cdot\|_{r,\partial\Omega}$. Identifying $L_2(\partial\Omega)$ with its dual, let $\langle\cdot,\cdot\rangle$ be pairing of $H^r(\partial\Omega)$ and $H^{-r}(\partial\Omega)$. The norm $\|\cdot\|_{r,\partial\Omega}$ can be defined equivalently by means of the Fourier series with respect to the eigenfunctions of the Laplace-Beltrami operator (see [2], Remark 7.5). In fact, if $v \in H^r(\partial\Omega)$ then

$$\|v\|_{r,\partial\Omega} \stackrel{\text{def}}{=} \Big(\sum_{n=0}^{\infty}(1+\lambda_n)^r \sum_{k=-n}^{n}\langle v, N_n^{(k)}\rangle^2\Big)^{1/2}\,. \tag{2.10}$$

We proceed with an extension of the operator $\boldsymbol{K}$ onto $H^r(\partial\Omega)$ by means of the spectral representation of $\boldsymbol{K}$, see (2.7): If $v \in H^r(\partial\Omega)$ then

$$\boldsymbol{K}v \stackrel{\text{def}}{=} \sum_{n=0}^{\infty}\frac{R}{2n+1}\sum_{k=-n}^{n}\langle v, N_n^{(k)}\rangle N_n^{(k)} \tag{2.11}$$

and, similarly,

$$\boldsymbol{K}^{-1}v \stackrel{\text{def}}{=} \sum_{n=0}^{\infty}\frac{2n+1}{R}\sum_{k=-n}^{n}\langle v, N_n^{(k)}\rangle N_n^{(k)}\,. \tag{2.12}$$

Definition (2.11) is a natural generalization of the original formula (2.3).

Lemma 2.1. *The operators*

$$\boldsymbol{K}\colon \quad H^r(\partial\Omega) \to H^{r+1}(\partial\Omega)\,,$$
$$\boldsymbol{K}^{-1}\colon H^r(\partial\Omega) \to H^{r-1}(\partial\Omega)$$

are bounded for each real r. Moreover, $\boldsymbol{K}\boldsymbol{K}^{-1} = \boldsymbol{K}^{-1}\boldsymbol{K} = $ *identity (in* $H^r(\partial\Omega)$*).*

Proof. Let $v \in H^r(\partial\Omega)$. Then, by virtue of (2.11) and (2.4),

$$\|\boldsymbol{K}v\|_{r+1,\partial\Omega}^2 = \sum_{n=0}^{\infty}(1+\lambda_n)^{r+1}\sum_{k=-n}^{n}\left(\frac{R}{2n+1}\right)^2\langle v, N_n^{(k)}\rangle^2\,.$$

With help of (2.9), the right hand side can be simply estimated as follows

$$\|\boldsymbol{K}v\|_{r+1,\partial\Omega}^2 \leqq \|v\|_{r,\partial\Omega}^2 \sup_{n=0,1,\ldots} R^2\frac{n^2+n+1}{(2n+1)^2} = R^2\|v\|_{r,\partial\Omega}^2\,.$$

Similarly, $\|K^{-1}v\|^2_{r-1,\partial\Omega} \leqq 4R^{-2}\|v\|^2_{r,\partial\Omega}$. The last statement immediately follows from the definition of K and K^{-1}. Q.E.D.

By virtue of Lemma 2.1, the boundary condition (2.4) is meaningful in the cases when $w \in H^r(\partial\Omega)$, $u \in H^r(\partial\Omega)$, $\partial u/\partial\nu \in H^{r-1}(\partial\Omega)$ for each real r. In the next step we formulate problem (1.1), (2.4) variationally on Ω.

