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THE DISPERSION OF GAS EXHALATIONS AND THE PROBLEM OF DISTRIBUTION OF NEW SOURCES ON A DRY HILLY SURFACE

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Summary. The process of gas exhalations in the lower layer of the atmosphere and the problem of distribution of new sources of exhalations in a hilly terrain are studied. Among other, the following assumptions are introduced: (1) the terrain is a hilly one, (2) the exhalations enter a chemical reaction with the atmosphere, (3) the process is stationary, (4) the vector of wind velocity satisfies the continuity equation. The mathematical formulation of the problem then is a mixed boundary value problem for an elliptic equation with the given distribution on its right-hand side. It is shown that the problem has a unique "very weak" solution which is sufficiently smooth if so are the coefficients of diffusion and the components of the wind velocity vector. Further, the problem of distribution of new sources of exhalations is discussed and a method of calculation of its solution is suggested.

Keywords: mixed boundary value problem; elliptic equation; weak solution of; gas exhalation, dispersion of; sources of exhalation, distribution of

AMS Subject class.: 35 J 25, 76 N 99.

INTRODUCTION

The main goal of this paper is to study the following two problems: 1. Existence, unicity and regularity of the "very weak" solution of the boundary value problem corresponding to the dispersion of gas exhalations over a dry hilly surface. 2. The (optimal) distribution of the source of exhalations on a dry hilly surface.

The problem of dispersion of gas exhalations over a flat surface was considered by many authors (see e.g. Berliand [1] or Sutton [10] etc.). The problem of a reasonable distribution of new sources of exhalations on a flat surface was considered by Marchuk [7] and Berliand and coll. [2]. Hino in [4] and Berliand in [1] considered the problem of dispersion of exhalations over a hilly surface (under some simplifying assumptions). In [9] the author considered existence, unicity and regularity of the solution of the boundary value problem corresponding to the process of dispersion of gas exhalations over a general wet hilly surface. As to the author's know ledge, the problem of dispersion of exhalations and the (optimal) distribution of new sources of exhalations on a general dry hilly surface have not been considered as yet.

Under the assumptions formulated in section I, the process of exhalation dispersion corresponds to the mixed boundary value problem for the elliptic equation of the second order with the Dirac distribution on its right-hand side. So we must seek the solution of the boundary value problem in a "very weak" sense (see Definition 3).

Existence, unicity and regularity of the "very weak" solution are proved in Section II.

Section III deals with the problem of the (optimal) distribution of new sources of exhalations.

I. FORMULATION OF THE PROBLEM OF DISPERSION OF EXHALATIONS OVER A DRY HILLY SURFACE

The general continuity equation has the form

$$\frac{\partial c}{\partial t} - \operatorname{div}(K \operatorname{grad} c) + (\mathbf{v}, \operatorname{grad} c) + \sigma c = f(t, \xi), \tag{1}$$

where $c = c(t, \xi)$ is the concentration of the exhalations,

$$K = \begin{pmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{pmatrix}$$

is the matrix of the coefficients of turbulent diffusion, $\mathbf{v} = [v_x, v_y, v_z]$ is the vector of the wind velocity, $f = f(t, \xi)$ is the density of the given source of exhalations and $(. , .)$ is the inner product in $\mathbf{R}^3$.

Assume that during the process of dispersion of exhalations the following conditions are satisfied:

1. The hilly earth surface over which the exhalations spread is described by a twice continuously differentiable function $z = z(x, y)$.

2. The exhalated gas reacts chemically with the atmosphere. Its loss due to the chemical reaction is characterized by a non-negative constant σ.

3. The source of exhalations is situated at the point $\xi_0 = [0, 0, h]$, where h is its effective height, and Q is its emission for a time unit. Thus the right-hand side of the equation (1) is given by $Q . \delta_{\xi_0}$, where δ_{ξ_0} is the Dirac distribution with its support in ξ_0.

4. The process is stationary, i.e. $c = c(x, y, z)$.

5. The surface $z = z(x, y)$ consists of two parts P_1 and P_2, where P_1 corresponds to all water surfaces and bogs, part P_2 corresponds to dry earth surface. Experimentally it was shown that the wet surface P_1 absorbs almost all exhalations, while on the dry part P_2 total repulsion takes place. Mathematically it means that

$$c(x, y, z)|_{P_1} = 0$$

and

$$\frac{\partial c}{\partial \nu}_{|P_2} = n_x k_x \frac{\partial c}{\partial x} + n_y k_y \frac{\partial c}{\partial y} + n_z k_z \frac{\partial c}{\partial z}_{|P_2} = 0\,,$$

where $\boldsymbol{n} = [n_x, n_y, n_z]$ is a vector of the external normal to the surface $z = z(x, y)$.

