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ON THE SOLUTION OF BOUNDARY VALUE PROBLEMS FOR SANDWICH PLATES

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Summary. A mathematical model of the equilibrium problem of elastic sandwich plates is established. Using the theory of inequalities of Korn's type for a general class of elliptic systems the existence and uniqueness of a variational solution is proved.

Key words: sandwich plates, elliptic systems

AMS Subject class.: 73 K 10, 73 C 35, 35 J 55

INTRODUCTION

The high strength characteristics of composite materials make it possible to construct plates and shells with both great strength and low weight. The walls of the structure which guarantee the required carrying capacity are thin, but their bending rigidity is not sufficient. One can increase the bending rigidity of the wall while preserving the weight of the structure if one uses the three-layer structure of the wall with a filling. The filling of the wall guarantees the cooperation of the outer layers with high strength. Since light materials of the "foam plastic" type or "ribs elements" are used for the core layer, the field of deformation of the core layer affects the work of the whole structure essentially. Hence when establishing a mathematical model, we have to take into account the above mentioned specific features of the sandwich structure.

In the books [5], [6] the equations of equilibrium and boundary conditions are derived on the basis of the "broken line" hypothesis from the principle of minimum potential energy. The solution of these equations is based on the Fourier and Ritz methods, respectively. It is the aim of the present paper to establish the mathematical model and prove the existence and uniqueness of a variational solution. To this end we apply the theory of inequalities of Korn's type for a general class of elliptic systems [1], [2].

1. ASSUMPTIONS

Let us consider a thin three-layer plate, which has two outer stiff layers — faces — of a thickness e with a high strength. These layers are connected by a core layer of a thickness $2h^0$, made of a material with an essentially lower strength. We shall assume that the core layer can transmit an essential part of horizontal forces and

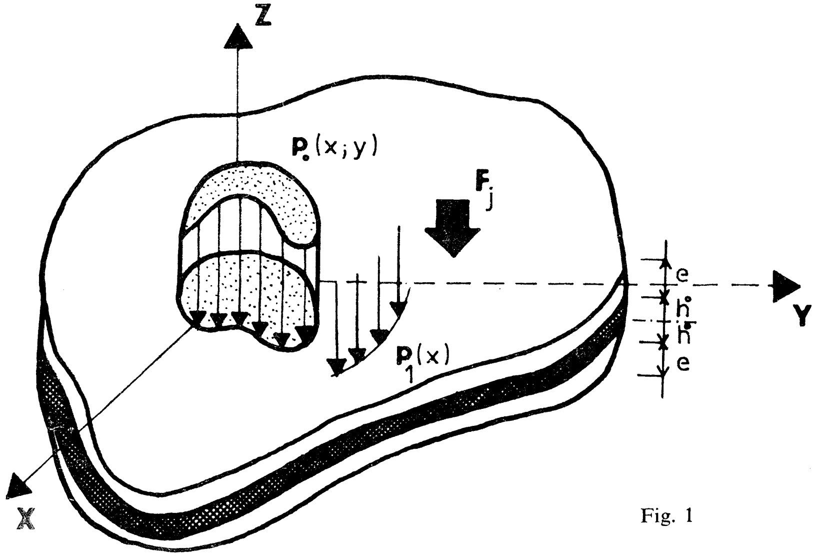

Fig. 1

bending moments. We assume that all three layers are elastic, working togehter without any shearing on the interlayer boundaries. Let the material of the stiff layers be isotropic and that of core layer transversally isotropic, the axis of isotropy being perpendicular to the middle plane of the plate.

Basic hypotheses:

(1.1) 1° Kirchhoff's hypothesis for the outer stiff layers,

2° the shear deformations γ_{xz}, γ_{yz} of the core layer are functions of x, y only,

3° the relative extension ε_z of the middle layer will be neglected,

4° the normal stress σ_z is negligible compared with σ_x, σ_y and therefore will be neglected.

The system of hypotheses enables us to take into account the total deformation of the core layer, because the straight element, perpendicular to the middle plane before the deformation, remains straight after the deformation (but not perpendicular to the deformed middle plane, due to the nonzero shear deformations). As the hypothesis of normal preservation has been accepted for the faces, the graph of the displacement along the thickness of the plate is piecewise linear (see Fig. 2).

