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NUMERICAL ANALYSIS FOR OPTIMAL SHAPE DESIGN IN ELLIPTIC BOUNDARY VALUE PROBLEMS

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Summary. Shape optimization problems are optimal design problems in which the shape of of the boundary plays the role of a design, i.e. the unknown part of the problem. Such problems arise in structural mechanics, acoustics, electrostatics, fluid flow and other areas of engineering and applied science. The mathematical theory of such kind of problems has been developed during the last twelve years. Recently the theory has been extended to cover also situations in which the behaviour of the system is governed by partial differential equations with unilateral boundary conditions. In the paper an efficient method of nonlinear programming for solving optimal shape design problems is presented. The effectiveness of the technique proposed is demonstrated by numerical examples.

Keywords: optimization, elliptic boundary value problems, nonlinear programming, finite element method.

AMS Subject classification: 49A22, 49A29, 49D37, 65N30.

1. INTRODUCTION

Shape optimization problems are optimal design problems in which the shape of the boundary plays the role of a design. Such problems arise in structural mechanics, acoustics, electrostatics, fluid flow and other areas of engineering and applied science.

The mathematical theory of such kind of problems has been developed during the last twelve years [7], [8], [10]. Recently the theory has been extended to cover also situations in which the behaviour of the system is governed by partial differential equations with unilateral boundary conditions [1], [3], [4], [9], [11], [14]. The related problems are treated in [13], [15], [16].

In the present paper we focus our attention to finding an efficient method of nonlinear programming for solving the optimal shape design problems. A numerical study is presented for several types of optimal shape design problems. Some comparison is made of the exact and the numerical initial values of the cost functionals.

2. OPTIMIZATION PROBLEMS

Let us consider the following model problems. Let $\Omega(v) \subset \mathbb{R}^2$ be a domain (see Fig. 1) with the following geometrical structure:

$$\Omega(v) = \{0 < x_1 < v(x_2),\ 0 < x_2 < 1\},$$
$$\partial\Omega(v) = \Gamma_1 \cup \Gamma(v), \quad \text{the boundary of } \Omega(v) \text{ with}$$
$$\Gamma_1 = \partial\Omega(v) - \Gamma(v),$$
$$\Gamma(v) = \{x \in \mathbb{R}^2,\ x_1 = v(x_2),\ 0 \leqq x_2 \leqq 1\}$$

where the function $v \in C^{0,1}(\langle 0, 1\rangle)$, i.e. a Lipschitz function, is to be determined from one of the domain optimization problems

$$(\mathrm{P}_i) \qquad \min_{v \in U_{ad}} J_i(y(v), v), \quad i = 1, 2, 3, 4.$$

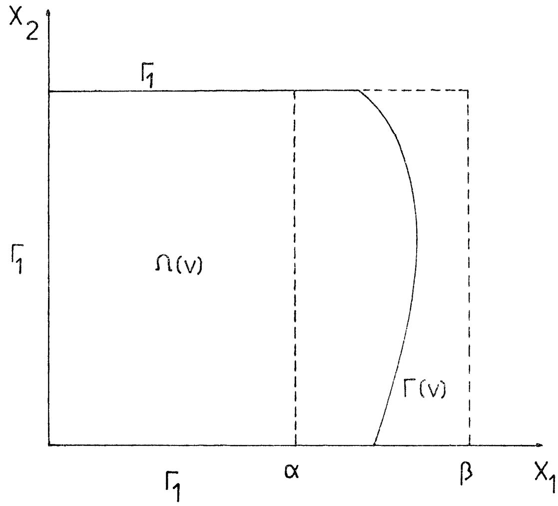

Fig. 1

Here

$$U_{ad} = \{v \in C^{0,1}(\langle 0, 1\rangle),\ 0 < \alpha \leqq v(x_2) \leqq \beta,$$
$$|\mathrm{d}v/\mathrm{d}x_2| \leqq C_1,\ \textstyle\int_0^1 v(x_2)\,\mathrm{d}x_2 = C_2\}$$

where α, β, C_1, C_2 are given positive constants, and the cost functionals are

$$J_1(y(v), v) = \int_{\Omega(v)} (y(v))^2\,\mathrm{d}x,$$
$$J_2(y(v), v) = 1/2 \int_{\Gamma(v)} (y(v))^2\,\mathrm{d}s,$$
$$J_3(y(v), v) = 1/2 \int_{\Omega(v)} \nabla y(v)^2\,\mathrm{d}x,$$
$$J_4(y(v), v) = \|\partial y(v)/\partial n\|^2_{-1/2,\Gamma(v)} = \|u(y(v))\|^2_{1,\Omega(v)}.$$