We define the bilinear form

$$\mathfrak{a}(w, v) \overset{\text{def}}{=} \sum_{i,j=1}^{3} \int_\Omega a_{ij} \frac{\partial w}{\partial x_j} \frac{\partial v}{\partial x_i} \, dx$$

for each w and v from $H^1(\Omega)$. If $u \in H^1(\Omega)$ then the condition (1.1) on Ω is equivalent to

$$\mathfrak{a}(u, v) = 0 \quad \text{for each} \quad v \in H^1_0(\Omega)\,. \tag{2.13}$$

The conormal derivative $\partial u/\partial\nu$ can be defined variationally as follows: $\partial u/\partial\nu \in H^{-1/2}(\partial\Omega)$,

$$\left\langle \frac{\partial u}{\partial \nu}, v \right\rangle = \mathfrak{a}(u, v) \quad \text{for each} \quad v \in H^1(\Omega)\,. \tag{2.14}$$

Due to (2.4)

$$-\frac{\partial u}{\partial \nu} = \frac{1}{2R} u + \tfrac{1}{2} K^{-1}u - K^{-1}w \quad \text{in} \quad H^{-1/2}(\partial\Omega)\,.$$

Substituting into (2.14), we obtain a variational condition on $u \in H^1(\Omega)$:

$$\mathfrak{a}(u, v) + \tfrac{1}{2}\langle K^{-1}u, v\rangle + \frac{1}{2R}\langle u, v\rangle = \langle K^{-1}w, v\rangle \quad \text{for each} \quad v \in H^1(\Omega)\,. \tag{2.15}$$

It motivates the following definition: We call $u \in H^1(\Omega)$ a *weak solution* to (1.1) in Ω if the variational condition (2.15) holds. Clearly, each smooth solution being restricted to Ω is a weak solution. The trace $w \in H^{1/2}(\partial\Omega)$ of the incident wave is the only data of the problem (2.15).

Theorem 2.1. *For each $w \in H^{1/2}(\partial\Omega)$ there exists a unique weak solution $u \in H^1(\Omega)$.*

Proof. We verify the assumption of the Lax-Milgram theorem:

The bilinear form $\mathfrak{a}(\cdot,\cdot) + \tfrac{1}{2}\langle K^{-1}\,\cdot, \cdot\rangle + 1/2R\langle\cdot, \cdot\rangle$ is continuous in $H^1(\Omega) \times H^1(\Omega)$. We should note perhaps that the continuity of the term $\langle K^{-1}\,\cdot, \cdot\rangle$ follows from Lemma 2.1 and from the well known continuous embedding $H^1(\Omega) \subset H^{1/2}(\partial\Omega)$.

Moreover, the bilinear form is $H^1(\Omega)$-elliptic. Indeed, we estimate

$$\mathfrak{a}(v, v) + \tfrac{1}{2}\langle K^{-1}v, v\rangle + \frac{1}{2R}\langle v, v\rangle \geqq c\sum_{i=1}^{3}\int_\Omega \left(\frac{\partial v}{\partial x_i}\right)^2 dx + \frac{1}{R}\langle v, v\rangle \tag{2.16}$$

for each $v \in H^1(\Omega)$, where we have used the ellipticity assumption (ii) in order to estimate $\mathfrak{a}(v, v)$ and employed the definition formula (2.12) in order to estimate

$\langle \boldsymbol{K}^{-1}v, v\rangle \geqq 1/R\langle v, v\rangle$. The right hand side of (2.16) is the square of an equivalent norm in $H^1(\Omega)$.

As we have shown above, $\langle \boldsymbol{K}^{-1}\cdot, \cdot\rangle$ is a continuous bilinear form on $H^1(\Omega)$. Thus the right hand side of (2.15) is a bounded linear functional (of v) in $H^1(\Omega)$ for each fixed $w \in H^{1/2}(\partial\Omega)$.

This completes the verification of the assumptions, which means that the Lax-Milgram theorem implies the assertion of Theorem 2.1. Q.E.D.

3. APPROXIMATION

The variational definition (2.14) of the weak solution u suggests the Ritz-Galerkin approximation of u. Let S^h be a finite dimensional subspace of $H^1(\Omega)$; let S^h be spanned by a basis $\{\varphi_1, \ldots, \varphi_N\}$. Then we define $u^h \in S^h$ to be the Ritz-Galerkin approximation of u in S^h if

$$(3.1) \qquad a(u^h, v) + \tfrac{1}{2}\langle \boldsymbol{K}^{-1}u^h, v\rangle + \frac{1}{2R}\langle u^h, v\rangle = \langle \boldsymbol{K}^{-1}w, v\rangle$$

for each $v \in S^h$. Naturally, u^h solves (3.1) if and only if $u^h = \sum_{i=1}^{N} \alpha_i\varphi_i$, where $\alpha = (\alpha_1, \ldots, \alpha_N)^{\mathsf{T}} \in \mathbb{R}^N$ satisfies a set of linear algebraic equations

