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OPTIMAL CONTROL PROBLEMS FOR VARIATIONAL INEQUALITIES WITH CONTROLS IN COEFFICIENTS AND IN UNILATERAL CONSTRAINTS

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Summary. We deal with an optimal control problem for variational inequalities, where the monotone operators as well as the convex sets of possible states depend on the control parameter. The existence theorem for the optimal control will be applied to the optimal design problems for an elasto-plastic beam and an elastic plate, where a variable thickness appears as a control variable.

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1. EXISTENCE THEOREM

Let U with a norm $\|\cdot\|_U$ be a Banach space of controls, $U_{ad} \subset U$ a compact set of admissible controls. We shall consider a family $\{A(e)\}$, $e \in U_{ad}$ of monotone operators. We follow the papers [1], [3], [4], introducing in addition a system of convex sets $\{K(e)\}$, $e \in U_{ad}$. In order to characterize the dependence $e \to K(e)$ we recall the special type of convergence of sequences of sets introduced by Mosco in [8].

Definition 1.1. *A sequence $\{K_n\}$ of subsets of a normed space V converges to a set $K \subset V$, if K contains all weak limits of sequences $\{u_k\}$, $u_k \in K_{n_k}$, where $\{K_{n_k}\}$ are arbitrary subsequences of $\{K_n\}$ and every element $v \in K$ is the (strong) limit of some sequence $\{v_n\}$, $v_n \in K_n$.*

Notation: $K = \operatorname{Lim}_{n\to\infty} K_n$.

Let V be a reflexive Banach space with a norm $\|\cdot\|$, V^* its dual space with a norm $\|\cdot\|_*$ and a duality pairing $\langle\cdot,\cdot\rangle$ between V^* and V. Further, we introduce the systems $\{K(e)\}$, $\{A(e)\}$, $e \in U_{ad}$ of convex closed sets $K(e) \subset V$ and operators $A(e)$: $V \to V^*$ satisfying the following assumptions:

$$e_n \to e_0 \quad \text{in} \quad U \Rightarrow K(e_0) = \operatorname{Lim}_{n\to\infty} K(e_n)\,, \tag{1.1}$$

$$\langle A(e)\,u - A(e)\,v\,,\ u - v\rangle > 0 \tag{1.2}$$

$$\text{for all} \quad e \in U_{ad}\,,\quad u, v \in V\,,\quad u \neq v\,,$$

(1.3) $$\lim_{t \to \infty} \langle A(e) [u + t(v - u)], w \rangle = \langle A(e) u, w \rangle$$
for all $e \in U_{ad}$; $u, v, w \in V$,

(1.4) $$\|v\| \leqq M \Rightarrow \|A(e) v\|_* \leqq C(M) \quad \text{for all} \quad e \in U_{ad},$$

(1.5) there exists a function r: $[0, \infty) \to R$ and elements $w(e) \in K(e)$ such that $\lim_{t \to 0} r(t) = \infty$, $\|w(e)\| \leqq C$, $\langle A(e) v, v - w(e) \rangle \geqq \|v\| r(\|v\|)$ for all $v \in V$, $e \in U_{ad}$.

The assumptions (1.2)–(1.5) mean that the system $\{A(e)\}$ is strictly monotone, hemicontinuous and uniformly bounded and uniformly coercive with respect to $e \in U_{ad}$. We assume further that the operator $A(\cdot) v$: $U_{ad} \to V$ is continuous for all $v \in V$

(1.6) $$e_n \to e_0 \quad \text{in} \quad U \Rightarrow A(e_n) v \to A(e_0) v \quad \text{in} \quad V^* \quad \text{for all} \quad v \in V.$$

Let a continuous operator B: $U_{ad} \to V^*$ and a functional $f \in V^*$ be given. Under the assumptions (1.2), (1.3), (1.4) the operator $A(e)$: $V \to V^*$ is pseudomonotone for every $e \in U_{ad}$ (see [7] – Proposition 2.5, Chapt. 2). Then there exists (due to [7], Theorem 8.2, Chapt. 2) a unique solution $u(e) \in K(e)$ of the following variational inequality for any $e \in U_{ad}$:

(1.7) $$\langle A(e) u(e), v - u(e) \rangle \geqq \langle f + B(e), v - u(e) \rangle \quad \text{for all} \quad v \in K(e).$$

Let us consider a functional j: $U \times V \to R$ fulfilling the condition

(1.8) $$e_n \to e \quad \text{in} \quad U \quad \text{and} \quad v_n \to v \quad (\text{weakly}) \text{ in} \quad V \Rightarrow j(e, v) \leqq \liminf_{n \to \infty} j(e_n, v_n).$$

Our aim is to solve the following optimal control problem:

Problem P. To find a control $e_0 \in U_{ad}$ such that

(1.9) $$\langle A(e_0) u(e_0), v - u(e_0) \rangle \geqq \langle f + B(e_0), v - u(e_0) \rangle \quad \text{for all } v \in K(e_0),$$

(1.10) $$J(e_0) = j(e_0, u(e_0)) \leqq j(e, u(e)) = J(e) \quad \text{for all} \quad e \in U_{ad},$$

where $u(e) \in K(e)$ is a solution of the state inequality (1.7) uniquely defined for every $e \in U_{ad}$.

