NOTE ON THE ESTIMATION OF PARAMETERS OF THE MEAN AND THE VARIANCE IN n-STAGE LINEAR MODELS

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Summary. The paper deals with the estimation of the unknown vector parameter of the mean and the parameters of the variance in the general n-stage linear model. Necessary and sufficient conditions for the existence of the uniformly minimum variance unbiased estimator (UMVUE) of the mean-parameter under the condition of normality are given. The commonly used least squares estimators are used to derive the expressions of UMVUE-s in a simple form.

Keywords: Variance-components model, n-stage linear model, estimation of parameters.
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INTRODUCTION

An n-stage linear model is frequently modelled as follows:

$$(1)\qquad \begin{aligned} \mathbf{Y}_1 &= \mathbf{X}_1\boldsymbol{\beta}_1 + \varepsilon_1\,, \\ \mathbf{Y}_2 &= \mathbf{C}_{2,1}\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2 + \varepsilon_2\,, \\ &\dots \\ \mathbf{Y}_n &= \mathbf{C}_{n,1}\boldsymbol{\beta}_1 + \mathbf{C}_{n,2}\boldsymbol{\beta}_2 + \dots + \mathbf{C}_{n,n-1}\boldsymbol{\beta}_{n-1} + \mathbf{X}_n\boldsymbol{\beta}_n + \varepsilon_n\,. \end{aligned}$$

The vectors $\mathbf{Y}_i$, $i = 1, \dots, n$ are n_i-dimensional normally distributed vectors of measurements, each at the i-th stage. The vectors $\boldsymbol{\beta}_i$ are k_i-dimensional, unknown, and are to be estimated. The $n_i \times k_i$-matrices $\mathbf{X}_i$ are considered to be known and of full column rank; the matrices $\mathbf{C}_{i,j}$ are of the type $n_i \times k_j$ and are known — they represent the connections between the parameters of different stages. The vectors ε_i are uncorrelated with $\varepsilon_i \sim N_{n_i}(\mathbf{O}, \sigma_i^2 \mathbf{I}_{n_i})$, where the parameters σ_i^2, $i = 1, \dots, n$ are unknown, such that we assume $\sigma_i^2 \neq \sigma_j^2$, $i \neq j$, and the matrices $\mathbf{I}_{n_i}$ denote the identity matrices of the types $n_i \times n_i$.

Kubáček in [2] considers this type of the model in the case that the covariance matrix of ε_i is Σ_i, $i = 1, \dots, n$, and it is known at each stage.

The aim of this paper is to give necessary and sufficient conditions for the existence of UMVUE (uniformly minimum variance unbiased estimator) for $\boldsymbol{\beta}_i$, $i = 1, \dots, n$, based on the measurements of all stages; to give explicit formulae for UMVUE's. Apart from that, to give UBUE's (uniformly best unbiased estimators) for σ_i^2, $i = 1, \dots, n$.

SOLUTION OF THE PROBLEM

The model (1) can be obviously expressed in the form which leads to the commonly known variance–components model:

$$\mathbf{Y}^* = \mathbf{X}\boldsymbol{\beta}^* + \boldsymbol{\varepsilon}^* \tag{2}$$

where $\mathbf{Y}^* = (\mathbf{Y}_1', \ldots, \mathbf{Y}_n')'$, $\boldsymbol{\beta}^* = (\boldsymbol{\beta}_1', \ldots, \boldsymbol{\beta}_n')'$, $\boldsymbol{\varepsilon}^* = (\boldsymbol{\varepsilon}_1', \ldots, \boldsymbol{\varepsilon}_n')'$ and