$$(3.2) \qquad \boldsymbol{B}\alpha + \boldsymbol{M}\alpha = \boldsymbol{f},$$

$$\boldsymbol{B} \overset{\text{def}}{=} \{b_{ij}\}_{i,j=1,\ldots,N}, \quad b_{ij} = a(\varphi_j, \varphi_i) + \frac{1}{2R}\langle\varphi_j, \varphi_i\rangle,$$

$$\boldsymbol{f} \overset{\text{def}}{=} (f_1, \ldots, f_N)^{\mathsf{T}}, \quad f_j = \langle \boldsymbol{K}^{-1}w, \varphi_j\rangle,$$

$$\boldsymbol{M} \overset{\text{def}}{=} \{m_{ij}\}_{i,j=1,\ldots,N}, \quad m_{ij} = \tfrac{1}{2}\langle \boldsymbol{K}^{-1}\varphi_j, \varphi_i\rangle.$$

The operator $\boldsymbol{K}^{-1}$ is defined via (2.11) which means that the evaluation of f_j and m_{ij} requires the Fourier expansions of w and each φ_j into spherical functions $\{\{N_n^{(k)}\}_{k=-n}^{n}\}_{n=0}^{\infty}$ on $L_2(\partial\Omega)$. In the actual implementation, we are able to evaluate a few first terms of these expansions only. In fact, we replace the operator $\boldsymbol{K}^{-1}$ by an operator $\boldsymbol{K}_p^{-1}$ which is defined as follows: p is a positive integer,

$$(3.3) \qquad \boldsymbol{K}_p^{-1}v \overset{\text{def}}{=} \sum_{n=0}^{p} \frac{2n+1}{R} \sum_{k=-n}^{n} \langle v, N_n^{(k)}\rangle\, N_n^{(k)}$$

for each $v \in H^r(\partial\Omega)$; r is arbitrary.

Our aim is to estimate the error in calculation of u^h when replacing $\boldsymbol{K}^{-1}$ by $\boldsymbol{K}_p^{-1}$ in the formulas for $\boldsymbol{f}$ and $\boldsymbol{B}$ in (3.2). The impact of other factors (numerical integration, approximation of the domain Ω by isoparametric elements, etc.) on the total error can be studied by standard techniques and thus it is ommited here.

Notation (convergence norms). If $u \in H^1(\Omega)$ then

$$|u|_{1,\Omega} \stackrel{\text{def}}{=} \left(\sum_{i=1}^{3} \int_{\Omega} \left(\frac{\partial u}{\partial x_i} \right)^2 \mathrm{d}x \right)^{1/2}$$

(i.e., $|\cdot|_{1,\Omega}^2$ is the Dirichlet integral) and

$$\|u\|_{\Omega} \stackrel{\text{def}}{=} (|u|_{1,\Omega}^2 + \|u\|_{0,\partial\Omega}^2)^{1/2} .$$

We start with two simple embedding statements:

Lemma 3.1. *There exists a linear mapping* $\mathscr{J}: H^{1/2}(\partial\Omega) \to H^1(\Omega)$ *such that*
a) *if* $v \in H^{1/2}(\partial\Omega)$ *then* $\mathscr{J}v = v$ *a.e. on* $\partial\Omega$,
b) $|\mathscr{J}v|_{1,\Omega} \leqq \|v\|_{1/2,\partial\Omega}$.

Proof. For a given $v \in H^{1/2}(\partial\Omega)$ let $z \in H^1(\Omega)$ be the solution of the problem

$$\Delta z = 0 \quad \text{in} \quad \Omega \,, \quad z = v \quad \text{on} \quad \partial\Omega \,.$$

We set $\mathscr{J}v \stackrel{\text{def}}{=} z$.