6. The wind velocity satisfies the mass conservation law:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0\,.$$

On the earth surface, the vector of the wind velocity lies in the tangent plane to the surface $z = z(x, y)$, i.e.

$$n_x v_x + n_y v_y + n_z v_z = 0\,.$$

7. The concentration of exhalations vanishes in the infinity, i.e.

$$\lim_{\substack{|x|+|y|+z\to\infty \\ z>0}} c(x, y, z) = 0\,.$$

Under these assumptions we can formulate the corresponding boundary value problem

$$-\operatorname{div}(K \operatorname{grad} c) + (\mathbf{v} \operatorname{grad} c) + \sigma c = Q\delta_{\xi_0}(\xi)\,,$$

$$\frac{\partial c}{\partial \nu}_{|P_2} = 0\,,$$

$$c_{|\Gamma_1} = 0\,,$$

$$\lim_{\substack{|x|+|y|+z\to\infty \\ z>0}} c(x, y, z) = 0\,.$$

II. EXISTENCE, UNICITY AND REGULARITY OF THE SOLUTION

We consider our boundary value problem in the bounded domain

$$\Omega \subset \{\xi = [x, y, z] \in \mathbb{R}^3;\ z > z(x, y)\}\,.$$

Suppose that the boundary $\partial\Omega$ is twice continuously differentiable and put $\Gamma_2 = (\partial\Omega \cap P_2)^0$ and $\Gamma_1 = \partial\Omega \setminus \bar{\Gamma}_2$. (Both the sets Γ_2 and Γ_1 are open in $\partial\Omega$).

In the domain Ω our boundary value problem is the mixed boundary value problem in the form

(2) $$Lc = -\operatorname{div}(K \operatorname{grad} c) + (\mathbf{v} \operatorname{grad} c) + \sigma c = Q\delta_{\xi_0}(\xi) \quad \text{in} \quad \Omega\,,$$

(3) $$c_{|\Gamma_1} = 0\,,$$

(4) $$\frac{\partial c}{\partial \nu}_{|\Gamma_2} = 0\,.$$

Remark. Physical considerations make us choose Ω in such a way that, roughly speaking, (i) it adheres to the earth surface (i.e., some non-void part of $\partial\Omega$ is described by the function $z = z(x, y)$; (ii) it contains the point ξ_0 where the source of exhalations is situated; (iii) it is sufficiently large so that we can put approximately $c = 0$ on the part of $\partial\Omega$ which does not adhere to the earth surface.

By the symbols $W^{k,2}(\Omega)$ and $\mathring{W}^{k,2}(\Omega)$ we denote the Sobolev spaces (see [3]). Moreover, put

$$\mathfrak{U} = \{u \in C^\infty(\bar{\Omega}) \colon \operatorname{supp}(u) \subset \Omega \cup \Gamma_2\}$$

and denote by V the closure of the set $\mathfrak{U}$ in the space $W^{1,2}(\Omega)$. Denote by $\mathcal{D}(\Omega)$ the set of infinitely smooth functions on Ω such that $\operatorname{supp}(u) \subset \Omega$.

Definition 1. *Let Q be a space such that $\mathcal{D}(\Omega)$ is dense in Q and the imbedding of V into Q is continuous. Denote by Q^* the dual space to Q. We say that a function $u \in W^{1,2}(\Omega)$ (or $u \in V$) is a weak solution of the differential equation*

$$Au = -\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{3} b_i \frac{\partial u}{\partial x_i} + a(\xi)\, u = f \tag{5}$$

with $f \in W^{-1,2}(\Omega)$ (or of the mixed boundary value problem

$$Au = f\,, \tag{6}$$

$$u_{|\Gamma_1} = 0\,, \tag{7}$$

$$\frac{\partial u}{\partial \nu}_{|\Gamma_2} = 0\,, \tag{8}$$

with the right-hand side $f \in Q^*$, respectively) if for all $v \in \mathring{W}^{1,2}(\Omega)$ ($v \in V$) the relation

$$((u, v)) = \sum_{i,j=1}^{3} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{i=1}^{3} \int_\Omega b_i \frac{\partial u}{\partial x_i} v + \int_\Omega auv = \langle f, v\rangle \tag{9}$$

holds, where the symbol $\langle .\,, .\rangle$ denotes the duality between $W^{-1,2}(\Omega)$ and $\mathring{W}^{1,2}(\Omega)$ (Q^* and Q, respectively).

Proposition 1. *Let the surface Γ_1 have a positive measure. Then the norm in the space V is equivalent to the norm*

$$\|u\|_V = \Big(\sum_{|\beta|=1} \|D^\beta u\|^2_{L_2(\Omega)}\Big)^{1/2}\,.$$

Proof. See [5] or [8], Theorem 1.9, Chapter 1.