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1; & \mathbf{O}; & \ldots; & \mathbf{O} \\ \mathbf{C}_{2,1}; & \mathbf{X}_2; & \ldots; & \mathbf{O} \\ \ldots & & & \\ \mathbf{C}_{n,1}; & \mathbf{C}_{n,2}; & \ldots; & \mathbf{X}_n \end{pmatrix};$$

the covariance matrix of the vector $\boldsymbol{\varepsilon}^*$ can be expressed in the form

$$\Sigma_{\boldsymbol{\theta}} = \sum_{i=1}^{n} \sigma_i^2 \mathbf{V}_i\,, \quad \boldsymbol{\theta} = (\sigma_1^2, \ldots, \sigma_n^2)'\,.$$

Considering each stage separately we see that there exist matrices $\mathbf{Q}_i$, $i = 1, \ldots, n$, of the types $k_i \times n_i$ for which $\mathbf{Q}_i\mathbf{X}_i = \mathbf{I}_{k_i}$, and $\hat{\boldsymbol{\beta}}_i = \mathbf{Q}_i\mathbf{Y}_i$ is a commonly used least squares estimator for $\boldsymbol{\beta}_i$ based on the measurements restricted only for the i-th stage.

Lemma 1. *The LBUE (locally best unbiased estimator) for $\boldsymbol{\beta}^*$ based on the measurements of all stages in model (2) is given by*

$$\hat{\mathbf{B}} = (\mathbf{X}'\Sigma_{\boldsymbol{\theta}}^{-1}\mathbf{X})^{-1}\, \mathbf{X}'\Sigma_{\boldsymbol{\theta}}^{-1}\mathbf{Y}^*\,. \tag{3}$$

Following Kleffe [1] we can check the necessary and sufficient conditions for the existence of UMVUE with respect to $\boldsymbol{\theta} = (\sigma_1^2, \ldots, \sigma_n^2)'$ of each linear unbiasedly estimable function of $\boldsymbol{\beta}^*$. We shall use the following notation: $\mathbf{I} = \Sigma_0 = \Sigma_{\boldsymbol{\theta}}$ is the identity matrix for $\boldsymbol{\theta} = (1, \ldots, 1)'$; $\mathbf{M} = \mathbf{I} - \mathbf{X}\mathbf{X}^+$, where $\mathbf{X}^+$ is the Moore-Penrose inverse of the matrix $\mathbf{X}$.

Theorem 1. *The necessary and sufficient condition for the existence of UMVUE of each linear unbiasedly estimable function of $\boldsymbol{\beta}^*$ in model (2) is*

$$\mathscr{R}(\mathbf{C}_{i,j}) \subset \mathscr{R}(\mathbf{X}_i) \quad \forall j = 1, \ldots, i - 1\,; \quad \forall i = 2, \ldots, n\,,$$

i.e. the column space of $\mathbf{C}_{i,j}$ is included in the column space of the matrix $\mathbf{X}_i$.

Proof. For simplicity, let us prove the statement of the theorem for $n = 4$. In general, the proof is based on the same idea and needs tedious calculations.

For $n = 4$ the matrix $\mathbf{X}$ is of the form:

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1; & \mathbf{O}; & \mathbf{O}; & \mathbf{O} \\ \mathbf{C}_{2,1}; & \mathbf{X}_2; & \mathbf{O}; & \mathbf{O} \\ \mathbf{C}_{3,1}; & \mathbf{C}_{3,2}; & \mathbf{X}_3; & \mathbf{O} \\ \mathbf{C}_{4,1}; & \mathbf{C}_{4,2}; & \mathbf{C}_{4,3}; & \mathbf{X}_4 \end{pmatrix}.$$