Since $-\int_{\Omega} \Delta z \, z \, \mathrm{d}x = |z|_{1,\Omega}^2 - \int_{\partial\Omega} (\partial z/\partial \nu) \, z \, \mathrm{d}\sigma$, ν being the outward normal, we have

$$|z|_{1,\Omega}^2 = \int_{\partial\Omega} \frac{\partial z}{\partial \nu} v \, \mathrm{d}\sigma \,.$$

A harmonic function z can be expanded by making use of the harmonic polynomials. Namely,

$$z = \sum_{n=0}^{\infty} \left(\frac{r}{R} \right)^n \sum_{k=-n}^{n} \langle v, N_n^{(k)} \rangle \, N_n^{(k)} \,.$$

Then we calculate

$$\int_{\partial\Omega} \frac{\partial z}{\partial \nu} v \, \mathrm{d}\sigma = \sum_{n=0}^{\infty} n \sum_{k=-n}^{n} \langle v, N_n^{(k)} \rangle^2$$

and estimate

$$\int_{\partial\Omega} \frac{\partial z}{\partial \nu} v \, \mathrm{d}\sigma \leq \sum_{n=0}^{\infty} (1 + n(n+1))^{1/2} \sum_{k=-n}^{n} \langle v, N_n^{(k)} \rangle^2 = \|v\|_{1/2,\partial\Omega}^2 \,.$$

Thus, $|z|_{1,\Omega}^2 \leq \|v\|_{1/2,\partial\Omega}^2$, which implies the last statement of the lemma. Q.E.D.

Lemma 3.2. *Each* $v \in H^1(\Omega)$ *satisfies*

$$\|v\|_{1/2,\partial\Omega} \leqq (|v|_{1,\Omega}^2 + \|v\|_{0,\partial\Omega}^2)^{1/2} = \|v\|_{\Omega} \,. \tag{3.4}$$

Proof. Since $\mathscr{J}v$ is harmonic, it minimizes the Dirichlet integral over the set of $H^1(\Omega)$ – functions having the same trace, i.e.

$$|\mathscr{J}v|_{1,\Omega}^2 \leqq |\tilde{v}|_{1,\Omega}^2$$

for each $\tilde{v} \in H^1(\Omega)$, $\tilde{v} = v$ on $\partial\Omega$. In particular,

$$|\mathscr{J}v|_{1,\Omega} \leqq |v|_{1,\Omega} . \tag{3.5}$$

In the same way as in the previous lemma, we find

$$|\mathscr{J}v|_{1,\Omega}^2 = \int_{\partial\Omega} \frac{\partial(\mathscr{J}v)}{\partial \nu} v \, d\sigma = \sum_{n=0}^{\infty} n \sum_{k=-n}^{n} \langle v, N_n^{(k)} \rangle^2$$

by using the expansion of $\mathscr{J}v$ into harmonic polynomials. Clearly, $(1 + n(n + 1))^{1/2} \leqq n + 1$ for each integer $n \geqq 0$, i.e.

$$\|v\|_{1/2,\partial\Omega}^2 \leqq \sum_{n=0}^{\infty} n \sum_{k=-n}^{n} \langle v, N_n^{(k)} \rangle^2 + \|v\|_{0,\partial\Omega}^2 = |\mathscr{J}v|_{1,\Omega}^2 + \|v\|_{0,\partial\Omega}^2 .$$

Taking into account (3.5), we immediately obtain the estimate (3.4). Q.E.D.

Notation. We introduce variants of problems (2.15) and (3.1) where the operators $\boldsymbol{K}^{-1}$ are replaced by the "truncated" versions $\boldsymbol{K}_p^{-1}$, see (3.3).

Let $u_p \in H^1(\Omega)$ satisfy

$$a(u_p, v) + \tfrac{1}{2}\langle \boldsymbol{K}^{-1} u_p, v \rangle + \frac{1}{2R} \langle u_p, v \rangle = \langle \boldsymbol{K}_p^{-1} w, v \rangle \quad \text{for each} \quad v \in H^1(\Omega) . \tag{3.6}$$

Let $u_p^h \in S^h$ solve

$$a(u_p^h, v) + \tfrac{1}{2}\langle \boldsymbol{K}^{-1} u_p^h, v \rangle + \frac{1}{2R} \langle u_p^h, v \rangle = \langle \boldsymbol{K}_p^{-1} w, v \rangle \quad \text{for each} \quad v \in S^h . \tag{3.7}$$

Let $u_{p,m}^h \in S^h$ solve

$$a(u_{p,m}^h, v) + \tfrac{1}{2}\langle \boldsymbol{K}_m^{-1} u_{p,m}^h, v \rangle + \frac{1}{2R} \langle u_{p,m}^h, v \rangle = \langle \boldsymbol{K}_p^{-1} w, v \rangle \quad \text{for each} \quad v \in S^h . \tag{3.8}$$

We note that the above problems (3.6)–(3.8) are uniquely solvable for each choice of $w \in H^{1/2}(\partial\Omega)$; the proof of this statement would follow the argument of the proof of Theorem 2.1.