Proposition 2. *Let Ω be a domain with a twice continuously differentiable boundary and let $\Omega_1 \subset \bar{\Omega}_1 \subset \Omega \cup \Gamma$, where Γ is any open part of $\partial\Omega$. Further, let the following conditions be satisfied:*

(i) *There exists $\gamma > 0$ such that*

$$\gamma^{-1}\|\eta\|^2 \leqq \sum_{i,j=1}^{3} a_{ij}(\xi)\, \eta_i \eta_j \leqq \gamma \|\eta\|^2$$

for a.e. $\xi \in \mathbb{R}^3$ and every $\eta = [\eta_1, \eta_2, \eta_3] \neq 0$.

(ii) *The functions* a_{ij} *have Lipschitz derivatives of the first order and* b_i, a *are Lipschitz on* $\bar{\Omega}$.

Then every weak solution u *of the equation* (5) *with the right-hand side* $f \in L_2(\Omega)$ *which on* Γ *is given by some* u_0 *from* $W^{2,2}(\Omega_1)$, *i.e.*

$$Au = f\,,$$

$$u_{|\Gamma} = u_{0|\Gamma}\,, \quad u_0 \in W^{2,2}(\Omega_1)\,,$$

belongs to $W^{2,2}(\Omega_1)$, *while for every subdomain* $\Omega_2 \subset \bar{\Omega}_2 \subset \Omega_1 \cup \Gamma$ *there exists a constant* $M = M(\Omega_1, \Omega_2)$ *such that the following inequality holds*:

$$\|u\|_{W^{2,2}(\Omega_2)} \leqq M(\Omega_1, \Omega_2)\left(\|f\|_{L_2(\Omega_1)} + \|u\|_{L_2(\Omega_1)} + \|u_0\|_{W^{2,2}(\Omega_1)}\right).$$

Proof. See [6].

Theorem 1. (*Existence and unicity of the weak solution.*) *Let the following conditions be satisfied*:

(i) *There exists a constant* $\gamma > 0$ *such that the inequality*

$$\gamma^{-1}\|\eta\|^2 \leqq k_x^{(\xi)}\eta_x^2 + k_y^{(\xi)}\eta_y^2 + k_z(\xi)\,\eta_z^2 \leqq \gamma\|\eta\|^2$$

holds for every $\eta = [\eta_x, \eta_y, \eta_z] \neq 0$ *and a.e.* $\xi \in \mathbb{R}^3$.

(ii) k_x, k_y, k_z *have Lipschitz derivatives of the first order and* v_x, v_y, v_z *are Lipschitz functions on* $\bar{\Omega}$.

Then the mixed boundary problem

$$Lc = f\,, \tag{10}$$

$$c_{|\Gamma_1} = 0\,, \tag{11}$$

$$\frac{\partial c}{\partial \nu}_{|\Gamma_2} = 0 \tag{12}$$

has a unique weak solution c *from* V for each right-hand side $f \in L_2(\Omega)$.

If Ω_1 *is a subdomain such that* $\partial\Omega_1 \cap \partial\Omega \subset \Gamma_1$, *then* c *belongs to* $W^{2,2}(\Omega_1)$ *and for every subdomain* $\Omega_2 \subset \bar{\Omega}_2 \subset \Omega_1 \cup \{\partial\Omega_1 \cap \partial\Omega\}$ *there exists a constant* $M = M(\Omega_1, \Omega_2)$ *such that*

$$\|c\|_{W^{2,2}(\Omega_2)} \leqq M(\Omega_1, \Omega_2)\left(\|f\|_{L_2(\Omega_1)} + \|c\|_{L_2(\Omega_1)}\right), \tag{13}$$

where the constant $M(\Omega_1, \Omega_2)$ *is independent of* f.

Proof. In the special case of the operator L, the relation corresponding to (9) has the form

$$((c, w)) = \int_\Omega (K \operatorname{grad} c, \operatorname{grad} w) + \int_\Omega (\mathbf{v}, \operatorname{grad} c)\, w + \int_\Omega \sigma c w = \int_\Omega f w\,.$$

To prove existence and unicity of a weak solution in V by means of the Lax-Milgram theorem (see e.g. [8], Lemma 3.1, Chapter 1) we need to prove continuity of the bilinear form, which is trivial, and of validity the inequality $((w, w)) \geqq \alpha \|w\|_V^2$ for all $w \in V$ (with some $\alpha > 0$).