Moreover, we assume that the thickness of the core layer $2h^0$ is constant, whereas the thickness e of the faces is a continuous function of $(x, y) \in \bar{\Omega}$, $\|e\|_{C(\bar{\Omega})} \ll h^0$.

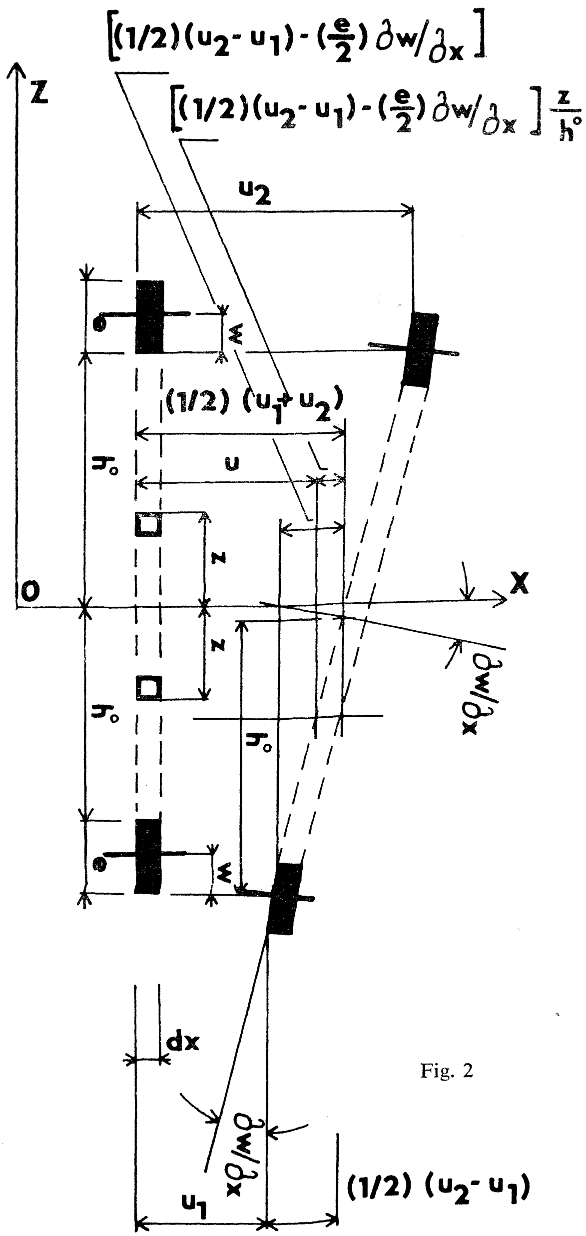

Fig. 2

On the basis of the hypotheses (1.1) (see also Fig. 2) the following relations for the vector of displacements can be formulated:

1. The stiff layers:

lower layer: $z \in [-h^0 - e, -h^0]$,

$$(1.2)\qquad \begin{aligned} u(x, y, z) &= u_1(x, y) - [(z + h^0) + (e(x, y)/2)]\, \partial w(x, y)/\partial x, \\ v(x, y, z) &= v_1(x, y) - [(z + h^0) + (e(x, y)/2)]\, \partial w(x, y)/\partial y; \end{aligned}$$

upper layer: $z \in [h^0, h^0 + e]$,

$$\begin{aligned} u(x, y, z) &= u_2(x, y) - [(z - h^0) - (e(x, y)/2)]\, \partial w(x, y)/\partial x, \\ v(x, y, z) &= v_2(x, y) - [(z - h^0) - (e(x, y)/2)]\, \partial w(x, y)/\partial y. \end{aligned}$$

2. The core layer:

$$(1.3)\qquad \begin{aligned} u(x, y, z) &= (1/2)\, \{(u_1 + u_2) - (z/h^0)\, [(u_1 - u_2) - e(x, y)\, \partial w/\partial x]\}, \\ v(x, y, z) &= (1/2)\, \{(v_1 + v_2) - (z/h^0)\, [(v_1 - v_2) - e(x, y)\, \partial w/\partial y]\}. \end{aligned}$$

Here u, v are displacements in the direction of the (positive) x- and y-axes, respectively, w is the deflection in the direction of the positive z-axis; u_i, v_i are the displacements of the middle planes of the lower ($i = 1$) and upper ($i = 2$) layers, respectively.