There exists a nonsingular matrix $\mathbf{T}$ of the form

$$\mathbf{T} = \begin{pmatrix} \mathbf{I}; & \mathbf{O}; & \mathbf{O}; & \mathbf{O} \\ -\mathbf{C}_{2,1}\mathbf{Q}_1; & \mathbf{I}; & \mathbf{O}; & \mathbf{O} \\ \mathbf{C}_{3,2}\mathbf{Q}_2\mathbf{C}_{2,1}\mathbf{Q}_1 - \mathbf{C}_{3,1}\mathbf{Q}_1; & -\mathbf{C}_{3,2}\mathbf{Q}_2; & \mathbf{I}; & \mathbf{O} \\ (-\mathbf{C}_{4,3}\mathbf{Q}_3\mathbf{C}_{3,2}\mathbf{Q}_2\mathbf{C}_{2,1}\mathbf{Q}_1 + \mathbf{C}_{4,3}\mathbf{Q}_3\mathbf{C}_{3,1}\mathbf{Q}_1 + & & & \\ + \mathbf{C}_{4,2}\mathbf{Q}_2\mathbf{C}_{2,1}\mathbf{Q}_1 - \mathbf{C}_{4,1}\mathbf{Q}_1); & \mathbf{C}_{4,3}\mathbf{Q}_3\mathbf{C}_{3,2}\mathbf{Q}_2 - \mathbf{C}_{4,2}\mathbf{Q}_2; & -\mathbf{C}_{4,3}\mathbf{Q}_3; & \mathbf{I} \end{pmatrix}$$

fulfilling the property

$$\mathbf{T} \, . \, \mathbf{X} = \begin{pmatrix} \mathbf{X}_1; & \mathbf{O}; & \mathbf{O}; & \mathbf{O} \\ \mathbf{O}; & \mathbf{X}_2; & \mathbf{O}; & \mathbf{O} \\ \mathbf{O}; & \mathbf{O}; & \mathbf{X}_3; & \mathbf{O} \\ \mathbf{O}; & \mathbf{O}; & \mathbf{O}; & \mathbf{X}_4 \end{pmatrix}.$$

Kleffe's condition in [1] states $\mathbf{M}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\boldsymbol{\Sigma}_{\mathbf{O}}^{-1}\mathbf{X} = \mathbf{O}$ for all $\boldsymbol{\theta}$. We transform model (2) by the nonsingular matrix $\mathbf{T}$ which yields the equivalent model

$$\mathbf{TY}^* = \mathbf{TX}\boldsymbol{\beta}^* + \mathbf{T}\boldsymbol{\varepsilon}^* . \tag{4}$$

The matrix $\mathbf{M} = \mathbf{I} - \mathbf{TX}(\mathbf{TX})^+$ in model (4) is of the form

$$\begin{pmatrix} \mathbf{I} - \mathbf{X}_1\mathbf{X}_1^+; & \ldots; & \mathbf{O} \\ \ldots & & \\ \mathbf{O}; & \ldots; & \mathbf{I} - \mathbf{X}_4\mathbf{X}_4^+ \end{pmatrix} = \begin{pmatrix} \mathbf{M}_1; & \ldots; & \mathbf{O} \\ \ldots & & \\ \mathbf{O}; & \ldots; & \mathbf{M}_4 \end{pmatrix}, \quad \text{where} \quad \mathbf{M}_i = \mathbf{I} - \mathbf{X}_i\mathbf{X}_i^+ .$$

Substituting for $\mathbf{M}$, $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}$, $\boldsymbol{\Sigma}_{\mathbf{O}}$ and $\mathbf{X}$ we get

$$\mathbf{M}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\boldsymbol{\Sigma}_{\mathbf{O}}^{-1}\mathbf{X} =$$