Theorem 1.1. *Let the assumptions (1.1)–(1.6) hold. Then there exists at least one solution e_0 of Optimal Control Problem* P.

Proof. Let $\{e_n\}$ be a minimizing sequence for the functional J:

(1.11) $$\lim_{n \to \infty} J(e_n) = \inf_{e \in U_{ad}} J(e)$$

(we put $\inf_{e \in U_{ad}} J(e) = -\infty$, if the set $\{J(e)\}$ is not lower bounded).

Since the set U_{ad} is compact in U, there exist $e_0 \in U_{ad}$ and a subsequence (denoted again by $\{e_n\}$) of $\{e_n\}$ such that

(1.12) $$\lim_{n\to\infty} e_n = e_0 \quad \text{in} \quad U\,.$$

Denoting $u(e_n) \equiv u_n \in K(e_n)$ we may write

(1.13) $$\langle A(e_n)\, u_n, v - u_n\rangle \geqq \langle f + B(e_n), v - u_n\rangle \quad \text{for all} \quad v \in K(e_n)\,,$$
$$n = 1, 2, \ldots$$

Inserting $v = w(e_n)$ from (1.5) we arrive at

(1.14) $$\langle A(e_n)\, u_n, u_n - w(e_n)\rangle \leqq \langle f + B(e_n), u_n - w(e_n)\rangle\,.$$

The uniform coercivity of $\{A(e)\}$ and the continuity of B imply

(1.15) $$\|u_n\|\, r(\|u_n\|) \leqq C_1\|u_n\| + C_2\,, \quad n = 1, 2, \ldots$$

Since $\lim_{t\to\infty} r(t) = \infty$, we have

(1.16) $$\|u_n\| \leqq C_3\,, \quad n = 1, 2, \ldots$$

Then there exists a subsequence (denoted again by $\{u_n\}$) such that

(1.17) $$u_n \rightharpoonup u \quad \text{(weakly) in } V, \quad u_n \in K(e_n)\,.$$

The assumption (1.1) implies

(1.18) $$u \in K(e_0)\,.$$

By virtue if (1.4) and (1.16) we obtain

(1.19) $$\|A(e_n)\, u_n\|_* \leqq C_4\,, \quad n = 1, 2, \ldots$$

Then there exists an element $\chi \in V^*$ and a subsequence (again denoted by $\{A(e_n)\, u_n\}$) such that

(1.20) $$A(e_n)\, u_n \rightharpoonup \chi \quad \text{(weakly) in } V^*\,.$$

Let $\{w_n\}$ be a sequence such that

(1.21) $$w_n \to u \quad \text{in} \quad V, \quad w_n \in K(e_n)\,, \quad n = 1, 2, \ldots$$

The existence of w_n is ensured by (1.1), (1.12) and (1.18). Combining (1.17) and (1.21) we have

(1.22) $$(u_n - w_n) \rightharpoonup 0 \quad \text{(weakly) in} \quad V.$$

Inserting $v = w_n$ into (1.13) and using (1.22) together with the continuity of the operator $B\colon U \to V^*$ we obtain

(1.23) $$\limsup_{n\to\infty} \langle A(e_n)\, u_n, u_n - w_n\rangle \geqq 0\,.$$

Combining the last inequality with (1.19) and (1.21) we arrive at

$$\limsup_{n\to\infty} \langle A(e_n)\, u_n, u_n - u\rangle \leqq 0\,, \tag{1.24}$$

and comparing it with (1.20) we have

$$\limsup_{n\to\infty} \langle A(e_n)\, u_n, u_n\rangle \geqq \langle \chi, u\rangle\,. \tag{1.25}$$

The monotonicity of $A(e_n)$ on V (assumption (1.2)) implies

$$\langle \chi, u\rangle \geqq \limsup_{n\to\infty} \left[\langle A(e_n)\, v, u_n - v\rangle + \langle A(e_n)\, u_n, v\rangle\right]\,, \quad n = 1, 2, \ldots$$

Taking into account the relations (1.6), (1.12), (1.17), (1.20) we obtain

$$\langle \chi - A(e_0)\, v, v - u\rangle \geqq 0 \quad \text{for all} \quad v \in V\,. \tag{1.26}$$

Let $v = u + t(w - u)$, $t \in R$, $w \in V$. Then we have

$$\langle \chi - A(e_0)\left[u + t(w - u)\right], u - w\rangle \geqq 0 \quad \text{for all} \quad w \in V\,.$$