For the strain tensor components we obtain the following strain-displacement relations:

1° The stiff layers (upper signs hold for the upper stiff layer, lower signs for the lower stiff layer):

$$(1.4)\qquad \begin{aligned} \varepsilon_x^{1,2} &= (1/2)\, \partial(u_1 + u_2)/\partial x - z\, \partial^2 w/\partial x^2 \pm h^0\, \partial\alpha_1/\partial x, \\ \varepsilon_y^{1,2} &= (1/2)\, \partial(v_1 + v_2)/\partial y - z\, \partial^2 w/\partial y^2 \pm h^0\, \partial\alpha_2/\partial y, \\ \gamma_{xy}^{1,2} &= (1/2)\, \partial(u_1 + u_2)/\partial y + (1/2)\, \partial(v_1 + v_2)/\partial x - \\ &\quad - 2z\, \partial^2 w/\partial x\, \partial y \pm h^0(\partial\alpha_1/\partial y + \partial\alpha_2/\partial x). \end{aligned}$$

2° The core layer:

$$(1.5)\qquad \begin{aligned} \varepsilon_x^0 &= (1/2)\, \partial(u_1 + u_2)/\partial x + z(-\partial^2 w/\partial x^2 + \partial\alpha_1/\partial x), \\ \varepsilon_y^0 &= (1/2)\, \partial(v_1 + v_2)/\partial y + z(-\partial^2 w/\partial y^2 + \partial\alpha_2/\partial y), \\ \gamma_{xy}^0 &= (1/2)\, \partial(u_1 + u_2)/\partial y + (1/2)\, \partial(v_1 + v_2)/\partial x + \\ &\quad + 2z[-\partial^2 w/\partial x\, \partial y + (1/2)\, (\partial\alpha_1/\partial y + \partial\alpha_2/\partial x)], \end{aligned}$$

where

$$(1.6)\qquad \begin{aligned} \alpha_1(x, y) &= -(u_1 - u_2)/2h^0 + (1 + (e/2h^0))\, \partial w/\partial x, \\ \alpha_2(x, y) &= -(v_1 - v_2)/2h^0 + (1 + (e/2h^0))\, \partial w/\partial y. \end{aligned}$$

The stress tensor components have the form

$$(1.7)\qquad \sigma_x^i = E_i/(1-\mu^2)(\varepsilon_x^i + \mu\varepsilon_y^i) = E_i/(2(1-\mu^2))\,[\partial(u_1+u_2)/\partial x + \\ + \mu\,\partial(v_1+v_2)/\partial y - 2z(\partial^2 w/\partial x^2 + \mu\,\partial^2 w/\partial y^2) + \\ + (-1)^i\,2h^0(\partial\alpha_1/\partial x + \mu\,\partial\alpha_2/\partial y)],$$

$$\sigma_y^i = E_i/(1-\mu^2)(\varepsilon_y^i + \mu\varepsilon_x^i) = E_i/(2(1-\mu^2))\,[\partial(v_1+v_2)/\partial y + \\ + \mu\,\partial(u_1+u_2)/\partial x - 2z(\partial^2 w/\partial y^2 + \mu\,\partial^2 w/\partial x^2) + \\ + (-1)^i\,2h^0(\partial\alpha_2/\partial y + \mu\,\partial\alpha_1/\partial x)],$$

$$\tau_{xy}^i = E_i/(2(1+\mu))\,\gamma_{xy}^i = G_i[\partial((u_1+u_2)/2)/\partial y + \partial((v_1+v_2)/2)/\partial x - \\ - 2z\,\partial^2 w/\partial x\,\partial y + (-1)^i\,h^0(\partial\alpha_1/\partial y + \partial\alpha_2/\partial x)]$$

for $i = 1, 2$;