The function $y(v)$ is the solution of the boundary value problem (the state problem)

$$(\mathrm{SP}_1) \qquad \begin{aligned} -\Delta y &= f \quad \text{in} \quad \Omega(v)\,, \\ y &= 0 \quad \text{on} \quad \Gamma(v)\,, \\ \partial y/\partial n &= 0 \quad \text{on} \quad \Gamma_1(v) = \partial\Omega(v) - \Gamma(v)\,, \end{aligned}$$

or

$$(\mathrm{SP}_2) \qquad \begin{aligned} -\Delta y &= f \quad \text{in} \quad \Omega(v)\,, \\ y &= 0 \quad \text{on} \quad \Gamma_1(v) = \partial\Omega(v) - \Gamma(v)\,, \\ y \geqq 0\,, \quad \partial y/\partial n &\geqq 0\,, \quad y\,\partial y/\partial n = 0 \quad \text{on} \quad \Gamma(v) \end{aligned}$$

for a given function $f \in L^2(\Omega_\beta)$, where $\Omega_\beta = (0, \beta) \times (0, 1)$ and $\partial y/\partial n$ denotes the derivative with respect to the outward normal to $\Gamma(v)$.

In variational formulation, (SP_1) reads:
Find $y = y(v) \in V(v)$ such that

$$(\mathrm{VP}_1) \qquad \int_{\Omega(v)} \nabla y \, \nabla w \, \mathrm{d}x = \int_{\Omega(v)} f w \, \mathrm{d}x \quad \forall w \in V(v)$$

where

$$V(v) = \{w \in H^1(\Omega(v)),\ w = 0 \text{ on } \Gamma(v)\}\,.$$

We denote by $H^k(\Omega)$ the Sobolev space $W_2^{(k)}(\Omega)$ with the usual norm $\|\cdot\|_{k,\Omega}$, $H^0 = L^2$, with the scalar product $(\cdot, \cdot)_{0,\Omega}$.

The symbol $\|\partial y/\partial n\|_{-1/2,\Gamma(v)}$ represents the norm of the boundary flux in the space $H^{-1/2}(\Gamma(v)) = [H^{1/2}(\Gamma(v))]'$ (dual space). For details we refer to the paper [2].

Remark 2.1. The problem (VP_1) has a unique solution for any $v \in U_{\mathrm{ad}}$.

The state problem (SP_2) can be formulated in terms of a variational inequality as follows.

Find $y = y(v) \in K(v)$ such that

$$(\mathrm{VP}_2) \qquad \int_{\Omega(v)} \nabla y \, \nabla(z - y) \, \mathrm{d}x \geqq \int_{\Omega(v)} f(z - y) \, \mathrm{d}x \quad \forall z \in K(v)$$

where

$$K(v) = \{z \in H^1(\Omega(v)),\ z = 0 \text{ on } \Gamma_1(v),\ z \geqq 0 \text{ on } \Gamma(v)\}\,.$$

Remark 2.2. The problem (VP_2) has a unique solution for any $v \in U_{\mathrm{ad}}$.

Theorem 2.1. *The problems* (P_i) *for the cost functionals* J_i $(i = 1, 2, 3)$ *with the state problem* (SP_2) *have at least one solution.*

Proof. See [1], Th. 1.

Theorem 2.2. *The problems* (P_i) *for the cost functionals* J_i $(i = 3, 4)$ *with the state problem* (SP_1) *have at least one solution.*

Proof. See [2], Th. 2.1.

Remark 2.3. The existence theorem for the problem (P_4) for the cost functional J_4 with the state unilateral problem (SP_2) has not been proved yet. A closely related problem has been considered in [12]. Nonetheless, the numerical results of this optimal shape problem are presented in Sec. 4.