Using the hemicontinuity of $A(e_0)$ (see (1.3)) we obtain after $t \to 0$

$$\langle \chi - A(e_0)\, u, u - w\rangle \geqq 0 \quad \text{for all} \quad w \in V\,,$$

and hence

$$\chi = A(e_0)\, u\,, \tag{1.27}$$

$$A(e_n)\, u_n \rightharpoonup A(e_0)\, u \quad \text{(weakly) in} \quad V^*\,. \tag{1.28}$$

Using again the monotonicity of $A(e_n)$ we have

$$\langle A(e_n)\, u_n, u_n - u\rangle \geqq \langle A(e_n)\, u, u_n - u\rangle\,, \quad n = 1, 2, \ldots$$

The convergences (1.12), (1.17), the assumption (1.6) and the last inequality imply

$$\liminf_{n\to\infty} \langle A(e_n)\, u_n, u_n - u\rangle \geqq 0$$

which compared with (1.24) leads to

$$\lim_{n\to\infty} \langle A(e_n)\, u_n, u_n - u\rangle = 0\,. \tag{1.29}$$

Combining (1.28) and (1.29) we arrive at the relation

$$\langle A(e_0)\, u, u - v\rangle = \lim_{n\to\infty} \langle A(e_n)\, u_n, u_n - v\rangle \quad \text{for all} \quad v \in V\,. \tag{1.30}$$

Let $v \in K(e_0)$ be an arbitrary element and let $\{v_n\}$ be such a sequence that

$$v_n \to v \quad \text{in} \quad V, \quad v_n \in K(e_n)\,, \quad n = 1, 2, \ldots \tag{1.31}$$

Using (1.30), (1.31), (1.28), (1.13) and the continuity of $B(\cdot)$ we arrive at

$$(1.32)\quad \langle A(e_0)\,u, u - v\rangle = \lim_{n\to\infty} \langle A(e_n)\,u_n, u_n - v_n\rangle \leqq \lim_{n\to\infty} \langle f + B(e_n), u_n - v_n\rangle =$$

$$= \langle f + B(e_0), u - v\rangle \quad \text{for all} \quad v \in K(e_0)\,.$$

Hence u is a solution of the state inequality (1.9) and

$$(1.33)\qquad u \equiv u(e_0),\, u(e_n) \rightharpoonup u(e_0) \quad \text{(weakly) in} \quad V\,.$$

It follows from (1.8) that

$$J(e_0) = j(e_0, u(e_0)) \leqq \liminf_{n\to\infty} j(e_n, u(e_n)) = \liminf_{n\to\infty} J(e_n) = \inf_{e\in U_{ad}} J(e)\,,$$

which completes the proof of (1.10) and thus of Theorem 1.1.

2. APPLICATIONS

We shall investigate some optimal control problems connected with the optimal design of a beam and a plate with respect to a variable thickness. Henceforth we shall denote by $L_2(\Omega)$ the space of all functions $f: \Omega \to R$, $\Omega \subset R^m$ Lebesgue integrable with their second power on Ω, and by $H^k(\Omega)$ the Sobolev space of all functions from $L_2(\Omega)$ with distributive derivatives up to the order k in $L_2(\Omega)$. $H^k(\Omega)$ is the Hilbert space with the scalar product

$$((u, v))_k \sum_{|p|\leqq k} \int_\Omega D^p u D^p v \,\mathrm{d}x\,, \quad |p| = p_1 + \ldots + p_m\,,$$

and the norm $\|u\|_k = ((u, u))^{1/2}$.

Further, we denote

$$H_0^k(\Omega) = \{v \in H^k(\Omega): v = D^p v = 0 \quad \text{on} \quad \partial\Omega \quad \text{for} \quad |p| \leqq k - 1\}\,.$$

It is well known that $H_0^k(\Omega)$ is the Hilbert space with the scalar product

$$(u, v)_k = \sum_{|p|=k} \int_\Omega D^p u D^p v \,\mathrm{d}x$$

and the norm $|u|_k = (u, u)_k^{1/2}$. For $k = 0$ we have the scalar product and the norm in $L_2(\Omega)$.

I. Optimal design of a beam. Let us consider an elasto-plastic beam of a length a with a variable thickness expressed by a function $e: \Omega \to R$, $\Omega = (0, a)$.