Subtracting (3.8) and (3.7) yields

$$a(u_{p,m}^h - u_p^h, v) + \tfrac{1}{2}\langle \boldsymbol{K}_m^{-1}(u_{p,m}^h - u_p^h), v \rangle + \langle (\boldsymbol{K}_m^{-1} - \boldsymbol{K}^{-1}) u_p^h, v \rangle + \\ + \frac{1}{2R} \langle u_{p,m}^h - u_p^h, v \rangle = 0 ;$$

we set $v = u_{p,m}^h - u_p^h$. Due to the assumption (ii), the first term can be estimated by $c|u_{p,m}^h - u_p^h|_{1,\Omega}^2$.

The second term is nonnegative since $\boldsymbol{K}_m^{-1}$ is clearly positive definite, i.e. $\langle \boldsymbol{K}_m^{-1} v, v \rangle \geqq 0$.

Thus, we easily deduce the estimate

$$C_1 \|u_{p,m}^h - u_p^h\|_\Omega^2 \leqq \|(\boldsymbol{K}_m^{-1} - \boldsymbol{K}^{-1}) u_p^h\|_{-1/2,\partial\Omega} \|u_{p,m}^h - u_p^h\|_{1/2,\partial\Omega} ,$$

where $C_1 \overset{\text{def}}{=} 2 \min(c, 1/(2R))$. By virtue of (3.4) we find

$$C_1 \|u^h_{p,m} - u^h_p\|_\Omega \leqq \|(\boldsymbol{K}_m^{-1} - \boldsymbol{K}^{-1})\, u^h_p\|_{-1/2,\partial\Omega}\,. \tag{3.9}$$

We note that

$$(\boldsymbol{K}^{-1} - \boldsymbol{K}_m^{-1})\, v = \sum_{n=m+1}^{\infty} \frac{2n+1}{R} \sum_{k=-n}^{n} \langle v, N_n^{(k)} \rangle\, N_n^{(k)}$$

for each $v \in H^{1/2}(\partial\Omega)$. One can simply derive an estimate

$$\|(\boldsymbol{K}_m^{-1} - \boldsymbol{K}^{-1} v\|^2_{-1/2,\partial\Omega} \leqq \frac{4}{R^2} \sum_{n=m+1}^{\infty} (1 + n(n+1))^{1/2} \sum_{k=-n}^{n} \langle v, N_n^{(k)} \rangle^2\,.$$

In order to interpret its right hand side, we introduce

Notation. If $v \in H^r(\partial\Omega)$, r arbitrary, let $v = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \langle v, N_n^{(k)} \rangle\, N_n^{(k)}$ be the relevant Fourier expansion. For each positive integer m we define the projection Π_m: $H^r(\partial\Omega) \to H^r(\partial\Omega)$ as follows:

$$\Pi_m v \overset{\text{def}}{=} \sum_{n=0}^{m} \sum_{k=-n}^{n} \langle v, N_n^{(k)} \rangle\, N_n^{(k)}\,, \tag{3.10}$$

i.e., Π_m truncates the Fourier expansion of v.

Making use of the projection Π_m, we can estimate

$$\|(\boldsymbol{K}_m^{-1} - \boldsymbol{K}^{-1})\, v\|_{-1/2,\partial\Omega} \leqq \frac{2}{R} \|v - \Pi_m v\|_{1/2,\partial\Omega} \tag{3.11}$$

for each $v \in H^{1/2}(\partial\Omega)$.