3. APPROXIMATE SOLUTION

3.1. The Primal Finite Element Method

The problems (P_i), $i = 1, 2, 3$ with the state problem (SP_2) or (VP_2) can be solved by the "displacement" finite elements making use of the primal variational formulation. To this end we follow the approach of [1], transforming each of the problems (P_i), $i = 1, 2, 3$ into an equivalent one with the state problem defined on a fixed square domain and then employing bilinear finite elements on a uniform mesh. The unknown part of the boundary is sought among continuous piecewise linear functions. Thus, let N be a positive integer and $h = 1/N$. Denote by e_j, $j = 1, \ldots, N$ the interval $\langle (j-1)\,h, jh \rangle$ and introduce the set

$$U_{ad}^h = \{w_h \in U_{ad},\ w_h|_{e_j} \in P_1,\ \forall j\}$$

where P_1 denotes the space of linear polynomials.

Let Ω_h denote the domain bounded by the graph Γ_h of the function $w_h \in U_{ad}^h$, i.e. $\Omega_h = \Omega(w_h)$.

We define

$$\begin{aligned}
\hat{\Omega} &= (0,1) \times (0,1)\,,\\
\hat{K}_{ij} &= \langle (i-1)\,h, ih \rangle \times \langle (j-1)\,h, jh \rangle\,,\\
\mathscr{K}_{\hat{h}} &= \{\hat{K}_{ij}\}_{i,j=1}^{N}\,,\\
F_h &: \hat{\Omega} \to \Omega_h\,, \quad F_h = (F_{1h}, F_{2h})\,,\\
F_{1h}(\hat{x}_1, \hat{x}_2) &= \hat{x}_1\, w_h(\hat{x}_2)\,,\\
F_{2h}(\hat{x}_1, \hat{x}_2) &= \hat{x}_2\,,\\
K_{ij} &= F_h(\hat{K}_{ij}) \quad \forall i, j\,,\\
\mathscr{K}_h &= \{K_{ij}\}_{i,j=1}^{N}\,.
\end{aligned} \tag{3.1}$$

Note that each K_{ij} is a trapezoid and

$$F_h|_{K_{ij}} \in Q_1 \times Q_1$$

where $Q_1 = \{p,\ p = p(\hat{x}_1, \hat{x}_2) = a_{00} + a_{10}\hat{x}_1 + a_{01}\hat{x}_2 + a_{11}\hat{x}_1\hat{x}_2\}$ denotes the space of bilinear polynomials.

Let us consider the problem (VP_2) on the domain Ω_h. To approximate $K(w_h)$ we introduce the set

$$K_h = \{z_h, z_h \in K(w_h) \cap C(\Omega_h), z_h \circ F_h|_{R_{ij}} \in Q_1 \ \forall i, j\} .$$

Let us define the solution of the approximate state problem as

$(SP_2)^h$ the solution $y_h \in K_h$ of (VP_2) on Ω_h for any $z_h \in K_h$.

Instead of (VP_2), however, it is more suitable to solve numerically an equivalent problem on $\hat{\Omega}$, which is obtained by the transformation (3.1) of the integrals in (VP_2) [1].

The cost functionals J_i, $i = 1, 2, 3$ will be replaced by the approximate functionals J_i^h, $i = 1, 2, 3$. Then we will solve the problem

$(P_i)^h$ $$\min_{v_h \in U_{ad}^h} J_i^h(y(v_h), v_h) , \quad i = 1, 2, 3 .$$

Remark 3.1. A subsequence of solutions of $(P_i)^h$ exists and converges in some sense to a solution of the continuous problem (P_i), i = 1, 2, 3, if h tends to zero (see Th. 3.1 in [1]).

3.2. The Dual Finite Element Method

Since the cost functionals J_i, $i = 3, 4$ are expressed in terms of the gradient ∇y and not in terms of the function y itself, it seems to be of advantage to employ the dual variational formulation of the state problem. Thus we shall calculate the gradient ∇y directly.

To this aim we introduce the space of solenoidal (divergence-free) vector functions

$$Q_0(v) = \{\boldsymbol{q} \in [L^2(\Omega(v))]^2, \operatorname{div} \boldsymbol{q} = 0 \text{ in } \Omega(v), \boldsymbol{q} . \nu = 0 \text{ on } \partial\Omega(v) - \Gamma(v)\} =$$
$$= \{\boldsymbol{q} \in [L^2(\Omega(v))]^2, \textstyle\int_{\Omega(v)} \boldsymbol{q} . \nabla w \, dx = 0 \ \forall w \in V(v)\} .$$

Let us construct the vector field $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)$:

(3.2) $$\bar{\lambda}_1(x_1, x_2) = - \textstyle\int_0^{x_1} f(t, x_2) \, dt , \qquad \bar{\lambda}_2 = 0$$

assuming that the integral has sense for $x_2 = 0$, $x_2 = 1$ and almost all $x_2 \in (0, 1)$.