$$= \begin{pmatrix} \mathbf{M}_1; & \mathbf{O}; & \mathbf{O}; & \mathbf{O} \\ \mathbf{O}; & \mathbf{M}_2; & \mathbf{O}; & \mathbf{O} \\ \mathbf{O}; & \mathbf{O}; & \mathbf{M}_3; & \mathbf{O} \\ \mathbf{O}; & \mathbf{O}; & \mathbf{O}; & \mathbf{M}_4 \end{pmatrix} . \begin{pmatrix} \sigma_1^2\mathbf{X}_1 & \mathbf{O}; & \mathbf{O}; & \mathbf{O} \\ (\sigma_2^2 - \sigma_1^2)\,\mathbf{C}_{2,1} & \sigma_2^2\mathbf{X}_2; & \mathbf{O}; & \mathbf{O} \\ (\sigma_1^2 - \sigma_2^2)\,\mathbf{C}_{3,2}\mathbf{Q}_2\mathbf{C}_{2,1} + (\sigma_3^2 - \sigma_1^2)\,\mathbf{C}_{3,1} & (\sigma_3^2 - \sigma_2^2)\,\mathbf{C}_{3,2}; & \sigma_3^2\mathbf{X}_3; & \mathbf{O} \\ (\sigma_2^2 - \sigma_1^2)\,\mathbf{C}_{4,3}\mathbf{Q}_3\mathbf{C}_{3,2}\mathbf{Q}_2\mathbf{C}_{2,1} + (\sigma_1^2 - \sigma_2^2)\,\mathbf{C}_{4,2}\mathbf{Q}_2\mathbf{C}_{2,1} + & & & \\ + (\sigma_1^2 - \sigma_3^2)\,\mathbf{C}_{4,3}\mathbf{Q}_3\mathbf{C}_{3,1} + (\sigma_4^2 - \sigma_1^2)\,\mathbf{C}_{4,1} & (\sigma_2^2 - \sigma_3^2)\,\mathbf{C}_{4,3}\mathbf{Q}_3\mathbf{C}_{3,2} + (\sigma_2^2 - \sigma_3^2)\,\mathbf{C}_{4,2}; & (\sigma_4^2 - \sigma_3^2)\,\mathbf{C}_{4,3}; & \sigma_4^2\mathbf{X}_4 \end{pmatrix}.$$

The resulting matrix equals the zero matrix if and only if each block is a zero-block, and this holds if and only if

$$\mathbf{M}_i\mathbf{C}_{i,j} = \mathbf{O} \quad \forall j = 1, \ldots, i - 1 \, ; \quad \forall i = 2, 3, 4$$

which is equivalent to the condition stated in the theorem. Q.E.D.

In the sequel, the explicit expression for UMVUE will be treated provided the only admissible conditions are those between the neighbouring stages, i.e., the model (2) can be expressed in the form

$$(5) \qquad \begin{pmatrix} \mathbf{Y}_1 \\ \vdots \\ \mathbf{Y}_n \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1; & \mathbf{O}; & \ldots; & \mathbf{O} \\ \mathbf{C}_{2,1}; & \mathbf{X}_2; & \ldots; & \mathbf{O} \\ \mathbf{O}; & \mathbf{C}_{3,2}; & \ldots; & \mathbf{O} \\ \mathbf{O}; & \ldots; & \mathbf{C}_{n,n-1}; & \mathbf{X}_n \end{pmatrix} \boldsymbol{\beta}^* + \varepsilon^* \, .$$

Theorem 2. *The UMVUE for $\boldsymbol{\beta}_i$, $i = 1, \ldots, n$ based on the measurements of all stages in model (5) is given by*

$$^\wedge\mathbf{B}_i = -\mathbf{Q}_i\mathbf{C}_{i,i-1}\hat{\boldsymbol{\beta}}_{i-1} + \hat{\boldsymbol{\beta}}_i \, , \quad i = 1, \ldots, n \, ,$$

where $\hat{\boldsymbol{\beta}}_i = \mathbf{Q}_i\mathbf{Y}_i$, $i = 1, \ldots, n$ is the least squares estimator for $\boldsymbol{\beta}_i$ based on the measurements at the i-th stage only.