Applying (3.11) to (3.9) for $v = u_p$, we conclude

$$\|(\boldsymbol{K}_m^{-1} - \boldsymbol{K}^{-1})\, u^h_p\|_{-1/2,\partial\Omega} \leqq$$

$$\leqq \frac{2}{R} \|u_p - \Pi_m u_p\|_{1/2,\partial\Omega} + \|(\boldsymbol{K}_m^{-1} - \boldsymbol{K}^{-1})\,(u_p - u^h_p)\|_{-1/2,\partial\Omega} \leqq$$

$$\leqq \frac{2}{R} \|u_p - \Pi_m u_p\|_{1/2,\partial\Omega} + \frac{4}{R} \|u_p - u^h_p\|_{1/2,\partial\Omega}\,;$$

the last inequality follows from the fact that

$$\|\boldsymbol{K}_m^{-1} v\|_{-1/2,\partial\Omega} \leqq \|\boldsymbol{K}^{-1} v\|_{-1/2,\partial\Omega} \leqq \frac{2}{R} \|v\|_{1/2,\partial\Omega}$$

for each $v \in H^{1/2}(\partial\Omega)$, see the proof of Lemma 2.1. Thus, using (3.9) and the embedding (3.4), we obtain

$$\|u^h_{p,m} - u^h_p\|_\Omega \leqq \frac{2}{C_1 R} \|u_p - \Pi_m u_p\|_{1/2,\partial\Omega} + \frac{4}{C_1 R} \|u_p - u^h_p\|_\Omega\,. \tag{3.12}$$

The complete error $u - u^h_{p,m}$ is then estimated by means of the triangle inequality:

(3.13) $$\|u - u^h_{p,m}\|_\Omega \leqq \|u - u_p\|_\Omega + \|u_p - u^h_p\|_\Omega + \|u^h_{p,m} - u^h_p\|_\Omega \leqq$$
$$\leqq \|u - u_p\|_\Omega + C_2\|u_p - \Pi_m u_p\|_{1/2,\partial\Omega} + C_3\|u_p - u^h_p\|_\Omega\,,$$

where $C_2 = 2(C_1 R)^{-1}$, $C_3 = 1 + 2C_2$. The contribution $\|u - u_p\|_\Omega$ is investigated in

Lemma 3.3. *The following inequalities hold*:

(3.14) $$\|u - u_p\|_\Omega \leqq C_4\|(\boldsymbol{K}^{-1} - \boldsymbol{K}_p^{-1})\, w\|_{-1,2,\partial\Omega} \leqq \frac{2C_4}{R}\,\|w - \Pi_p w\|_{1/2,\partial\Omega}\,,$$

where $C_4 = (\min(c, 1/(2R)))^{-1}$.

Proof. We set $v = u - u_p$. Subtracting (2.15) and (3.6) yields $a(v, v) + 1/2R\langle v, v\rangle + \frac{1}{2}\langle K^{-1}v, v\rangle = \langle(\boldsymbol{K}^{-1} - \boldsymbol{K}_p^{-1})\, w, v\rangle$. We note that $a(v, v) \geqq c|v|^2_{1,\Omega}$, $\langle \boldsymbol{K}^{-1}v, v\rangle \geqq 0$ and due to (3.4), $|\langle(\boldsymbol{K}^{-1} - \boldsymbol{K}_p^{-1})\, w, v\rangle| \leqq \|(\boldsymbol{K}^{-1} - \boldsymbol{K}_p^{-1})\,.\, w\|_{-1/2,\partial\Omega}\, \|v\|_\Omega$. The estimate (3.14) immediately follows. Q.E.D.

Remark. It can be shown that

(3.15) $$\langle \boldsymbol{K}^{-1}v, v\rangle \geqq \frac{1}{R}\,\|v\|^2_{1/2,\partial\Omega} \quad \text{for each} \quad v \in H^{1/2}(\partial\Omega)\,.$$

Using this inequality in the above proof, we can obtain (3.14) with a slightly different constant $C_4 = (2/R\min(c, 1/(2R)))^{-1/2}$ which is better then the former one if R is small.

We can conclude the question of convergence. According to (3.14), we can make the error $\|u - u_p\|_\Omega$ arbitratrily small by taking p large enough. The error $\|u_p - \Pi_m u_p\|_{1/2,\partial\Omega}$ can be controlled by the choice of m. The contribution $\|u_p - u^h_p\|_\Omega$ can be estimated by making suitable assumptions on the family of spaces S^h. In standard situations, $\|u_p - u^h_p\|_\Omega \to 0$ as $h \to 0$. Thus we resume that the error $\|u - u^h_{p,m}\|_\Omega$ can be made arbitrarily small by taking $p \to \infty$, $m \to \infty$ and $h \to 0$.