$$(1.8)\qquad \sigma_x^0 = E_0/(2(1-\mu^2))\,[\partial(u_1+u_2)/\partial x + \mu\,\partial(v_1+v_2)/\partial y - \\ - 2z(\partial^2 w/\partial x^2 + \mu\,\partial^2 w/\partial y^2) + z(\partial\alpha_1/\partial x + \mu\,\partial\alpha_2/\partial y)],$$

$$\sigma_y^0 = E_0/(2(1-\mu^2))\,[\partial(v_1+v_2)/\partial y + \mu\,\partial(u_1+u_2)/\partial x - 2z(\partial^2 w/\partial y^2 + \\ + \mu\,\partial^2 w/\partial x^2) + z(\partial\alpha_2/\partial y + \mu\,\partial\alpha_1/\partial x)],$$

$$\tau_{xy}^0 = G_0[1/2\,\partial(u_1+u_2)/\partial y + 1/2\,\partial(v_1+v_2)/\partial x + \\ + z(-2\,\partial^2 w/\partial x\,\partial y + \partial\alpha_1/\partial y + \partial\alpha_2/\partial x)],$$

$$\tau_{xz}^0 = G_0\alpha_1,\quad \tau_{yz}^0 = G_0\alpha_2,$$

where E_i is Young's modulus of the i-th layer, μ is Poisson's ratio, the same for all three layers,

$$G_i = E_i/(2(1+\mu)),\quad E_1 = E_2 = E_e,\quad E_0 = E_h\,.$$

We assume that

$$0 < \mu < 1\,.$$

The strain energy of the i-th layer is given by the formula

$$U_i = \frac{1}{2}\int_{\Omega\times\Delta_i} (\sigma_x^i\varepsilon_x^i + \sigma_y^i\varepsilon_y^i + \tau_{xy}^i\gamma_{xy}^i + \tau_{xz}^i\gamma_{xz}^i + \tau_{yz}^i\gamma_{yz}^i)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,,\quad i = 0, 1, 2\,,$$

where

$$\Delta_i = \begin{cases} (-h^0 - e, -h^0) & \text{for}\quad i = 1\,,\\ (-h^0, h^0) & \text{for}\quad i = 0\,,\\ (h^0, h^0 + e) & \text{for}\quad i = 2\,. \end{cases}$$

Then we can write the strain energy of the whole plate in the form

$$(1.9)\qquad U = \sum_{i=0}^{2} U_i\,.$$

Let us denote

$$u^1 = u_1 + u_2\,,\quad u^2 = v_1 + v_2\,,\quad u^3 = u_1 - u_2\,,\quad u^4 = v_1 - v_2\,,\quad u^5 = w$$

and consider the vector field

$$\boldsymbol{u} = (u^1, u^2, u^3, u^4, u^5)\,.$$

We shall need the following system of strain operators:

$$(1.10)\quad \begin{aligned}
&\mathcal{N}_1(\boldsymbol{u}) = \partial u^1/\partial x\,, \quad \mathcal{N}_2(\boldsymbol{u}) = \partial u^2/\partial y\,,\\
&\mathcal{N}_3(\boldsymbol{u}) = 1/4(\partial u^1/\partial y + \partial u^2/\partial x)\,, \quad \mathcal{N}_4(\boldsymbol{u}) = \partial^2 u^5/\partial x^2\,,\\
&\mathcal{N}_5(\boldsymbol{u}) = \partial^2 u^5/\partial y^2\,, \quad \mathcal{N}_6(\boldsymbol{u}) = \partial u^5/\partial x\,\partial y\,,\\
&\mathcal{N}_7(\boldsymbol{u}) = (-1/2h^0)(\partial u^3/\partial x) - (\mu/2h^0)(\partial u^4/\partial y) + [1 + (e/2h^0)]\,.\\
&\qquad .\,(\partial^2 u^5/\partial x^2) + \mu[(1 + e/2h^0)]\,\partial^2 u^5/\partial y^2 + 1/2h^0[(\partial e/\partial x)\,.\\
&\qquad .\,(\partial u^5/\partial x) + \mu(\partial e/\partial y)(\partial u^5/\partial y)]\,,\\
&\mathcal{N}_8(\boldsymbol{u}) = (-\mu/2h^0)(\partial u^3/\partial x) - (1/2h^0)(\partial u^4/\partial y) + \mu[(1 + e/2h^0)]\,.\\
&\qquad .\,(\partial^2 u^5/\partial x^2) + [1 + (e/2h^0)](\partial^2 u^5/\partial y^2) +\\
&\qquad + 1/2h^0[\mu(\partial e/\partial x)(\partial u^5/\partial x) + (\partial e/\partial y)(\partial u^5/\partial y)]\,,\\
&\mathcal{N}_9(\boldsymbol{u}) = (-1/2h^0)[(\partial u^3/\partial y) + (\partial u^4/\partial x)] + (2 + e/h^0)(\partial^2 u^5/\partial x\,\partial y) +\\
&\qquad + (1/2h^0)[(\partial e/\partial y)(\partial u^5/\partial x) + (\partial e/\partial x)(\partial u^5/\partial y)]\,.
\end{aligned}$$