Proof. The best linear unbiased estimator for $\boldsymbol{\beta}^*$ from Lemma 1 is

$$^\wedge\mathbf{B} = (\mathbf{X}'\Sigma_\theta^{-1}\mathbf{X})^{-1} \, \mathbf{X}'\Sigma_\theta^{-1}\mathbf{Y}^* \, .$$

Using the fact that $\mathscr{R}(\mathbf{C}_{i,i-1}) \subset \mathscr{R}(\mathbf{X}_i)$ we denote

$$\mathbf{D} = \begin{pmatrix} \mathbf{I}; & \mathbf{O}; & \ldots; & & \mathbf{O} \\ -\mathbf{Q}_2\mathbf{C}_{2,1}; & \mathbf{I}; & \ldots; & & \mathbf{O} \\ \ldots & & & & \\ \mathbf{O}; & & \ldots; & -\mathbf{Q}_n\mathbf{C}_{n,n-1}; & \mathbf{I} \end{pmatrix} .$$

It is obvious that $\mathbf{X}_i\mathbf{Q}_i\mathbf{C}_{i,i-1} = \mathbf{C}_{i,i-1}$ and this yields the equality

$$\begin{pmatrix} \mathbf{X}_1; & \mathbf{O}; & \ldots; & \mathbf{O} \\ \mathbf{C}_{2,1}; & \mathbf{X}_2; & \ldots; & \mathbf{O} \\ \ldots & & & \\ \mathbf{O}; & \ldots; & \mathbf{C}_{n,n-1}; & \mathbf{X}_n \end{pmatrix} \cdot \begin{pmatrix} \mathbf{I}; & \mathbf{O}; & \ldots; & & \mathbf{O} \\ -\mathbf{Q}_2\mathbf{C}_{2,1}; & \mathbf{I}; & \ldots; & & \mathbf{O} \\ \ldots & & & & \\ \mathbf{O}; & & \ldots; & -\mathbf{Q}_n\mathbf{C}_{n,n-1}; & \mathbf{I} \end{pmatrix} =$$

$$= \begin{pmatrix} \mathbf{X}_1; & \ldots; & \mathbf{O} \\ \ldots & & \\ \mathbf{O}; & \ldots; & \mathbf{X}_n \end{pmatrix} = \mathbf{X}^* \, ,$$

which implies the model (5) in the form

$$\mathbf{Y}^* = \mathbf{X}^* \,.\, \mathbf{D}^{-1}\beta^* + \varepsilon^* \,. \tag{6}$$

From (6) we get

$$\hat{}\mathbf{D}^{-1}\beta^* = \begin{pmatrix} \mathbf{Q}_1; & \mathbf{O}; & \dots; & \mathbf{O} \\ \dots & & & \\ \mathbf{O}; & & \dots; & \mathbf{Q}_n \end{pmatrix} \begin{pmatrix} \mathbf{Y}_1 \\ \vdots \\ \mathbf{Y}_n \end{pmatrix}$$

and finally

$$\hat{}\mathbf{B} = \begin{pmatrix} \mathbf{Q}_1\mathbf{Y}_1 \\ -\mathbf{Q}_2\mathbf{C}_{2,1}\mathbf{Q}_1\mathbf{Y}_1 + \mathbf{Q}_2\mathbf{Y}_2 \\ \dots \\ -\mathbf{Q}_n\mathbf{C}_{n,n-1}\mathbf{Q}_{n-1}\mathbf{Y}_{n-1} + \mathbf{Q}_n\mathbf{Y}_n \end{pmatrix}$$

which implies the statement of the theorem. Q.E.D.

Owing to its simplicity, the following lemma is stated without any proof.