In the end we would like to make some remarks on the estimate of $\|u - u_p\|_\Omega$. The bound which Lemma 3.3 offers might be misleading in the case when $\|w\|_{1/2,\partial\Omega}$ itself is small. In other words, the only reasonable quantity to be estimated is the ratio

$$\frac{\|u - u_p\|_\Omega}{\|u\|_\Omega}\,.$$

Lemma 3.4. *There exists a constant* C_5 *such that*

(3.16) $$\|\boldsymbol{K}^{-1}w\|_{-1/2,\partial\Omega} \leqq C_5\|u\|_\Omega$$

for each $w \in H^{1/2}(\partial\Omega)$; *u is the relevant weak solution. Let C be a constant satisfying*

(iv) $$\Big|\sum_{i,j=1}^{3} a_{ij}\xi_i\eta_j\Big| \leqq C\Big(\sum_{i=1}^{3}\xi_i^2\Big)^{1/2}\Big(\sum_{i=1}^{3}\eta_i^2\Big)^{1/2}$$

for each $\xi, \eta \in \mathbb{R}^3$ a.e. on Ω (see the assumption (i)). *Then the constant C_5 can be taken as*

$$C_5 = \left(C^2 + \frac{1}{4R^2}\right)^{1/2} + \frac{1}{R}.$$

Proof. For a given $z \in H^{1/2}(\partial\Omega)$ we substitute $v = \mathscr{J}z$ into (2.15). Then

$$\|\boldsymbol{K}^{-1}w\|_{-1/2,\partial\Omega} = \sup_{\|z\|_{1/2,\partial\Omega}=1} \langle \boldsymbol{K}^{-1}w, z\rangle \leqq$$

$$\leqq \sup_{\|z\|_{1/2,\partial\Omega}=1} \mathfrak{a}(u, \mathscr{J}z) + 1/(2R)\,\|u\|_{-1/2,\partial\Omega} + \tfrac{1}{2}\,\|\boldsymbol{K}^{-1}u\|_{-1/2,\partial\Omega}.$$

By making use of (iv) we estimate $|\mathfrak{a}(u, \mathscr{J}z)| \leqq C|u|_{1,\Omega}\,|\mathscr{J}z|_{1,\Omega}$. According to Lemma 3.1, $|\mathscr{J}z|_{1,\Omega} \leqq \|z\|_{1/2,\partial\Omega} = 1$. Thus, $|\mathfrak{a}(u, \mathscr{J}z)| \leqq C|u|_{1,\Omega}$.

We have shown (in the proof of Lemma 2.1) that

$$\|\boldsymbol{K}^{-1}u\|_{-1/2,\partial\Omega} \leqq 2R^{-1}\|u\|_{1/2,\partial\Omega}.$$

By virtue of the embedding (3.4),

$$\frac{1}{2}\,\|\boldsymbol{K}^{-1}u\|_{-1/2,\partial\Omega} \leqq \frac{1}{R}\,\|u\|_{\Omega}.$$

Finally, we note that $\|u\|_{-1/2,\partial\Omega} \leqq \|u\|_{0,\partial\Omega}$. Combining the above inequalities, we easily derive (3.16). Q.E.D.

Lemmas 3.3 and 3.4 yield the estimate

$$\frac{\|u - u_p\|_{\Omega}}{\|u\|_{\Omega}} \leqq C_4 C_5^{-1}\,\frac{\|(\boldsymbol{K}^{-1} - \boldsymbol{K}_p^{-1})\,w\|_{-1/2,\partial\Omega}}{\|\boldsymbol{K}^{-1}w\|_{-1/2,\partial\Omega}}. \tag{3.17}$$

As an illustration, we estimate the above error in the important case of a point charge, i.e., we assume

$$w(x) = \frac{Q}{|x - y|} \quad \text{for each} \quad x \in \mathbb{R}^3,$$

where Q is a constant and $y \in \mathbb{R}^3 - \bar{\Omega}$ is fixed. Without loss of generality, let $y = (0, 0, \varrho)$. Then $1/|x - y| = (\varrho^2 + r^2 - 2r\varrho\cos\alpha)^{-1/2}$ in the spherical coordinates, $x = (r, \alpha, \vartheta)$. Expanding $(1 + \xi^2 - 2\xi\cos\alpha)^{-1/2} = \sum_{n=0}^{\infty} P_n(\cos\alpha)\,\xi^n$ we find $w(x) = Q/\varrho \sum_{n=0}^{\infty} P_n(\cos\alpha)\,(r/\varrho)^n$.