It is readily seen that

$$\operatorname{div} \bar{\lambda} = \partial\lambda_1/\partial x_1 = -f \quad \text{in} \quad \Omega_\beta ,$$
$$\bar{\lambda} . \nu = \bar{\lambda}_1 \nu_1 = 0 \quad \text{on} \quad \partial\Omega_\beta - \Gamma_\beta$$

where

$$\Gamma_\beta = \{(x_1, x_2), x_1 = \beta, x_2 = (0, 1)\} .$$

Then a suitable dual formulation of the problem (SP_1) or (VP_1) is

(3.3) to find $\boldsymbol{q}(v) \in Q_0(v)$ such that

$$(\boldsymbol{q}(v), \boldsymbol{p})_{0,\Omega(v)} = -(\bar{\lambda}, \boldsymbol{p})_{0,\Omega(v)} \quad \forall \boldsymbol{p} \in Q_0(v) .$$

Remark 3.2. There exists a unique solution of (3.3) and

$$\bar{\lambda} + \boldsymbol{q}(v) = \nabla y(v)$$

holds. Henceforth $\bar{\lambda}$ denotes everywhere the restriction of the vector field (3.2) onto the domain under consideration and $y(v)$ is the solution of (VP_1).

The cost functionals J_i, $i = 3, 4$, can be rewritten as

$$J_3(y(v), v) = \|\bar{\lambda} + \boldsymbol{q}(v)\|_{0,\Omega(v)}^2 = J_3^*(\boldsymbol{q}(v))$$

and

$$J_4(y(v), v) = \|u(\boldsymbol{q}(v))\|_{1,\Omega(v)}^2 = J_4^*(\boldsymbol{q}(v)) \tag{3.4}$$

where

$u = u(\boldsymbol{q}(v))$ is the solution of the auxiliary problem

$$u \in V^C(v) ,$$

$$\int_{\Omega(v)} (\nabla u \cdot \nabla w + uw) \, \mathrm{d}x = \int_{\Omega(v)} ((\bar{\lambda} + \boldsymbol{q}(v)) \cdot \nabla w - wf) \, \mathrm{d}x \quad \forall w \in V^C(v)$$

where

$$V^C(v) = \{w \in H^1(\Omega), \gamma w = 0 \text{ on } \partial\Omega - \Gamma_0 ,$$

γ is the trace operator, Γ_0 is an "extension" of Γ such that $\bar{\Gamma} \subset \Gamma_0 \subset \partial\Omega$, Γ_0 is connected and open in $\partial\Omega\}$.

For the proof of (3.4) see [2].

Remark 3.3. Theorem 2.1 in [2] yields the existence of a solution of the equivalent optimization problem

$$\min_{v \in U_{ad}} J_i^*(\boldsymbol{q}(v)) , \quad i = 3, 4 .$$

The domain Ω_h will be divided into triangles by the moving mesh technique as follows (Fig. 2).

We choose $\alpha_0 \in (0, \alpha)$ and introduce a uniform triangulation of the rectangle $R = \langle 0, \alpha_0 \rangle \times \langle 0, 1 \rangle$, independent of v_h if h is fixed. In the remaining part $\Omega_h - R$ let the nodal points divide the intervals $\langle \alpha_0, v_h(jh) \rangle$ into M uniform segments, where $M = 1 + \operatorname{int}((\beta - \alpha_0) N)$ ("int" denotes the integer part of the number). One can easily find that then the segments parallel to the x_1-axis are not longer than h and shorter than $h(\alpha - \alpha_0)/(\beta - \alpha_0)$. One also deduces the following estimate for the interior angles ω of the triangulation:

$$\operatorname{tg} \omega \geqq (\alpha - \alpha_0)/(\beta - \alpha_0)(1 + C_1 + C_1^2)^{-1} .$$