We set $U = H^2(\Omega)$ – a reflexive Banach space with a norm $\|\cdot\|_U = \|\cdot\|_2$. Let us introduce the set of admissible controls – thickness functions

$$U_{ad} = \{e \in H^3(\Omega): 0 < e_{\min} \leqq e(x) \leqq e_{\max} \quad \text{for all} \quad x \in \Omega\,,$$

$$\|e\|_3 \leqq C_1\,, \quad \int_\Omega e(x)\,\mathrm{d}x = C_2\,, \quad e(0) = C_3\,, \quad e'(0) = C_4\,,$$

$$e(a) = C_5\,, \quad e'(a) = C_6\}\,.$$

It results from the compact imbeddings $H^3(\Omega) \hookrightarrow H^2(\Omega)$, $H^2(\Omega) \hookrightarrow C^1(\Omega)$ that the set U_{ad} is compact in U.

We assume the beam to be clamped at both ends and put $V = H_0^2(\Omega)$. We further suppose the beam to be forced to lie over an obstacle represented by a function $\Phi: \Omega \to R$. Hence the function describing the deflection of the beam belongs to the set

$$K(e) = \{v \in H_0^2(\Omega): v(x) \geqq \Phi(x) + \tfrac{1}{2}\, e(x),\, x \in \Omega\}\,.$$

We recall that the function v expresses the deflection of the middle line of the beam. We assume

$$\Phi \in C(\bar{\Omega})\,, \quad \Phi(0) < -\frac{C_3}{2}\,, \quad \Phi(a) < -\frac{C_5}{2}\,, \tag{2.1}$$

where the constants C_3, C_5 appear in the definition of the set U_{ad}. The condition (2.1) ensures that the set $K(e)$ is nonempty for every $e \in U_{ad}$. It can be easily seen that $K(e)$ is convex and closed. The system $\{K(e)\}$ fulfils the condition (1.1). Indeed, if $\lim_{n\to\infty} e_n = e_0$ in $U = H^2(\Omega)$, $e_n \in U_{ad}$, then there exists a subsequence $\{e_m\}$ weakly convergent in $H^3(\Omega)$ to the element $e_0 \in U_{ad}$. Let $w_k \rightharpoonup w$, $w_k \in K(e_k)$, $k = 1, 2, \ldots$; $w \in V = H_0^2(\Omega)$. We then have

$$w_k(x) \geqq \Phi(x) + \frac{e_k(x)}{2} \quad \text{for all} \quad x \in \Omega$$

which implies, with respect to the compact imbedding $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$,

$$w(x) \geqq \Phi(x) + \frac{e_0(x)}{2} \quad \text{for all} \quad x \in \Omega\,, \tag{2.2}$$

and hence $w \in K(e_0)$.

If $v \in K(e_0)$, then we put $v_m = v + \frac{1}{2}(e_m - e_0)$. The elements $\{v_m\}$ satisfy the conditions

$$v_m \in K(e_m)\,, \quad \lim_{m\to\infty} v_m = v \quad \text{(strongly) in} \quad V\,.$$

Hence the condition (1.1) holds.

Now we define the system $\{A(e)\}$ of operators corresponding to the bending of elasto-plastic beams in the same way as it was done in [3] or in [6], Chapt. III. Let $\varrho \in C^1[0, \infty)$ be a material function fulfilling the following conditions for all $\xi \in [0, \infty)$

$$0 < \varrho_0 \leqq \varrho(\xi) \leqq \varrho_1\,, \tag{2.3}$$

$$0 < \psi_0 \leqq \frac{d}{d\xi}\,[\xi \varrho(\xi^2)] \leqq \psi_1 \tag{2.4}$$

with positive constants ϱ_0, ϱ_1, ψ_0, ψ_1.

We define the function

$$g_e(x, t) = \int_{-\frac{1}{2}e(x)}^{\frac{1}{2}e(x)} \varrho(z^2 t)\, z^2 \,\mathrm{d}z\,, \quad t \geqq 0 \tag{2.5}$$

and an operator $A(e)\colon V \to V^*$ by

$$\langle A(e)\, u, v \rangle = 2 \int_\Omega g_e(x, (u'')^2)\, u'' v'' \,\mathrm{d}x\,; \quad u, v \in V. \tag{2.6}$$

The constant function $\varrho \equiv E/2$ corresponds to the linear elasticity. Them

$$2g_e(x, t) = \frac{E}{12}\, e^3(x)\,, \quad \langle A(e)\, u, v\rangle = \frac{E}{12} \int_\Omega e^3(x)\, u'' v'' \,\mathrm{d}x\,.$$

Lemma 2.1. *The family* $\{A(e)\}$, $e \in U_{ad}$ *of operators defined by* (2.5), (2.6) *satisfies the assumptions* (1.2)–(1.6).

Proof. Using the Lagrange theorem and the inequalities (2.4) we obtain

$$\langle A(e)\, u - A(e)\, v, u - v\rangle \geqq c_0 \|u - v\|^2\,, \tag{2.7}$$

$$\|A(e)\, u - A(e)\, v\| \leqq c_1 \|u - v\| \quad \text{for all} \quad u, v \in V, \tag{2.8}$$

where $c_0 = 2e_{\min}\psi_0$, $c_1 = 2e_{\max}\psi_1$, and

$$\|v\| = \left(\int_\Omega (v'')^2 \,\mathrm{d}x\right)^{1/2}, \quad v \in V \tag{2.9}$$

is the norm in the space V.