Observe first that

$$\int_\Omega (\mathbf{v}, \operatorname{grad} w)\, w = \frac{1}{2} \int_\Omega (\mathbf{v}, \operatorname{grad}(w^2)) = \frac{1}{2} \int_{\partial\Omega} v_n w^2 \, dS - \frac{1}{2} \int_\Omega w^2 \operatorname{div} \mathbf{v} = 0 .$$

Here we used the assumption 6 (Section I) and the fact that $w = 0$ on Γ_1. Further, $\int_\Omega \sigma w^2 \geqq 0$ by the assumption 2 (Section I). So we finally obtain

$$((w, w)) \geqq \int_\Omega (K \operatorname{grad} w, \operatorname{grad} w) \geqq \gamma^{-1} \|w\|_V^2 .$$

The inequality (13) follows from Proposition 2.

Analogously, we get the following result:

Theorem 1*. *Let the conditions of Theorem 1 be satisfied. Then adjoint problem*

$$(10^*) \qquad L^* c^* = - \operatorname{div}(K \operatorname{grad} c^*) - (\mathbf{v} \operatorname{grad} c^*) + \sigma c^* = f^* ,$$

$$(11^*) \qquad c^*_{|\Gamma_1} = 0 ,$$

$$(12^*) \qquad \frac{\partial c^*}{\partial \nu}_{|\Gamma_2} = 0$$

has a unique weak solution c^* from V for an arbitrary $f^* \in L_2(\Omega)$. If Ω_1, Ω_2 are the subdomains from Theorem 1, then the solution $c^* \in W^{2,2}(\Omega_1)$ and the inequality

$$(13^*) \qquad \|c^*\|_{W^{2,2}(\Omega_2)} \leqq M(\Omega_1, \Omega_2) \left(\|f^*\|_{L_2(\Omega_1)} + \|c^*\|_{L_2(\Omega_1)} \right)$$

holds, where the constant $M(\Omega_1, \Omega_2)$ is independent of f^*.

Remark. According to Theorem 1*, the Green operator

$$G^* : L_2(\Omega) \to V \cap W^{2,2}(\Omega_1)$$

is defined by the relation

$$G^* f^* = c^* ,$$

where c^* is a weak solution of the problem (10*)–(12*).

Let Ω_0 be a subdomain of Ω such that $\xi_0 \in \Omega_0 \subset \bar{\Omega}_0 \subset \Omega$. Denote

$$W(\Omega_0) = W^{2,2}(\Omega_0) \cap L_2(\Omega) ,$$

introducing the norm

$$\|u\|_{W(\Omega_0)} = \|u\|_{W^{2,2}(\Omega_0)} + \|u\|_{L_2(\Omega)} .$$

Definition 2. *The boundary value problem is called W-correct, if for every $f \in L_2(\Omega)$ there exists one and only one weak solution u from $W(\Omega_0)$.*

Definition 3. *Let the boundary value problem*

$$(6^*) \qquad A^* v^* = - \sum_{i,j=1}^{3} \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial v^*}{\partial x_i} \right) - \sum_{i=1}^{3} \frac{\partial (b_i v^*)}{\partial x_i} + a(\xi)\, v^* = f^* ,$$

(7*) $$v^*_{|\Gamma_1} = 0\,,$$

(8*) $$\frac{\partial v^*}{\partial \nu}_{|\Gamma_2} = 0$$

be W-correct. The function $u \in L_2(\Omega)$ is called a "very weak" solution of the mixed boundary value problem (6)–(8) with the right-hand side $f \in W^*(\Omega_0)$, if for every $f^* \in L_2(\Omega)$ the inequality

(14) $$(u, f^*) = \langle f, G^* f^* \rangle$$

holds, where the symbol $\langle \cdot, \cdot \rangle$ means the duality between $W^*(\Omega_0)$ and $W(\Omega_0)$ and $(\cdot, \cdot)$ is the scalar product in $L_2(\Omega)$.

Proposition 3. *If Ω is a domain with a Lipschitz boundary, then the imbedding of the space $W^{2,2}(\Omega)$ into $C(\bar{\Omega})$ is continuous.*

Proof. The assertion follows as a special case from the imbedding theorem – see [8], Theorem 3.8, Chapter 2.

Lemma 1. *The Dirac distribution δ_{ξ_0} belongs to $W^*(\Omega_0)$.*

Proof. The assertion follows from the continuity of the imbedding of $W^{2,2}(\Omega_0)$ into $C(\bar{\Omega}_0)$.