Lemma 2. *The covariance matrix of the estimator from Theorem 2 is*

$$\Sigma_{\hat{}\mathbf{B}} = \begin{pmatrix} \sigma_1^2\mathbf{Q}_1\mathbf{Q}_1'; & -\sigma_1^2\mathbf{Q}_1\mathbf{Q}_1'\mathbf{C}_{2,1}'\mathbf{Q}_2'; & \mathbf{O}; \dots; \mathbf{O} \\ -\sigma_1^2\mathbf{Q}_1\mathbf{Q}_1'\mathbf{C}_{2,1}'\mathbf{Q}_2'; & \sigma_1^2\mathbf{Q}_2\mathbf{C}_{2,1}\mathbf{Q}_1\mathbf{Q}_1'\mathbf{C}_{2,1}'\mathbf{Q}_2' + \sigma_2^2\mathbf{Q}_2\mathbf{Q}_2'; & -\sigma_2^2\mathbf{Q}_2\mathbf{Q}_2'\mathbf{C}_{3,2}'\mathbf{Q}_3'; \ \mathbf{O}; \dots; \mathbf{O} \\ \dots & & \dots \\ \mathbf{O}; & \dots; & \mathbf{O}; \dots; \mathbf{O}; \ \sigma_{n-1}^2\mathbf{Q}_n\mathbf{C}_{n,n-1}\mathbf{Q}_{n-1}\mathbf{Q}_{n-1}'\mathbf{C}_{n,n-1}'\mathbf{Q}_n' + \sigma_n^2\mathbf{Q}_n\mathbf{Q}_n' \end{pmatrix}.$$

The next theorem solves the problem of estimation of σ_i^2, $i = 1, \dots, n$, under the conditions stated in Theorem 1.

Theorem 2. *The UMVUIE* (uniformly minimum variance unbiased invariant *estimator*) *for σ_i^2 in model* (2) *under the conditions $\mathscr{R}(\mathbf{C}_{i,j}) \subset \mathscr{R}(\mathbf{X}_i)$ $\forall j = 1, \dots, i-1$, $\forall i = 2, \dots, n$ is*

$$\hat{\sigma}_i^2 = \frac{1}{n_i - k_i} \mathbf{Y}_i'(\mathbf{I} - \mathbf{X}_i\mathbf{Q}_i)\,\mathbf{Y}_i \,.$$

Proof. The notation

$$\Sigma_i = \begin{pmatrix} \mathbf{O}, & \dots, & \mathbf{O} & & \\ \dots & & & & \\ \mathbf{O}, & \dots, & \mathbf{I}_i, & \dots, & \mathbf{O} \\ \dots & & & & \\ \mathbf{O}, & \dots, & \mathbf{C} & & \end{pmatrix}$$

for $i = 1, \ldots, n$ will be used. The locally best unbiased invariant estimator for σ_i^2 exists (as stated in [1]) if and only if the criterion matrix $\mathbf{Q}$ with $q_{ij} = \operatorname{tr}(\mathbf{M}\Sigma_\theta\mathbf{M})^+ \Sigma_i(\mathbf{M}\Sigma_\theta\mathbf{M})^+ \Sigma_j$ is nonsingular.

Consider the transformation matrix $\mathbf{T}$ from the proof of Theorem 1 and the model (4). In that case we have $\mathbf{M} \,.\, \mathbf{T} = \mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_1, & \mathbf{O}, & \ldots, & \mathbf{O} \\ \ldots & & & \\ \mathbf{O}, & \ldots, & \mathbf{O}, & \mathbf{M}_n \end{pmatrix} \quad \text{with} \quad \mathbf{M}_i = \mathbf{I} - \mathbf{X}_i\mathbf{X}_i^+ .$$

Then

$$\operatorname{tr}(\mathbf{MT}\Sigma_\theta\mathbf{T}'\mathbf{M}')^+ \mathbf{T}\Sigma_i\mathbf{T}'(\mathbf{MT}\Sigma_\theta\mathbf{T}'\mathbf{M}')^+ \mathbf{T}\Sigma_j\mathbf{T}' = O \quad \text{for} \quad i \neq j ,$$

$$\operatorname{tr}(\mathbf{MT}\Sigma_\theta\mathbf{T}'\mathbf{M}')^+ \mathbf{T}\Sigma_i\mathbf{T}'(\mathbf{MT}\Sigma_\theta\mathbf{T}'\mathbf{M}')^+ \mathbf{T}\Sigma_i\mathbf{T}' = \operatorname{tr} \sigma_i^{-4}\mathbf{M}_i = \sigma_i^{-4}(n_i - k_i)$$

for $i = 1, \ldots, n$.