The properties (1.2), (1.3) follow from (2.7), (2.8). The boundedness (1.4) is a consequence of the upper estimates for the functions $e \in U_{ad}$ and ϱ. With respect to the assumptions (2.1) we can take in (1.5) a function $w(e) \in K(e)$ such that $w(e) \in C_0^\infty(\Omega)$, $w(e) \geqq 0$ on Ω and

$$w(e)\,(x) = \max_{x \in \bar{\Omega}} |\Phi(x)| + \tfrac{1}{2} e_{\max} \quad \text{for} \quad x \in (\delta, a - \delta)\,, \quad \delta > 0\,.$$

We see that this function $w(e) \equiv w$ does not depend on e. Thus we obtain (1.5) with a function r of the form $r(t) = c_0 t - c_1 \|w\|$, $t \geqq 0$, which completes the proof.

The load on the beam is represented by the functional

$$\langle f, v\rangle = \sum_{j=1}^{N} [P_j\, v(X_j) + M_j\, v(X_j)] + \int_\Omega f_0 v \,\mathrm{d}x\,, \quad v \in V,$$

where P_j, M_j are given constants, $X_j \in \Omega$, $j = 1, \ldots, N$, and f_0 is integrable in the sense of Lebesque on Ω. We have $f \in V^*$ due to the continuous imbedding $H^2(\Omega) \subset$ $\subset C^1(\Omega)$.

The operator $B: U_{ad} \to V^*$ and the cost functional $j: U \times V \to R$ can be chosen in the same way as in [3]:

$$\langle B(e), v \rangle = -k \int_\Omega e(x) v(x) \, dx, \quad k > 0, \quad v \in V, \tag{2.10}$$

which represents the load caused by the own weight of the beam. Further, we set

$$j_1(e, v) = \|v - z_d\|_2^2, \quad z_d \in V \tag{2.11}$$

or

$$j_2(e, v) = \int_\Omega e^2(x) (v''(x))^2 \, dx, \quad v \in V. \tag{2.12}$$

The functional j_2 expresses the intensity of the normal stress in the extreme fiber of the elastic beam. Both functionals satisfy the assumption (1.8) (see [1], Chapt. 2), and taking into account Lemma 2.1 and Theorem 1.1 we see that there exists an optimal thickness function $e_0: [0, q] \to R$ solving the optimal design problem for the elasto-plastic beam lying over an obstacle.

II. Optimal design of a plate. We consider an elastic plate, whose middle surface is a bounded region $\Omega \subset R^2$ with a Lipschitz boundary. Again we set $U = H^2(\Omega)$. The set of admissible controls has the form

$$U_{ad} = \Big\{ e \in H^3(\Omega): 0 < e_{min} \leqq e(x) \leqq e_{max} \quad \text{for all} \quad x \in \Omega,$$
$$\|e\|_3 \leqq C_1, \quad \int_\Omega e(x) \, dx = C_2, \quad e|_{\partial\Omega} = \varphi_0, \quad \frac{\partial e}{\partial n}\Big|_{\partial\Omega} = \varphi_1 \Big\}.$$

We assume the plate to be clamped on the boundary and put $V = H_0^2(\Omega)$. The set of possible deflections of a plate is

$$K(e) = \{ v \in H_0^2(\Omega): v(x) \geqq \Phi(x) + \tfrac{1}{2} e(x) \quad \text{for all} \quad x \in \Omega \},$$

where the function $\Phi: \Omega \to R$ representing the obstacle lying under the plate has to satisfy the conditions

$$\Phi \in C(\bar{\Omega}), \quad \Phi(s) < -\tfrac{1}{2} \varphi_0(s) \quad \text{for all} \quad s \in \partial\Omega. \tag{2.13}$$

It can be shown in the same way as in the case of the beam that $\{K(e)\}$, $e \in U_{ad}$ is a system of nonempty closed convex subsets of V satisfying the assumption (1.1).

The system of operators $A(e): V \to V^*$, $e \in U_{ad}$ is defined by

$$\langle A(e) u, v \rangle = \frac{E}{12(1 - \sigma^2)} \int_\Omega e^3(x) [u_{11}v_{11} + (u_{11}v_{22} + u_{22}v_{11}) + \tag{2.14}$$
$$+ 2(1 - \sigma) u_{12}v_{12} + u_{22}v_{22}] \, dx,$$

where

$$u_{ij} = \frac{\partial^2 u}{\partial x_i \, \partial x_j}, \quad i, j = 1, 2 \, ; \quad E > 0 \, , \quad 0 < \sigma < 1$$

The operators $A(e)$: $V \to V^*$ are linear bounded and strongly monotone uniformly with respect to $e \in U_{ad}$, and they satisfy the assumptions (1.2)–(1.6) as was verified for more general nonlinear operators in [4], Lemma 1.1.