Consequently, one obtains a regular family $\{\mathcal{T}_h(v_h)\}$ of triangulations, with

$$\max_{K \in \mathcal{T}_h(v_h)} (\operatorname{diam} K) \leqq h/\sin \omega_0 ,$$

$$\omega \geqq \omega_0 = \operatorname{arctg} ((\alpha - \alpha_0)/(\beta - \alpha_0) (1 + C_1 + C_1^2)^{-1} .$$

Let us consider the space $\mathcal{N}_h(v_h)$ of piecewise linear solenoidal (divergence-free) functions on the triangulation $\mathcal{T}_h$ and define ([17])

$$S_h = \mathcal{N}_h(v_h) \cap Q_0(v_h) = \{\boldsymbol{q}_h \in \mathcal{N}(v_h),\ \boldsymbol{q}_h \cdot \nu = 0 \text{ on } \partial\Omega_h - \Gamma_h\} .$$

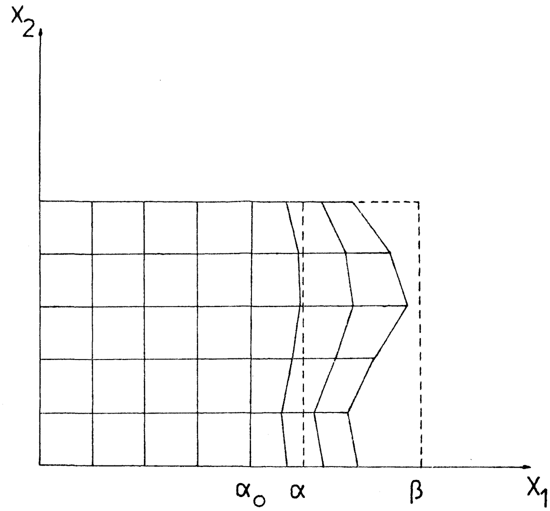

Fig. 2

Instead of the problem (SP_1) or (3.3) we solve the aproximate problem

$(SP_1)^h$ to find $\boldsymbol{q}^h(v_n) \in S_h$ such that

$$(\boldsymbol{q}^h(v_h), \boldsymbol{p}^h)_{0,\Omega_h} = -(\bar{\lambda}, \boldsymbol{p}^h)_{0,\Omega_h} \quad \forall \boldsymbol{p}^h \in S_h .$$

Remark 3.4. There exists a unique solution of $(SP_1)^h$ for any h and any $v_h \in U^h_{\text{ad}}$.

Remark 3.5. The cost functionals J_i, $i = 3, 4$, will be replaced by approximate functionals J_i^{*h}, $i = 3, 4$, and we then solve the problem

$(P_i)^h$ $$\min_{v_h \in U_{\text{ad}}{}^h} J_i^{*h}(\boldsymbol{q}(v_h), v_h) .$$

The approximate solutions of $(P_i)^h$ converge in some sense to a solution of the continuous problem (P_i), i = 3, 4 (see Th. 5.1, 6.1 in [2]).

3.3. Algorithm

The problems $(P_i)^h$ have to be solved iteratively. Let $v^k \in U^h_{\text{ad}}$ be a given function. We will denote the functions v^k and the vectors $\{v^k(jh)\}_{j=0}^N$, where $h = 1/N$, by the same symbol.

We want to find a new iteration, say $v^{k+1} \in U^h_{ad}$, such that

$$F(y(v^{k+1}), v^{k+1}) < F(y(v^k), v^k)$$

where

$$F = J^h_i \quad \text{or} \quad J^{*h}_i .$$

More precisely, the algorithm for solving the problems $(P_i)^h$ reads:

Algorithm 3.1. (Method of feasible direction [5], [6].)

Step 0. Choose $v^0 \in U^h_{ad}$ arbitrarily. Set $k = 0$, $\Omega^k = \Omega(v^k)$.

Step 1. Solve the state $y^k = y(v^k)$ in the domain Ω^k from the problem $(SP_j)^h$, $j = 1$ or 2.

Step 2. Find a feasible direction of descent $d \in C^{0,1}(\langle 0, 1 \rangle)$. If this is not possible stop, otherwise go to Step 3.