The criterion matrix $\mathbf{Q}$ is of the form

$$\mathbf{Q} = \begin{pmatrix} \sigma_1^{-4}(n_1 - k_1), & O, & \ldots, & O \\ \ldots & & & \\ O, & \ldots, & O, & \sigma_n^{-4}(n_n - k_n) \end{pmatrix} .$$

The LMVUIE of σ_i^2 is

$$\hat{\sigma}_i^2 = \frac{1}{\sigma_i^{-4}(n_i - k_i)} \mathbf{Y}^*(\mathbf{MT}\Sigma_\theta\mathbf{T}'\mathbf{M}')^+ \mathbf{T}\Sigma_i\mathbf{T}'(\mathbf{MT}\Sigma_\theta\mathbf{T}'\mathbf{M}')^+ \mathbf{Y}^* =$$

$$= \frac{1}{n_i - k_i} \mathbf{Y}_i'(\mathbf{I} - \mathbf{X}_i\mathbf{X}_i^+)\,\mathbf{Y}_i . \quad \text{Q.E.D.}$$

The LBUE for $\boldsymbol{\beta}^*$ in model (2) is given in Lemma 1. Under the conditions stated in Theorem 1 the explicit formula for UMVUE for $\boldsymbol{\beta}^*$ in model (2) can be derived. The statement for $n = 4$ reads as follows.

Theorem 3. *The UMVUE for $\boldsymbol{\beta}^*$ in model (2) for $n = 4$ is given by*

$$\hat{}\mathbf{B} = \begin{pmatrix} \hat{\boldsymbol{\beta}}_1 \\ \hat{\boldsymbol{\beta}}_2 - \mathbf{Q}_2\mathbf{C}_{2,1}\hat{\boldsymbol{\beta}}_1 \\ \hat{\boldsymbol{\beta}}_3 - \mathbf{Q}_3\mathbf{C}_{3,2}\hat{\boldsymbol{\beta}}_2 + (\mathbf{Q}_3\mathbf{C}_{3,2}\mathbf{Q}_2\mathbf{C}_{2,1} - \mathbf{Q}_3\mathbf{C}_{3,1})\,\hat{\boldsymbol{\beta}}_1 \\ \hat{\boldsymbol{\beta}}_4 - \mathbf{Q}_4\mathbf{C}_{4,3}\hat{\boldsymbol{\beta}}_3 + (-\mathbf{Q}_4\mathbf{C}_{4,2} + \mathbf{Q}_4\mathbf{C}_{4,3}\mathbf{Q}_3\mathbf{C}_{3,2})\,\hat{\boldsymbol{\beta}}_2 + (-\mathbf{Q}_4\mathbf{C}_{4,1} + \\ + \mathbf{Q}_4\mathbf{C}_{4,2}\mathbf{Q}_2\mathbf{C}_{2,1} - \mathbf{Q}_4\mathbf{C}_{4,3}\mathbf{Q}_3\mathbf{C}_{3,2}\mathbf{Q}_2\mathbf{C}_{2,1} + \mathbf{Q}_4\mathbf{C}_{4,3}\mathbf{Q}_3\mathbf{C}_{3,1})\,\hat{\boldsymbol{\beta}}_1 \end{pmatrix} .$$

Proof. Analogously as in Theorem 2 the conditions $\mathscr{R}(\mathbf{C}_{i,j}) \subset \mathscr{R}(\mathbf{X}_i)$ $\forall j =$