Let us introduce the matrix

$$\boldsymbol{K} = \begin{bmatrix} K^* & 0 \\ 0 & K^{**} \end{bmatrix}$$

where

$$K^* = \begin{bmatrix}
(B_e + B_h); & (B_e + B_h)\,\mu; & 0 & 0 & 0 & 0\\
(B_e + B_h)\,\mu; & (B_e + B_h) & 0 & 0 & 0 & 0\\
0 & 0 & 4(B_e + B_h)(1-\mu); & 0 & 0 & 0\\
0 & 0 & 0 & 2(D_e^* + D_h + D_{eh}); & 2(D_e^* + D_h + D_{eh})\,\mu; & 0\\
0 & 0 & 0 & 2(D_e^* + D_h + D_{eh})\,\mu; & 2(D_e^* + D_h + D_{eh}) & 0\\
0 & 0 & 0 & 0 & 0 & 2(D_e^* + D_h + D_{eh})(1-\mu)
\end{bmatrix}_{(6,6)},$$

$$K^{**} = \begin{bmatrix}
2[D_{eh} + (B_e + B_h/3)\,h^{02}] & 0 & 0\\
0 & 2[D_{eh} + (B_e + B_h/3)\,h^{02}] & 0\\
0 & 0 & 2[D_{eh} + (B_e + B_h/3)\,h^{02}](1-\mu)
\end{bmatrix}_{(3,3)},$$

$$D_h = E_h h^{03}/[3(1-\mu^2)]\,, \quad B_h = E_h h^0/(1-\mu^2)\,,$$
$$D_e = E_e e^3/[12(1-\mu^2)]\,, \quad B_e = eE_e/(1-\mu^2)\,,$$
$$D_e^* = D_e + B_e(h^0 + e/2)^2\,, \quad D_{eh} = D_h + B_h h^0(h^0 + e/2)\,.$$

Moreover, we define the bilinear form

$$a(e; \mathbf{u}, \mathbf{v}) = \int_\Omega \sum_{i,j=1}^{9} K_{ij}\, \mathcal{N}_i(\mathbf{u})\, \mathcal{N}_j(\mathbf{v})\, \mathrm{d}x\, \mathrm{d}y\,. \tag{1.11}$$

Integrating in (1.9) along the z-coordinate and using (1.4)–(1.8), we can show that

$$a(e; \mathbf{u}, \mathbf{u}) = 2U(\mathbf{u})\,. \tag{1.12}$$

Assume that the loading is determined only by transversal loads in the z-direction. Then the potential of external forces is

$$L(\mathbf{u}) = \int_\Omega p_0 w\, \mathrm{d}x\, \mathrm{d}y + \sum_{i=1}^{I} P_i w(x_i, y_i) + \int_\gamma p_1 w\, \mathrm{d}\gamma\,, \quad (w = u^5)\,, \tag{1.13}$$

where γ is a given (rectifiable) curve, $\gamma \subset \Omega$, $p_1 \in L^1(\gamma)$ and $p_0 \in L^1(\Omega)$ given functions, P_i given constants, $(x_i, y_i) \in \Omega$ given points.