As the perpendicular load we take the functional

$$\langle f, v \rangle = \sum_{j=1}^{N} P_j \, v(X_j) + \int_{\Omega} f_0 v \, dx \, , \quad v \in V \, ,$$

where P_j are given constants, $X_j \in \Omega$, $j = 1, \ldots, N$; $f_0 \in L_2(\Omega)$. We can again include the own weight of the plate represented by the operator

$$\langle B(e), v \rangle = -k \int_{\Omega} e(x) \, v(x) \, \mathrm{d}x \, , \quad e \in U_{ad} \, , \quad v \in V \, .$$

As the cost functional we can choose j_1 defined in (2.11), or

$$j_3(e, v) = \int_{\Omega} e^2(x) \, S[v, v] \, \mathrm{d}x \, , \quad e \in U_{ad} \, , \quad v \in V \, ,$$

where

$$S[v, v] = (v_{11}^2 + v_{22}^2)(1 - \sigma + \sigma^2) + v_{11} v_{22}(-1 + 4\sigma - \sigma^2) + 3(1 - \sigma)^2 \, v_{12}^2 \, ,$$

which corresponds to the minimization of the intensity of the shear stress at the extreme fibers of the plate.

As all the assumptions from Part 1 are fulfilled (see [4], III, Chapt. 1) there exists at least one optimal thickness-function of the plate with respect to the cost functionals j_1 or j_3.

3. OPTIMAL DESIGN OF A PLATE DEFORMED BY SHEAR FORCES

In the next problem we do not directly apply Theorem 1.1. We shall use the result about the dependence of the deflection of the plate on the form of the obstacle. It enables us to weaken the assumptions on the admissible set U_{ad}.

Again we consider a plate with a middle surface Ω. We assume that its vertical displacement is influenced by an obstacle. The boundary of the plate is supposed to be sufficiently smooth.

The set of admissible thickness-functions (controls) has the form

$$\text{(2.15)} \qquad U_{ad} = \Big\{ e \in C^{0,1}(\bar{\Omega}) \colon 0 < e_{min} \leqq e(x) \leqq e_{max} \text{ on } \Omega \, ,$$

$$\left|\frac{\partial e}{\partial x_i}\right| \leqq C_i\,, \quad i = 1, 2\,; \quad \text{a.e. on} \quad \Omega\,, \quad \int_\Omega e(x)\,\mathrm{d}x = C_3\,,$$

$$\left. e(s) = \varphi(s) \quad \text{on} \quad \partial\Omega,\ \varphi \in C(\partial\Omega) \right\}\,,$$

where C_1, C_2, C_3 are given constants, φ is a given function and $C^{0,1}(\bar{\Omega})$ is the set of all functions Lipschitz-continuous on $\bar{\Omega}$. The derivatives $\partial e/\partial x_i$ exist almost everywhere on Ω. Due to the Ascoli-Arzela theorem the set U_{ad} is compact in the Banach space $U = C(\bar{\Omega})$ of all continuous functions e: $\bar{\Omega} \to R$ with the norm

$$\|e\|_U = \max_{x\in\bar{\Omega}} |e(x)|\,, \quad e \in U\,. \tag{2.16}$$

We suppose that the boundary of the plate is not deformed and we set

$$V = H_0^1(\Omega)\,.$$

Assume again an obstacle lying under the plate, analytically described by a function $\Phi \in H^1(\Omega) \cap C(\bar{\Omega})$ fulfilling the condition

$$\Phi(s) + \tfrac{1}{2}\,\varphi(s) \leqq 0 \quad \text{for all} \quad s \in \partial\Omega\,, \tag{2.17}$$

where the function φ is defined in (2.15).

Now we introduce the system of sets $\{K(e)\}$, $e \in U_{\mathrm{ad}}$:

$$K(e) = \{v \in V\colon v(x) \geqq \Phi(x) + \tfrac{1}{2}\,e(x) \ \text{a.e. on} \ \Omega\}\,. \tag{2.18}$$

$K(e)$ is nonempty for every $e \in U_{\mathrm{ad}}$ due to the assumption (2.17). Indeed, we have

$$w \in K(e)\,, \quad w = \max\,\{0, \Phi + e/2\} \quad \text{(see [5], II, Chapt. 5)}\,.$$

It can be easily seen that $K(e)$ is convex and closed in V.