It is easy to project w to $N_n^{(k)}$:

$$\langle w, N_n^{(k)}\rangle = 0 \quad \text{for} \quad k \neq 0,$$

$$\langle w, N_n^{(0)}\rangle = QR\left(\frac{4\pi}{2n + 1}\right)^{1/2}\frac{R^n}{\varrho^{n+1}}, \quad n = 0, 1, \ldots$$

Then a simple manipulation yieds the estimate

$$(3.18) \qquad \frac{\|(\mathbf{K}^{-1} - \mathbf{K}_p^{-1}) w\|_{-1/2,\partial\Omega}}{\|\mathbf{K}^{-1} w\|_{-1/2,\partial\Omega}} \leqq \sqrt{2} \left(\frac{R}{\varrho}\right)^{p+1} .$$

Let w_1 be the Taylor expansion of w of the first order at the origin, i.e.

$$w_1(x) = \frac{Q}{\varrho} + \frac{Q}{\varrho^3} \sum_{i=1}^{3} x_i y_i ;$$

the function w_1 is called a plane wave approximation of w in a neighborhood of the origin.

One can check that in fact

$$w_1 = \sum_{n=0}^{1} \sum_{k=-n}^{n} \langle w, N_n^{(k)} \rangle N_n^{(k)} .$$

Then (3.17) and (3.18) yield the estimate

$$\frac{\|u - u_1\|_\Omega}{\|u\|_\Omega} \leqq \sqrt{(2)}\, C_4 C_5^{-1} \left(\frac{R}{\varrho}\right)^2$$

which gives a qualitatively meaning to the intuitive claim that a plane wave is a good approximation of w if the source is "far enough", i.e. if $|y| = \varrho$ is large.

References

[1] *V. Drápalík, V. Janovský:* On a potential problem with incident wave as a field source. Aplikace matematiky 33 (1988), 443—455

[2] *J. L. Lions, E. Magenes:* Problèmes aux limites non homogènes et applications. Dunod, Paris 1968.

[3] *G. C. Hsiao, P. Kopp, W. L. Wendland:* A Galerkin collocation method for some integral equations of the first kind. Computing 25 (1980), 89—130.

[4] *G. C. Hsiao, W. L. Wendland:* A finite element method for some integral equations of the first kind. J. Math. Appl. Anal. 58 (1977), 449—481.

[5] *C. Johnson, J. C. Nedelec:* On the coupling of boundary integral and finite element methods. Math. Comp. 35 (1980), 1063—1079.

Souhrn

NUMERICKÉ ŘEŠENÍ TŘÍDIMENZIONÁLNÍ POTENCIÁLNÍ ÚLOHY

VLADIMÍR DRÁPALÍK, VLADIMÍR JANOVSKÝ

Řeší se třírozměrný model difrakce elektrostatického pole na omezeném nehomogenním tělese. Pomocí vhodné nelokální okrajové podmínky lze úlohu formulovat na kouli, obsahující zadanou nehomogenitu.

Je ukázána existence a jednoznačnost řešení redukované úlohy. Tato úloha je potom aproximována metodou konečných prvků s tím, že nelokální hraniční podmínka je nahrazena částeč-

ným Fourierovým rozvojem do vlastních funkcí hraničního integrálního operátoru. Je analyzována konvergence metody.

Резюме

ЧИСЛЕННОЕ РЕШЕНИЕ ТРЕХМЕРНОЙ ЗАДАЧИ ТЕОРИИ ПОТЕНЦИАЛА

VLADIMÍR DRÁPALÍK, VLADIMÍR JANOVSKÝ

Рассматривается дифракция электростатического поля в заданной ограниченной среде. При помощи интегрального граничного условия задача формулируется на шаре, окружающем заданное тело.

Предлагается численное решение редуцированной задачи методом конечных элементов. Граничное условие аппроксимируется частичной суммой разложения фурье по собственным функциям интегрального оператора. Показывается сходимость метода.

Authors' addres: RNDr. *Vladimír Drápalík*, doc. RNDr. *Vladimír Janovský*, CSc., MFF UK, Malostranské nám. 25, 118 00 Praha 1.