We shall consider the classical boundary conditions of a partially clamped plate. Namely, we prescribe

$$\begin{aligned} u_i &= 0\,, \quad v_i = 0\,, \quad i = 1, 2\,, \\ w &= 0\,, \quad \partial w/\partial n = 0 \end{aligned} \tag{1.14}$$

on a part Γ_u of the boundary $\partial\Omega$, where $\partial w/\partial n$ denotes the normal derivative.

2. VARIATIONAL FORMULATION OF THE PROBLEM

The formulation of the problem will be based on the principle of virtual displacements.

Let Ω be a bounded domain in $\mathbb{R}^2$ with a Lipschitz boundary (see e.g. [2], Chapt. 1 for the definition).

We define the space of functions with finite energy

$$W = [H^1(\Omega)]^4 \times H^2(\Omega)\,,$$

where $H^k(\Omega)$, $k = 1, 2$, denotes the standard Sobolev space $W^{k,2}(\Omega)$. The norm in $H^k(\Omega)$ will be denoted by $\|\cdot\|_{k,\Omega}$, $H^0(\Omega) \equiv L^2(\Omega)$.

The vector fields $\mathbf{u} \in W$ will be called displacement functions with finite energy, since

$$\mathbf{u} \in W \Rightarrow U(\mathbf{u}) < \infty$$

follows from the fact that $\mathcal{N}_i(\mathbf{u}) \in L^2(\Omega)$ for all $i = 1, \ldots, 9$, and all the entries K_{ij} are bounded in Ω.

The boundary conditions (1.14) determine the space of virtual displacements.

Assume that Γ_u is an open part of the boundary $\partial\Omega$ and the length of Γ_u is positive. We define

$$V = [V_0]^4 \times H^2_{\Gamma_u}(\Omega) \subset W$$

where

$$V_0 = \{v \in H^1(\Omega): v = 0 \text{ on } \Gamma_u\}$$

and $H^2_{\Gamma_u}(\Omega)$ is the closure in $H^2(\Omega)$ of continuously differentiable functions, satisfying the conditions (1.14) for w, i.e.,

$$H^2_{\Gamma_u}(\Omega) = \overline{\mathscr{W}}, \quad \mathscr{W} = \{w \in C^\infty(\bar{\Omega}): w = \partial w/\partial n = 0 \text{ on } \Gamma_u\}.$$

Note that the boundary conditions

$$u_i = v_i = 0 \quad (i = 1, 2) \text{ on } \Gamma_u$$

are equivalent to

$$u^i = 0, \quad i = 1, 2, 3, 4 \text{ on } \Gamma_u.$$

The principle of virtual displacements

$$\delta U = \delta L$$

takes the form

(2.1) $$a(e; \mathbf{u}, \mathbf{v}) = L(\mathbf{v})$$

where

$$\mathbf{v} = \delta\mathbf{u} = (\delta u^1, \delta u^2, \delta u^3, \delta u^4, \delta u^5) \in V.$$

We say that $\mathbf{u} \in V$ is a *variational* (*weak*) *solution* of the boundary value problem under consideration, if (2.1) holds for any function $\mathbf{v} \in V$.

In order to prove the *existence* and *uniqueness of a variational solution*, we first have to prove the V – ellipticity of the form $a(e; \mathbf{u}, \mathbf{v})$, i.e., the existence of a positive constant a_0 such that

(2.2) $$a(e; \mathbf{u}, \mathbf{u}) \geqq a_0 \|\mathbf{u}\|^2_W \quad \forall \mathbf{u} \in V,$$

where

$$\|\mathbf{u}\|_W = \Big(\sum_{i=1}^{4} \|u^i\|^2_{1,\Omega} + \|u^5\|^2_{2,\Omega}\Big)^{1/2}.$$

The condition (2.2) is called an inequality of Korn's type and its proof is based on several results which will be taken from the literature (see the book [2] or the paper [1]).

First, we easily realize that the matrix $\mathbf{K}$ is positive definite, so that

(2.3) $$a(e; \mathbf{u}, \mathbf{u}) \geqq k_0 \sum_{i=1}^{9} \|\mathscr{N}_i(\mathbf{u})\|^2_{0,\Omega}$$

holds for any $\mathbf{u} \in W$.