Lemma 2. *Define a function*

(15) $$f_n = \frac{n^3}{\pi^{3/2}} \exp\{-n^2 \|\xi - \xi_0\|^2\}$$

for every $n \in \mathbb{N}$ and $\xi \in \mathbb{R}^3$. Then the sequence of functionals $F_n \in W^*(\Omega_0)$ defined as

(16) $$\langle F_n, \varphi \rangle = \int_\Omega f_n(\xi)\, \varphi(\xi)\, \mathrm{d}\xi\,, \quad \varphi \in W(\Omega_0)$$

converges weakly to δ_{ξ_0} in $W^*(\Omega_0)$.*

Proof. Let $\varphi \in W(\Omega_0)$. For $\varepsilon > 0$ let $\Delta > 0$ be such that $B \equiv B_\Delta(\xi_0) \subset \Omega_0$ and $|\varphi(\xi) - \varphi(\xi_0)| < \varepsilon$ for $\xi \in B$. It is easy to check that f_n tends to zero uniformly on $\Omega \setminus B$; this together with the Lebesgue Dominated Convergence Theorem yields that

$$\lim_{n\to\infty} \int_{\Omega\setminus B} f_n(\xi)\, \varphi(\xi)\, \mathrm{d}\xi = 0\,.$$

Using the fact that $\lim_{n\to\infty} \int_B f_n = 1$ we obtain that

$$\left| \lim_{n\to\infty} \int_B f_n(\xi)\, \varphi(\xi)\, \mathrm{d}\xi - \varphi(\xi_0) \right| < \varepsilon\,.$$

Combining these two results with the definition (16) we easily obtain

$$\lim_{n\to\infty} \langle F_n, \varphi \rangle = \varphi(\xi_0) = \delta_{\xi_0}(\varphi)\,.$$

Theorem 2. (*Existence and unicity of the "very weak" solution.*) *Let the conditions* (i)–(ii) *from Theorem* 1 *be satisfied. Then the mixed boundary value problem* (2)–(4) *has a unique "very weak" solution* $c \in L_2(\Omega)$.

Proof. Let $\{f_n\}$ be the sequence (15). According to Theorem 1, for every $n \in \mathbb{N}$ there exists a weak solution c_n of the problem

$$(17) \qquad Lc_n = Qf_n \text{ on } \Omega\,, \quad c_n|_{\Gamma_1} = 0\,, \quad \partial c_n/\partial \nu|_{\Gamma_2} = 0\,.$$

Since c_n is also a "very weak" solution of the problem (17), we have (for every $u \in \mathbb{N}$ and $f^* \in L_2(\Omega)$)

$$(18) \qquad (c_n, f^*) = \langle QF_n, G^*f^* \rangle\,,$$

where F_n is defined by (16).

From the estimate

$$(19) \qquad |\langle QF_n, G^*f^* \rangle| \leqq M\|F_n\|_{W^*(\Omega_0)}\,\|f^*\|_{L_2(\Omega)}\,,$$

from (18) and from the boundedness of $\{F_n\}_{n=1}^{\infty}$ which follows from Lemma 2 we obtain

$$(20) \qquad \|c_n\|_{L_2(\Omega)} \leqq M\|F_n\|_{W^*(\Omega_0)} \leqq M_1\,.$$

So we can find a subsequence $\{c_{n_k}\}$ converging weakly in $L_2(\Omega)$ to some $c \in L_2(\Omega)$ which is a "very weak" solution of our problem (2)–(4). The uniqueness of the "very weak" solution c follows from Definition 3.

Proposition 4. *Let* $\{\varphi_n^*\}$ *be a sequence of distributions such that* $\varphi_n^* \to \varphi^*$ *in* $\mathscr{D}'(\Omega)$ *and*

$$\|D^\beta \varphi_n^*\|_{L_2(\Omega)} \leqq C$$

for every $n \in \mathbb{N}$. *Then* $D^\beta \varphi^* \in L_2(\Omega)$ *and*

$$\|D^\beta \varphi^*\|_{L_2(\Omega)} \leqq C\,.$$

Proof. See [8] – Prop. 2.4, Chap. 2.

Theorem 3. (*Regularity of the "very weak" solution up to the boundary.*)

Let the assumptions (i)–(ii) *of Theorem* 1 *be satisfied. Then the "very weak" solution c belongs to* $W^{2,2}(\Omega_1) \cap W^{1,2}(\Omega_2)$ *for all subdomains* Ω_1 *and* Ω_2 *such that* $\bar{\Omega}_1 \subset \Omega \cup \Gamma_1$, $\bar{\Omega}_2 \subset \Omega \cup \Gamma_2$ *and* $\bar{\Omega}_1 \cap \bar{\Omega}_0 = \bar{\Omega}_2 \cap \bar{\Omega}_0 = \emptyset$.

Proof. Let c be a "very weak" solution of the problem (2)–(4) and $\Omega_1 \subset \bar{\Omega}_1 \subset \Omega \cup \Gamma_1$, $\bar{\Omega}_1 \cap \bar{\Omega}_0 = \emptyset$. Choose Ω_1' such that $\bar{\Omega}_1 \subset \Omega_1' \cup \{\partial\Omega_1' \cap \Gamma_1\} \subset \bar{\Omega}_1' \subset \Omega \cup \Gamma_1$, $\bar{\Omega}_1' \cap \bar{\Omega}_0 = \emptyset$.