$= 1, \ldots, i - 1, \forall i = 2, 3, 4$ are used. There exists a matrix $\mathbf{D}$ of the form

$$\mathbf{D} = \begin{pmatrix} \mathbf{I}; & \mathbf{O}; & \mathbf{O}; & \mathbf{O} \\ -\mathbf{Q}_2\mathbf{C}_{2,1}; & \mathbf{I}; & \mathbf{O}; & \mathbf{O} \\ \mathbf{Q}_3\mathbf{C}_{3,2}\mathbf{Q}_2\mathbf{C}_{2,1} - \mathbf{Q}_3\mathbf{C}_{3,1}; & -\mathbf{Q}_3\mathbf{C}_{3,2}; & \mathbf{I}; & \mathbf{O} \\ (-\mathbf{Q}_4\mathbf{C}_{4,1} + \mathbf{Q}_4\mathbf{C}_{4,2}\mathbf{Q}_2\mathbf{C}_{2,1} -; & -\mathbf{Q}_4\mathbf{C}_{4,2} + \mathbf{Q}_4\mathbf{C}_{4,3}\mathbf{Q}_3\mathbf{C}_{3,2}; & -\mathbf{Q}_4\mathbf{C}_{4,3}; & \mathbf{I} \\ -\mathbf{Q}_4\mathbf{C}_{4,3}\mathbf{Q}_3\mathbf{C}_{3,2}\mathbf{Q}_2\mathbf{C}_{2,1} + \mathbf{Q}_4\mathbf{C}_{4,3}\mathbf{Q}_3\mathbf{C}_{3,1}) & & & \end{pmatrix}$$

$$\mathbf{XD} = \begin{pmatrix} \mathbf{X}_1, & \mathbf{O}, & \mathbf{O}, & \mathbf{O} \\ \mathbf{O}, & \mathbf{X}_2, & \mathbf{O}, & \mathbf{O} \\ \mathbf{O}, & \mathbf{O}, & \mathbf{X}_3, & \mathbf{O} \\ \mathbf{O}, & \mathbf{O}, & \mathbf{O}, & \mathbf{X}_4 \end{pmatrix},$$

and the resulting equivalent model is

$$(6) \qquad \mathbf{Y}^* = \mathbf{X} \,.\, \mathbf{D} \,.\, \mathbf{D}^{-1}\boldsymbol{\beta}^* + \varepsilon^* \,.$$

The result stated in the theorem is a simple consequence. Q.E.D.

Remark. In [3] the UMVUE for $\boldsymbol{\beta}^*$ for $n = 2$ is derived. It can be shown that these two estimators, the one from Theorem 2 and the estimator derived in [3], coincide.

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Súhrn

POZNÁMKA K ODHADU PARAMETROV STREDNEJ HODNOTY A DISPERZIE V *n*-ETAPOVOM LINEÁRNOM MODELI

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Vo všeobecnom n-etapovom lineárnom modeli, ako je uvedený v (2), sú vektorový parameter $\boldsymbol{\beta}^*$ a parametre σ_i^2 $i = 1, \ldots, n$ neznáme. V práci je uvedená nutná a postačujúca podmienka pre existenciu rovnomerne najlepšieho nevychýleného odhadu $\hat{\mathbf{B}}$ pre $\boldsymbol{\beta}^*$, ako aj explicitné vzťahy pre $\hat{\mathbf{B}}$ založené na odhadoch metódou najmenších štvorcov. Uvedené sú rovnomerne najlepšie nevychýlené invariantné odhady pre parametre σ_i^2 $i = 1, \ldots, n$.

Резюме

ЗАМЕЧАНИЕ К ОЦЕНИВАНИЮ ПАРАМЕТРОВ СРЕДНЕГО И ДИСПЕРСИИ В n-ЭТАПНОЙ ЛИНЕЙНОЙ МОДЕЛИ

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В статье указано необходимое и достаточное условие для существования нелинейных оценок параметров среднего с равномерно минимальной дисперсией при условии нормального распределения. Выведены формулы для вычисления этих оценок, основанные на оценках полученных методом наименьших квадратов. Получены также равномерно наилучшие несмещенные оценки для параметров дисперсии.

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