A more valuable result is the so called *coerciveness* of *the system of operators* $\{\mathscr{N}_i(\mathbf{u})\}_{i=1}^9$ on the space W, i.e. the following inequality:

(2.4) $$\sum_{i=1}^{9} \|\mathscr{N}_i(\mathbf{u})\|^2_{0,\Omega} + \sum_{i=1}^{5} \|u^i\|^2_{0,\Omega} \geqq c\|\mathbf{u}\|^2_W \quad \forall \mathbf{u} \in W.$$

The latter inequality follows from Theorem 3.2 in [1]. In fact, we first have to write the operators $\mathcal{N}_i(\boldsymbol{u})$ in the form

$$\mathcal{N}_i(\boldsymbol{u}) = \sum_{s=1}^{5} \sum_{|\beta| \leqq \varkappa_s} n_{is\beta} D^\beta u^s , \quad i = 1, \ldots, 9 ,$$

where

$$D^\beta = \partial^{|\beta|} / \partial x^{\beta_1} \, \partial y^{\beta_2} ,$$

$$\varkappa_s = 1 \quad \text{for} \quad s = 1, 2, 3, 4 \quad \text{and} \quad \varkappa_s = 2 \quad \text{for} \quad s = 5 .$$

Note that if $e \in C(\bar{\Omega})$, then the coefficients satisfy

$$n_{is\beta} \in C(\bar{\Omega}) \quad \text{for all} \quad i = 1, \ldots, 9 , \quad s = 1, \ldots, 5 \quad \text{and} \quad |\beta| = \varkappa_s .$$

Thus we can apply the above mentioned theorem. Let us define the (9×5) matrix $[N_{is}\xi]$ with entries

$$N_{is}\xi = \sum_{|\beta| = \varkappa_s} n_{is\beta} \xi_1^{\beta_1} \xi_2^{\beta_2} , \quad i = 1, \ldots, 9 , \quad s = 1, \ldots, 5 .$$

Then the system $\{\mathcal{N}_i(\boldsymbol{u})\}_{i=1}^9$ is coercive on W (i.e., (2.4) holds) if and only if the rank of the matrix $[N_{is}\xi]$ is equal to 5 for

$$0 \neq \xi \in \mathbb{R}^2 \quad \text{if} \quad (x, y) \in \Omega ,$$
$$0 \neq \xi \in \boldsymbol{C}^2 \quad \text{if} \quad (x, y) \in \partial\Omega .$$

(Here $\boldsymbol{C}^2$ denotes the complex two-dimensional space.)

In our case the transposed matrix $[N_{is}\xi]^{\mathrm{T}}$ has the following form:

$$\begin{bmatrix} \xi_1 & 0 & \xi_2/4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi_2 & \xi_1/4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\xi_1/(2h^0) & -\mu\xi_1/(2h^0) & -\xi_2/(2h^0) \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mu\xi_2/(2h^0) & -\xi_2/(2h^0) & -\xi_1/(2h^0) \\ 0 & 0 & 0 & \xi_1^2 & \xi_2^2 & \xi_1\xi_2 & \left(1+\dfrac{e}{2h^0}\right)(\xi_1^2+\mu\xi_2^2); & \left(1+\dfrac{e}{2h^0}\right)(\mu\xi_1^2+\xi_2^2); & \left(2+\dfrac{e}{h^0}\right)\xi_1\xi_2 \end{bmatrix} .$$

It is not difficult to find that the rank of the matrix is 5 under the conditions imposed above. Therefore (2.4) holds.

Lemma 2.1. *Let us define the subspace*

$$P_V = \{\boldsymbol{u} \in V \colon \|\mathcal{N}_i(\boldsymbol{u})\|_{0,\Omega} = 0, \ i = 1, \ldots, 9\} .$$

Then

$$P_V = \{0\} , \tag{2.5}$$

i.e., P_V reduces to the zero element.