Step 3. Find $\lambda > \varepsilon_1$ such that

$$F(y(v^k + \lambda d), v^k + \lambda d) < F(y(v^k), v^k) - \varepsilon_2 , \quad v^k + \lambda d \in U^h_{ad}$$

where $\varepsilon_1, \varepsilon_2 > 0$ are given constants. If this is not possible then stop, otherwise set

$$v^{k+1} = v^k + \lambda d ,$$

$$\Omega^{k+1} = \Omega(v^{k+1}) ,$$

$$k = k + 1$$

and go to Step 1.

A natural choice for the direction of descent is the negative gradient of the cost functional.

Thus we are led to the following questions:

- Do the gradients of the cost functionals exist?
- How does one get the gradients of the cost functionals with respect to the function v^k?

Some answers to these questions are given in the paper [4], where the gradients are obtained in different ways.

In the present paper we compute an approximate gradient of F by finite differences

$$\partial F(y(v^k), v^k)/\partial v^k = (F(y(v^k + \lambda e_j), v^k + \lambda e_j) - F(y(v^k), v^k))/\lambda$$

where

$$e_j = (0, \ldots, \overset{j\text{-th}}{1}, \ldots, 0) ,$$

$$\lambda = 0{\cdot}001 \quad \text{(for example)} .$$

Remark 3.6. Note that the function $v^k \to J^h_i(y(v^k), v^k)$ need not be convex. Consequently, Algorithm 3.1 cannot guarantee obtaining more than a local minimum. However, in many practical applications this suffices to improve the performance of a system.

Remark 3.7. The approximate problem $(P_i)^h$, i = 1, 2, 3, 4, is a nonlinear programming problem with constraints. One possible approach to solve it is to use Algorithm 3.1. The state problem $(SP_2)^h$ can be solved by the primal finite element method (see sec. 3.1) with the *SSOR* and the projection method [18].

4. NUMERICAL TESTS

Several numerical tests carried out in order to study the performance of the method proposed.

The results of the shape optimization problems $(P_i)^h$, i = 1, 2, 3 or $(P_1)^h$ with the state problem $(SP_2)^h$ or $(SP_1)^h$ have been presented in [4] and [7], respectively. Therefore we focus our attention to the problems $(P_4)^h$ and $(P_3)^h$ with the state problem $(SP_2)^h$ and $(SP_1)^h$, respectively. The functional J_4^h has been replaced by

$$\hat{J}_4^h(y_h, v^k) = \int_0^1 (\partial y_h/\partial n)^2 \, dx_2 \, .$$

The results are presented in Table 4.1.

In the first five examples we solved the shape optimization problem $(P_4)^h$ with the state problem $(SP_2)^h$ using the primal finite element method (Sec. 3.1). The right hand side f of the $(SP_2)^h$ was either

$$f_1 = -1 \tag{3.5}$$

or

$$f_2 = 4 \sin 2\pi x_2 \, , \tag{3.6}$$

$$f_3 = 8 \sin 2\pi x_1 \sin 2\pi x_2 \, , \tag{3.7}$$

$$f_4 = 2((1/v + v' x_1^2/v^3)\, x_2(x_2 - 1) + + (1 - 2x_2)\, x_1^2 v'/v^2 + x_1(x_1/v - 1)) \, , \tag{3.8}$$

$$f_5 = 5 \sin 2\pi x_1 \sin \pi x_2 \, . \tag{3.9}$$

We chose the constants $\alpha = 0{\cdot}5$, $\beta = 1{\cdot}5$, $C_1 = 1$, $C_2 = 1$. The domain $\Omega(v^k)$ was divided into 128 ($h = 1/8$) triangles. The initial value of the unknown boundary was chosen to be $v^0 = 1$.

The final three applications test the shape optimization problem $(P_3)^h$ with the state problem $(SP_1)^h$. In this case we applied the dual finite element method (Sec. 3.2). The right hand side f of the $(SP_1)^h$ was either (3.5) or

$$f_6 = -x_2^2(1 - x_2)^2\,(2/v^2 + 6v'^2 x_1^2/v^4) + + 8v' x_1^2/v^3 (x_2 - x_2^2)\,(1 - 2x_2) + + 2(1 - 6x_2 + 6x_2^2)\,(1 - x_1^2/v^2) \, , \tag{3.10}$$

Table 4.1 Numerical Tests ($v_j^{\min} = v^{\min}(jh)$)