We assume that the *desk is deformed only by tangential stresses.* If $f_0 \in L_2(\Omega)$ represents the perpendicular load acting at the upper plane of the desk, then the deflection

$$u \equiv u(e) \in K(e)$$

is a solution of the variational inequality

$$\int_\Omega e(x)\,[\nabla u(x)\,.\,\nabla(v - u)\,(x)\,\mathrm{d}x] \geqq \int_\Omega \frac{1}{G}\,(f_0(x) - k\,e(x))\,(v - u)\,(x)\,\mathrm{d}x \tag{2.19}$$

$$\text{for all} \quad v \in K(e)\,,$$

which corresponds to the state inequality (1.7), where the operators $A(e)$, B and the functional f are of the form

(2.20) $$\langle A(e)\, u, v\rangle = \int_\Omega e(x)\,(\nabla u \,.\, \nabla v)\,\mathrm{d}x\,,$$

(2.21) $$\langle B(e), v\rangle = -\frac{k}{G}\int_\Omega e(x)\,v(x)\,\mathrm{d}x\,,$$

(2.22) $$\langle f, v\rangle = \frac{1}{G}\int_\Omega f_0(x)\,v(x)\,\mathrm{d}x \quad \text{for all} \quad u, v \in V, \quad e \in U_{\mathrm{ad}}\,.$$

The positive constant G is the shear modulus of elasticity.

Let us consider the cost functional of the form

(2.23) $$j(e, w) = \|w - z_d\|_1^2 + c_0\|e\|_1^2\,, \quad e \in U_{\mathrm{ad}}\,, \quad w \in V,$$

where $z_d \in H^1(\Omega)$, $c_0 \geqq 0$ and $\|\cdot\|_1$ is the norm in $H^1(\Omega)$.

In order to establish the existence of an optimal thickness-function we have to verify

(2.24) $$e_n \to e_0 \quad \text{in} \quad U \Rightarrow u(e_n) \rightharpoonup u(e_0) \quad \text{(weakly) in} \quad V.$$

Then there exists an optimal control $e_0 \in U_{\mathrm{ad}}$, because the cost functional $j\colon U_{\mathrm{ad}} \times V \to R$ is weakly lower semicontinuous with respect to $e \in U_{\mathrm{ad}} \subset H^1(\Omega)$, $w \in V$ and every sequence of elements of U_{ad} contains a subsequence weakly convergent in $H^1(\Omega)$ and strongly convergent in $U = C(\bar{\Omega})$.

We shall proceed in a similar way as in [2], Chapt. 3.7. First we recall an important result of F. Murat in [9]:

Lemma 2.2. *Let* $V = H_0^1(\Omega)$. If $\{g_n\} \subset V^*$ *is a sequence such that* $g_n \geqq 0$ *(in the distributional sense) and* $g_n \rightharpoonup g$ *(weakly) in* V^*, *then* $g_n \to g$ *(strongly) in* $W^{-1,q}(\Omega) = (W^{1,p}(\Omega))$ *for all* $q < 2$, $1/p + 1/q = 1$.

The inequality $g_n \geqq 0$ in V^* means

(2.25) $$\{g_n, \xi\rangle \geqq 0 \quad \text{for all} \quad \xi \in C_0^\infty(\Omega)\,, \quad \xi \geqq 0 \quad \text{on} \quad \Omega\,;$$

$W^{1,p}(\Omega)$ denotes the space of all functions from the space $L_p(\Omega)$, $p \geqq 1$, whose all distributive derivatives of the first order belong to $L_p(\Omega)$. We have $W^{1,2}(\Omega) \equiv H^1(\Omega)$.

Lemma 2.3. *Let* $u(e) \in K(e)$ *be a (unique) solution of the inequality* (2.19), $e \in U_{\mathrm{ad}}$. *Then the relation* (2.24) *holds.*

Proof. Let $\lim_{n\to\infty} e_n = e_0$ (strongly) in U, $e_n \in U_{\mathrm{ad}}$, $n = 0, 1, 2, \ldots$

It results from the form of the set U_{ad} that

(2.26) $$e_n \rightharpoonup e_0 \quad \text{(weakly) in} \quad W^{1,p}(\Omega) \quad \text{for every} \quad p \geqq 1\,.$$

Let us denote $u_n = u(e_n)$, $n = 0, 1, 2, \ldots$. We recall that the elements u_n are the solutions of the variational inequalities

$$(2.27)\qquad \langle A_n u_n, v - u_n\rangle \geqq \langle f + B_n, v - u_n\rangle \quad \text{for all} \quad v \in K(e_n)\,,$$

where we have denoted $A_n = A(e_n)$, $B_n = B(e_n)$, $n = 0, 1, \ldots$. In the same way as in Part 1 we can prove boundedness

$$(2.28)\qquad \|u_n\|_1 \leqq C\,, \quad n = 1, 2, \ldots$$

Hence there exists a subsequence of $\{u_n, e_n\}$ (still denoted by $\{u_n, e_n\}$) such that