Let c_n be a weak solution of the problem (15)–(17). According to Theorem 1, there exists a constant $M = M(\Omega_1, \Omega_1')$ independent of n such that the inequality

$$\|c_n\|_{W^{2,2}(\Omega_1)} \leqq M(\Omega_1, \Omega_1')\left(\|f_n\|_{L_2(\Omega_1')} + \|c_n\|_{L_2(\Omega_1')}\right)$$

holds for every n.

This estimate together with the boundedness of both $\{f_n\}$ and $\{c_n\}$ in $L_2(\Omega_1')$ (See Lemma 2 and the estimate (20)) yields the weak convergence of some subsequence $\{c_{n_k}\}$ in $W^{2,2}(\Omega_1)$ to some $d \in W^{2,2}(\Omega_1)$, which cannot differ from $c|_{\Omega_1}$. Thus $c|_{\Omega_1} \in W^{2,2}(\Omega_1)$.

Now, let $\Omega_2 \subset \bar{\Omega}_2 \subset \Omega \cup \Gamma_2$ and $\bar{\Omega}_2 \cap \bar{\Omega}_0 = \emptyset$. Choose Ω_2' such that $\bar{\Omega}_2 \subset \subset \Omega_2' \cup \{\partial\Omega_2' \cap \Gamma_2\} \subset \bar{\Omega}_2' \subset \Omega \cup \Gamma_2$ and $\bar{\Omega}_2' \cap \bar{\Omega}_2 = \emptyset$. Further, let ε be an infinitely differentiable function in $\bar{\Omega}$ such that

$$\varepsilon(\xi) = \begin{cases} 1 & \text{for} \quad \xi \in \Omega_2 \\ 0 & \text{for} \quad \xi \notin \Omega_2' . \end{cases}$$

From the coerciveness we obtain

$$(21) \qquad \|\varepsilon c_n\|^2_{W^{1,2}(\Omega)} \leqq \gamma((\varepsilon c_n, \varepsilon c_n)) \leqq$$
$$\leqq \gamma|((\varepsilon c_n, \varepsilon c_n)) - ((c_n, \varepsilon^2 c_n))| + \gamma|((c_n, \varepsilon^2 c_n))| .$$

The definition of the weak solution gives

$$(22) \qquad |((c_n, \varepsilon^2 c_n))| \leqq \|f_n\|_{L_2(\Omega_2')} \|c_n\|_{L_2(\Omega)}$$

for every $n \in \mathbb{N}$.

An easy computation yields

$$(23) \qquad |((\varepsilon c_n, \varepsilon c_n)) - ((c_n, \varepsilon^2 c_n))| \leqq M_1 \|c_n\|^2_{L_2(\Omega)} .$$

From (21)–(23) we obtain

$$\|\varepsilon c_n\|^2_{W^{1,2}(\Omega)} \leqq \gamma \|f_n\|_{L_2(\Omega_2 c)} \|c_n\|_{L_2(\Omega)} + M_1 \|c_n\|^2_{L_2(\Omega)} .$$

Reasoning as in the case of Ω_1 we conclude $c|_{\Omega_2} \in W^{1,2}(\Omega_2)$.

Proposition 5. *Let the assumption* (i) *from Proposition 2 be satisfied. Let, further, the functions a_{ij} have Lipschitz derivatives of the order k and functions b_i, a Lipschitz derivatives of the order $(k - 1)$ in $\bar{\Omega}$ (for $k \geqq 1$).*

Then every weak solution u of the equation (5) *with $f \in W^{k-1,2}(\Omega)$ belongs to $W^{k+1,2}(\Omega_1)$ for any $\Omega_1 \subset \bar{\Omega}_1 \subset \Omega$. There exists a constant $M = M(\Omega_1)$ such that*

$$\|u\|_{W^{k+1,2}(\Omega_1)} \leqq M(\Omega_1) \left(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{W^{k-1,2}(\Omega)}\right)$$

holds, where $M(\Omega_1)$ is independent of f.

Proof. See [8], Theorem 1.2, Chapter 4.

Theorem 4. (*Interior regularity of the "very weak" solution.*)

Let the condition (i) *from Theorem 1 be satisfied. Let, further, the functions k_x, k_y, k_z have Lipschitz derivatives of the order k and the functions v_x, v_y, v_z derivatives of the order $(k - 1)$ on $\bar{\Omega}$ $(k \geqq 1)$.*

Then the "very weak" solution c belongs to $W^{k+1,2}(\Omega_3)$ for every subdomain $\Omega_3 \subset \bar{\Omega}_3 \subset \Omega - \bar{\Omega}_0$.