Nro.	J_i^h	$(SP_i)^h$	f	$J_{i,\text{init}}^h$	$J_{i,\text{init}}^{\text{exact}}$	$J_{i,\min}^h$	CPU [s]
1	J_4^h	$(SP_2)^h$	(3·5)	0·0451	–	0·0451	12
	$v_j^{\min}$ = 1., 1., 1., 1., 1., 1., 1., 1., 1.						
2	J_4^h	$(SP_2)^h$	(3·6)	0·0351	–	0·0318	301
	$v_j^{\min}$ = 0·98, 0·99, 1·05, 1·04, 1·01, 0·97, 0·98, 0·98, 0·97						
3	J_4^h	$(SP_2)^h$	(3·7)	0·0918	–	0·0155	387
	$v_j^{\min}$ = 0·97, 1·10, 1·22, 1·10, 0·97, 0·93, 0·87, 0·88, 0·89						
4	J_4^h	$(SP_2)^h$	(3·8)	0·0259	1/30	0·0168	370
	$v_j^{\min}$ = 0·83, 0·95, 1·06, 1·10, 1·10, 1·07, 1·00, 0·88, 0·86						
5	J_4^h	$(SP_2)^h$	(3·9)	0·1872	$2/\pi^2$	0·1712	401
	$v_j^{\min}$ = 0·82, 0·94, 1·03, 1·09, 1·11, 1·08, 1·02, 0·93, 0·80						
6	J_3^h	$(SP_1)^h$	(3·5)	4·8234	–	2·7298	520
	$v_j^{\min}$ = 1·40, 1·31, 1·11, 0·91, 0·71, 0·51						
7	J_3^h	$(SP_1)^h$	(3·10)	0·0417	0·0123	0·0332	610
	$v_j^{\min}$ = 1·10, 1·01, 1·05, 1·04, 0·97, 0·77						
8	J_3^h	$(SP_1)^h$	(3·11)	0·0542	–	0·0520	605
	$v_j^{\min}$ = 0·92, 1·01, 1·06, 1·02, 0·98, 0·93						

$J_{i,\text{init}}^h$ resp. $J_{i,\min}^{\text{exact}}$ – the numerical resp. exact value of the cost functional for the initial value of v^0,

$J_{i,\min}^h$ – the numerical value of the cost functional for the value of $v_j^{\min}$.

$$(3.11) \qquad f_7 = -2x_2^2(3x_1 - 1)(2x_2 - 3) - 6x_1^2(x_1 - 1)(2x_2 - 1).$$

The domain $\Omega(v^k)$ was divided into 50 ($h = 1/5$) triangles. The other constants used were the same as in the first five tests.

By comparison we see that in the cases 4, 5 and 7 the method seems to provide reasonably good results of the exact and numerical values of the cost functionals for the initial value of v^0 even for relatively coarse element mesh.

Note that for other initial values of v^0 we can obtain lower values of the local minimum of J_i.

The tests were carried out with ICL 2958 computer.

5. CONCLUSIONS

From the numerical study we have seen a good performance of the method proposed. As a summary we may conclude that the numerical gradient gives valuable results. It is a most straightforward and widely used method in practice.

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Souhrn

NUMERICKÁ ANALÝZA PRO OPTIMÁLNÍ NÁVRH OBLASTI V ELIPTICKÝCH OKRAJOVÝCH PROBLÉMECH

Zdeněk Kestřánek

V článku je užito numerické analýzy k výběru vhodné nelineární optimalizační metody pro řešení návrhu oblasti v eliptických problémech s jednostrannými okrajovými podmínkami. K aproximaci úlohy je užito primární i duální formulace metody konečných prvků. Na numerických příkladech je ukázáno chování navržené metody.

Резюме

ЧИСЛЕННЫЙ АНАЛИЗ ДЛЯ ОПТИМИЗАЦИИ ФОРМЫ ОБЛАСТИ В ЭЛЛИПТИЧЕСКИХ КРАЕВЫХ ПРОБЛЕМАХ

Zdeněk Kestřánek

В статье применяется численный анализ к определению нелинейного метода оптимизации формы области для эллиптических задач с односторонними краевыми условиями. Для аппроксимации задачи используются первоначальная и двойственная формулировка метода конечных элементов. На численных примерах показаны свойства указанного метода.

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