$$(2.29)\qquad u_n \rightharpoonup u \quad \text{(weakly) in} \quad V\,,$$

$$(2.30)\qquad u_n \to u \quad \text{(strongly) in} \quad L_2(\Omega)\,,$$

$$(2.31)\qquad e_n \rightrightarrows e_0 \quad \text{(uniformly) in} \quad C(\bar{\Omega}) = U\,,$$

$$(2.32)\qquad e_n \rightharpoonup e_0 \quad \text{(weakly) in} \quad W^{1,p}(\Omega) \quad \text{for all} \quad p \geqq 1\,,$$

$$(2.33)\qquad A_n \to A_0 \quad \text{in} \quad L(V, V^*)\,,$$

where $L(V, V^*)$ is the normed space of all linear bounded operators from V into V^*.

As $u_n \in K(e_n)$, we have the inequalities

$$u_n \geqq \Phi + \tfrac{1}{2}e_n \quad \text{a.e. on} \quad \Omega\,, \quad n = 1, 2, \ldots,$$

and the relations (2.30), (2.31) imply $u \geqq \Phi + \frac{1}{2}e_0$ a.e. on Ω and hence

$$u \in K(e_0)\,.$$

Let us rewrite the inequality (2.27) in the form

$$(2.34)\qquad \langle A_n u_n - f - B_n, v - u_n\rangle \geqq 0 \quad \text{for all} \quad v \in K(e_n)\,.$$

Taking $v = u_n + \xi$, $\xi \geqq 0$, $\xi \in C_0^\infty(\Omega)$ we obtain

$$(2.35)\qquad \langle A_n u_n - f - B_n\rangle \geqq 0 \quad \text{in} \quad V^*\,, \quad n = 1, 2, \ldots$$

Using the forms (2.20), (2.21) and the limits (2.29), (2.30), (2.31), (2.33) we arrive at

$$(2.36)\qquad A_n u_n - f - B_n \rightharpoonup A_0 u - f - B_0 \quad \text{(weakly) in} \quad V^*\,.$$

Applying now Lemma 2.2 we obtain

$$(2.37)\quad A_n u_n - f - B_n \to A_0 u - f - B_0 \quad \text{(strongly) in} \quad W^{-1,q}(\Omega)\,, \quad q < 2\,.$$

Setting $v = w + \frac{1}{2}(e_n - e_0)$ in (2.34) for any $w \in K(e_0)$ we have the relations

$$\langle A_0 u_n, w\rangle = \langle (A_0 - A_n)\, u_n, w\rangle + \langle A_n u_n, w\rangle \geqq (\langle A_0 - A_n)\, u_n, w\rangle +$$
$$+ \langle A_n u_n, u_n\rangle + \langle f + B_n, w - u_n\rangle - \tfrac{1}{2}\langle A_n u_n - f - B_n, e_n - e_0\rangle\,.$$

Using the relations (2.29), (2.32), (2.33), (2.37) and the weak lower semicontinuity:

$$\langle A_0 u, u \rangle \leqq \liminf_{n \to \infty} \langle A_0 u_n, u_n \rangle ,$$

we arrive at the inequality

$$\langle A_0 u, w \rangle \geqq \langle A_0 u, u \rangle + \langle f + B_0, w - u \rangle \quad \text{for all} \quad w \in K(e_0)$$

and hence $u \equiv u(e_0)$ and the relation (2.24) is verified. This proves the existence of the optimal thickness-function $e_0 \in U_{\text{ad}}$ for the Optimal design problem with the cost functional (2.23) an the state inequality (2.19).

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Súhrn

ÚLOHY OPTIMÁLNEHO RIADENIA PRE VARIAČNÉ NEROVNICE S RIADENIAMI V KOEFICIENTOCH A V JEDNOSTRANNÝCH VÄZBÁCH

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Je študovaná úloha optimálneho riadenia variačnou nerovnicou s riadeniami v koeficientoch operátora nerovnice, v pravej strane a v konvexnej množine možných stavov. Dokazuje sa existencia optimálneho riadenia. Riešené sú úlohy optimálneho navrhovania pružne-plastického nosníka a pružnej dosky s prekážkou a premennou hrúbkou ako kontrolnou premennou.

Резюме

ЗАДАЧИ ОПТИМАЛЬНОГО УПРАВЛЕНИЯ ВАРИАЦИОННЫМИ НЕРАВЕНСТВАМИ С УПРАВЛЕНИЯМИ В КОЭФФИЦИЕНТАХ И В ОДНОСТОРОННИХ ОГРАНИЧЕНИЯХ

IGOR BOCK, JÁN LOVÍŠEK

В работе исследована задача оптимального управления вариационным неравенством с управлениями в коеффициентах оператора неравенства, в правой части и в выпуклом множестве возможных состояний.

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