Proof. From the conditions

$$\mathcal{N}_i(\boldsymbol{u}) = 0 , \quad i = 1, 2, 3$$

we conclude that (u^1, u^2) are components of a rigid body displacement, so that (see e.g. [2], Theorem 6.3.2 and its proof)

(2.6) $$u^1 = a_1 - by\,, \quad u^2 = a_2 + bx\,,$$

where a_i and b are arbitrary real constants.

The conditions

$$\mathcal{N}_i(\boldsymbol{u}) = 0\,, \quad i = 4, 5, 6$$

imply that u^5 is a linear polynomial. The boundary conditions on Γ_u, however, yield $u^5 \equiv 0$. Consequently, the conditions of vanishing of $\mathcal{N}_7(\boldsymbol{u})$, $\mathcal{N}_8(\boldsymbol{u})$ and $\mathcal{N}_9(\boldsymbol{u})$ reduce to

$$\begin{aligned} \partial u^3/\partial x + \mu\, \partial u^4/\partial y &= 0\,, \\ \mu\, \partial u^3/\partial x + \partial u^4/\partial y &= 0\,, \\ \partial u^3/\partial y + \partial u^4/\partial x &= 0\,. \end{aligned}$$

The first two conditions are equivalent to

$$\partial u^3/\partial x = 0\,, \quad \partial u^4/\partial y = 0\,.$$

Thus we may again conclude that (u^3, u^4) represent a rigid body displacement, i.e.,

$$u^3 = d_1 - cy\,, \quad u^4 = d_2 + cx\,.$$

Next, let us consider the two functions u^1, u^2 from (2.6). Since $\boldsymbol{u} \in V$, u^1 and u^2 vanish on Γ_u. Assume that

$$|a_1| + |b| > 0\,.$$

Then

(2.7) $$a_1 - by = 0$$

holds on Γ_u. If $b = 0$, then $a_1 = 0$ follows, which is a contradiction. Therefore $b \neq 0$ and (2.7) represents a straight line in $\mathbb{R}^2$. The second condition

$$a_2 + bx = 0 \quad \text{on} \quad \Gamma_u$$

represents another straight line. The two straight lines intersect in one point only. Consequently, Γ_u is contained in a one-point set, which contradicts the assumption on the set Γ_u. We arrive at the conclusion that a_1 and b vanish.

The case $|a_2| + |b| > 0$ can be treated in a parallel way. The same argument is applicable to the couple (u^3, u^4). Q. E. D.

Now we are able to prove the V-ellipticity of the form a. In fact, we may apply Lemma 11.3.2 of the book [2], since (2.3), (2.4) and (2.6) verify the assumptions of the lemma.

Theorem 2.1. *There exists a unique variational solution of the boundary value problem.*

Proof. Using the Sobolev Embedding Theorem, it is easy to show that

$$|L(\mathbf{v})| \leqq C\|\delta u^5\|_{2,\Omega} \leqq C\|\mathbf{v}\|_W\,,$$

so that the functional $L\colon V \to \mathbb{R}^1$ is linear and continuous.

The form $a(e; \boldsymbol{u}, \boldsymbol{v})$ being symmetric and V-elliptic, the existence and uniqueness of a solution $\boldsymbol{u} \in V$ of the problem (2.1) follows immediately from the Riesz-Fréchet Theorem ([3]).

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Souhrn

ŘEŠENÍ OKRAJOVÝCH ÚLOH PRO SENDVIČOVÉ DESKY

IGOR BOCK, IVAN HLAVÁČEK, JÁN LOVÍŠEK

Odvozují se rovnice a okrajové podmínky matematického modelu sendvičových desek. Na základě teorie nerovností Kornova typu pro jistou dosti obecnou třídu eliptických soustav rovnic je dokázána existence a jednoznačnost variačního řešení okrajové úlohy.

Резюме

О РЕШЕНИИ КРАЕВЫХ ЗАДАЧ ДЛЯ ТРЕХСЛОЙНЫХ ПЛАСТИНОК

IGOR BOCK, IVAN HLAVÁČEK, JÁN LOVÍŠEK

Устанавливается математическая модель равновесных задач упругих трехслойных пластинок. При помощи теории неравенств типа Корна для общего класса эллиптических систем доказываются существование и единственность вариационного решения.

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