Proof. Let Ω_3' be a subdomain such that $\bar{\Omega}_3 \subset \Omega_3' \subset \bar{\Omega}_3' \subset \Omega - \bar{\Omega}_0$ and let c_n

be a weak solution of the problem (15)–(17). According to Proposition 5, there exists a constant $M = M(\Omega_3, \Omega'_3)$ such that the inequality

$$\|c_n\|_{W^{k+1,2}(\Omega_3)} \leqq M(\Omega_3, \Omega'_3)\left(\|f_n\|_{W^{k-1,2}(\Omega'_3)} + \|c_n\|_{W^{1,2}(\Omega'_3)}\right) \tag{24}$$

holds for every n. An analogous consideration as in the proof of the preceding theorem leads to the estimate

$$\|c_n\|_{W^{1,2}(\Omega'_3)} \leqq \gamma \|f_n\|_{L_2(\Omega)} \|c_n\|_{L_2(\Omega)} + M_1 \|c_n\|^2_{L_2(\Omega)} . \tag{25}$$

Now (24)–(25) and Proposition 4 imply our assertion.

Corollary. *If the coefficients K and $\mathbf{v}$ are infinitely differentiable, then the "very weak" solution c is infinitely differentiable with the exception of the point where the source is located.*

Proof follows from Theorem 4.

III. PROBLEM OF DISTRIBUTION AND OPTIMAL DISTRIBUTION OF NEW SOURCES OF EXHALATIONS

Let Ω be the domain described at the beginning of Section II. Let $D_{xy}, D^1_{xy}, \ldots, D^n_{xy}$ be domains in $\mathbb{R}^2$ such that the sets $D = \{\xi = [x, y, z] \in \mathbb{R}^3;\ [x, y] \in D_{xy},\ z = z(x, y)\}$ and $D^i = \{\xi = [x, y, z] \in \mathbb{R}^3;\ [x, y] \in D^i_{xy},\ z = z(x, y)\}$ $(i = 1, 2, \ldots, k\}$ are contained in Γ_2.

Our aim is to locate in D n sources of exhalations (chimneys of factories, power plants etc.) with given intervals $\langle h_j, H_j \rangle$ of their heights above the surface and given emissions Q_j, in such a way that in each domain D^i the value of some exhalation functional Φ_i (for example, the quantity of exhalation in the domain $[D^i_{xy} \times \langle 0, \infty)] \cap \bar{\Omega}$) does not exceed a given value N_i $(i = 1, \ldots, k)$ (N_i have the meaning of hygienic norms in the domains D^i. The domains D^i stand for check points, as e.g. densely populated areas, sources of drinking water, agricultural land etc.)

Denote by C the set of all admissible points where the mouths of the chimneys can be situated, i.e.

$$C = C_1 \times C_2 \times \ldots \times C_n , \tag{26}$$

where

$$C_j = \{\xi = [x, y, z];\ [x, y] \in \bar{D}_{xy},\ z \in \langle z(x, y) + h_j,\ z(x, y) + H_j \rangle\} \cap \bar{\Omega} .$$

If the sources are situated at a point $\Theta = [\xi_1, \ldots, \xi_n] \in C$, then the total concentration $c_\Theta = c_1 + \ldots + c_n$ is computed as the solution of the mixed boundary value problem

$$\begin{cases} Lc_\Theta \equiv -\operatorname{div}(K \operatorname{grad} c_\Theta) + (\mathbf{v}, \operatorname{grad} c_\Theta) + \sigma c_\Theta = f_\Theta , \\ c_\Theta|_{\Gamma_1} = 0 , \quad (\partial c_\Theta / \partial v)|_{\Gamma_2} = 0 , \end{cases} \tag{27}$$

where

$$f_\Theta = \sum_{j=1}^{n} Q_j \delta_{j\xi} . \tag{28}$$

Let f_i^* be the function from $L_2(\Omega)$ representing the exhalation functional Φ_i, i.e.,

$$\langle \Phi_i, v \rangle = (f_i^*, v) = \int_\Omega f_i^* v\,, \quad i = 1, \ldots, k \tag{29}$$

for every $v \in L_2(\Omega)$.

The problem, which was roughly described above, can be formulated mathematically as follows: We seek for all points $\Theta \in C$ such that the "very weak" solution c_Θ of (27) satisfies the inequality

$$(f_i^*, c_\Theta) \leqq N_i\,, \quad i = 1, \ldots, k\,. \tag{30}$$

Define $S_i \subset C$ as the set of all points Θ for which $(f_i^*, c_\Theta) \leqq N_i$. The solution of our problem is performed in two steps:

Step 1. Calculate (f_i^*, c_Θ), $\Theta \in C$, $i = 1, \ldots, k$.

Step 2. Comparing the obtained values with the corresponding N_i determine the sets S_i, $i = 1, \ldots, k$.

One of two possibilities can occur: Either $S = \bigcap_{i=1}^{k} S_i \neq \emptyset$ — then each $\Theta \in S$ solves our problem — or $S = \emptyset$. In this case there is no solution of the problem.

Remark. If $S = \emptyset$ then we must somehow lessen the emissions Q_j to make our problem solvable.

To avoid the calculation of the solution c_Θ at each point $\Theta \in C$, which we need in Step 1, we use the duality method (see e.g. [7]). Using Theorem 1 and the definition of the "very weak" solution we get

Theorem 5. (*The adjoint expression of the exhalation functional.*) *If c_i^* is a weak solution of the adjoint boundary value problem*

$$\begin{cases} L^* c_i^* = -\operatorname{div}(K \operatorname{grad} c_i^*) - (\mathbf{v}, \operatorname{grad} c_i^*) + \sigma c_i^* = f_i^*\,, \\ c_i^*|_{\Gamma_1} = 0\,, \quad \partial c_i^*/\partial \nu|_{\Gamma_2} = 0\,, \end{cases} \tag{31}$$

then the value of (f_i^*, c_Θ) can be calculated from the relation

$$(f_i^*, c_\Theta) = \langle f_\Theta, c_i^* \rangle\,, \quad i = 1, 2, \ldots, k\,. \tag{32}$$

The continuity of c_i^* in C_j together with the definition of f_Θ enable us to calculate $\langle f_\Theta, c_i^* \rangle$ as $\sum_{j=1}^{n} Q_j\, c_*^i(\xi_j)$. So this method reduces Step 1 to the slution of k boundary value problems for c_i^*, $i = 1, \ldots, k$ and the easy calculations of the expressions $\sum_{j=1}^{n} Q_j\, c_i^*(\xi_j)$.

Together with the problem we have just discussed various problems of optimal distribution of sources can be formulated. For example, we want to find $\Theta \in S$

which on this set minimizes the expression

$$\max_{i\in\{1,\ldots,k\}} \frac{\langle \Phi_i, c_\Theta \rangle}{N_i}.$$

Supposing $S \neq \emptyset$, the problem has always a solution thanks to the continuity of the minimized function on the compact set S.

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Souhrn

ŠÍŘENÍ PLYNNÝCH EXHALÁTŮ A PROBLÉM ROZLOŽENÍ NOVÝCH ZDROJŮ NA SUCHÉM KOPCOVITÉM TERÉNU

TRAN DIEN HIEN

V článku je zkoumán proces šíření plynných exhalátů v přízemní vrstvě atmosféry a problém rozložení nových zdrojů exhalátů na kopcovitém terénu. Při zkoumání se mimo jiné předpokládá, že: 1. Povrch je kopcovitý, 2. exhaláty se účastní chemické reakce s atmosférou, 3. proces je stacionární, 4. vektor rychlosti větru splňuje rovnici kontinuity. Matematickou formulací je pak smíšená okrajová úloha pro elliptickou rovnici s pravou stranou zadanou distribucí. Ukazuje se, že úloha má právě jedno „velmi slabé" řešení, které je dostatečně hladké, jsou-li koeficienty difuze a složky vektoru rychlosti větru dostatečně hladké. Dále se zkoumá existence řešení problému rozložení nových zdrojů exhalátů na kopcovitém terénu a ukazuje se postup pro výpočet tohoto řešení.

Резюме

РАСПРОСТРАНЕНИЕ ГАЗОВЫХ ЭКЗГАЛАЦИЙ И ПРОБЛЕМА РАСПОЛОЖЕНИЯ ИХ НОВЫХ ИСТОЧНИКОВ В СУХОЙ ХОЛМИСТОЙ МЕСТНОСТИ

TRAN DIEN HIEN

В статье исследуется процесс распространения газовых эксгалаций в приземном слое атмосферы и проблема расположения новых источников загрязнения воздуха в холмистой местности. При этом предполагается, что 1. местность холмистая, 2. эксгалаты химически реагируют с атмосферой, 3. процесс стационарен и 4. вектор скорости ветра удовлетворяет уравнению непрерывности. Математической формулировкой задачи является смешанная краевая задача для эллиптического уравнения с правой частью заданной распределением. Оказывается, что задача обладает в точности одним ,,очень слабым“ решением, которое достаточно гладко, если достаточно гладки коэффициентны диффузии и компоненты вектора скорости ветра. Исследуется также существование решения проблемы расположения новых источников загрязнения в холмистой местности и указывается приём для вычисления этого решения.

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