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BUCKLING OF BEAM-COLUMN PROBLEM OF ANISOTROPIC CYLINDRICAL SHELLS

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Summary. The object of this paper is to formulate the differential equations in the beam-column problem of the buckling of anisotropic cylindrical shells placed between the plates of a testing machine subject to an axial load P and a radial load H of sufficient magnitude to bring the expansion without constraint of the edges produced by P to zero deflection. The solution is obtained with necessary boundary conditions and the corresponding results for the isotropic case are deduced.

1. INTRODUCTION

The solution of the buckling problem for cylindrical shells in the case of an isotropic material are known from the literature on shells, e.g. Flugge, W. [1] pp. 443–472. Singer and Fershscher [3] solved the buckling problem for orthotropic conical shells under external pressure. Singer [2] (1962) solved the buckling of orthotropic and stiffened conical shells. M. M. Lei and Shun Cheng [4] solved the buckling problem of composite and homogeneous isotropic cylindrical shells under axial and radial loading. M. T. Wu and Shun Cheng [5] studied non linear axisymmetric, buckling of truncated spherical shells. De [6] formulated the differential equations of the buckling problem for anisotropic cylindrical shells, found the solution for anisotropic cylindrical shells without shear load in the case of two way compression from the differential equation formulated and deduced the corresponding result for the isotropic case. De [7] found the solution of the differential equation of the buckling problem for anisotropic cylindrical shells with shear load in the case of torsion of a long tube. The critical values of shear load and the total torque were also found and the corresponding results for the isotropic case were deduced as a special case.

The object of this paper is to study the beam-column problem in the case of axisymmetric anisotropic cylindrical shells.

A bar is said to be a beam when it carries a lateral load and thus is subject to bending, and it is called a column when it carries an axial compressive load. The column has a stability problem, the beam has none. When loads of both kinds act at the same time, a new problem arises. The bending load produces a lateral deflection, and this deflection provides a lever arm for the axial load which now produces additional bending. It is well-known that in this case stresses and deformations increase linearly with the lateral load but that they increase faster than linearly when the axial load approaches the buckling load of the column. This stress problem is known as the beam-column problem and we shell discuss it for anisotropic cylindrical shells.

An anisotropic shell is put between the plates of a testing machine. When a load is applied the length of the cylinder decreases and consequently, the diameter increases. This increase is prevented at the edges because of the friction between the cylinder and the plates of press. The deformation may be considered to be produced in two steps. At first the edge of the cylinder can expand without constraint and then a radial load of sufficient magnitude is applied to bring the edge to zero deflection. Under these assumptions the differential equations are formulated and the solutions are obtained under necessary boundary conditions.

2. BASIC EQUATIONS

The differential equations of the buckling problem for anisotropic shells, see De [6], are given by

$$\text{(1a)}\qquad u'' + A_1 u^{\bullet\bullet} + A_2 v'^{\bullet} + A_3 w' + k_1\{A_4(u^{\bullet\bullet} + w'^{\bullet\bullet}) - w'''\} - \\ - q_1(u^{\bullet\bullet} - w') - q_2 u'' - 2q_3 u'^{\bullet} = 0\,,$$

$$\text{(1b)}\qquad A_5 u'^{\bullet} + v^{\bullet\bullet} + A_6 v'' + w^{\bullet} + k_1\{3A_7 v'' - A_8 w''^{\bullet}\} - \\ - A_9[q_1(v^{\bullet\bullet} + w^{\bullet}) + q_2 v'' + 2q_3(v'^{\bullet} + w')] = 0\,,$$

$$\text{(1c)}\qquad A_{10} u' + v^{\bullet} + w + k_1\{A_7 u'^{\bullet\bullet} - A_9 u''' - A_8 v''^{\bullet} + A_9 w'''' + \\ + 2A_{11} w''^{\bullet\bullet} + A_{12}(w^{\bullet\bullet\bullet\bullet} + 2w^{\bullet\bullet} + w)\} + \\ + A_9[q_1(u' - v^{\bullet} + w^{\bullet\bullet}) + q_2 w'' - 2q_3(v' - w'^{\bullet})] = 0\,,$$

where $(\ )'$ and $(\ )^{\bullet}$ indicate $a\,\partial/\partial x\,(\ )$ and $\partial/\partial\varphi\,(\ )$, respectively, a is the radius of the shell (Fig. 1),

$$\text{(2)}\quad A_1 = \frac{D_{x\varphi}}{D_x}\,,\quad A_2 = \frac{D_\nu + D_{x\varphi}}{D_x}\,,\quad A_3 = \frac{D_\nu}{D_x}\,,\quad A_4 = \frac{K_{x\varphi}}{K_x}\,.\quad A_5 = \frac{D_\nu + D_{x\varphi}}{D_\varphi}\,,$$

$$A_6 = \frac{D_{x\varphi}}{D_\varphi}\,,\quad A_7 = \frac{D_x K_{x\varphi}}{D_\varphi K_x}\,,\quad A_8 = \frac{D_x(3k_{x\varphi} + K_\nu)}{D_\varphi K_x}\,,\quad A_9 = \frac{D_x}{D_\varphi}\,,\quad A_{10} = \frac{D_\nu}{D_\varphi}\,,$$

$$A_{11} = \frac{D_x(2K_{x\varphi} + k_\nu)}{D_\varphi K_x}\,,\quad A_{12} = \frac{D_x K_\varphi}{D_\varphi K_x}\,,$$

and

(3) $$k_1 = \frac{K_x}{a^2 D_x}, \quad q_1 = \frac{pa}{D_x}, \quad q_2 = \frac{P}{D_x}, \quad q_3 = \frac{T}{D_x},$$

where the rigidities D and K are given by

(i) extensional rigidities:

(4a) $$D_x = E_1 t_1 + 2E_2 t_2 ,$$
$$D_\varphi = E_2 t_1 + 2E_1 t_2 ,$$
$$D_v = E_v t ;$$

(ii) shear rigidity:

(4b) $$D_{x\varphi} = Gt ;$$

(iii) bending rigidities

(4c) $$K_x = \tfrac{1}{12}[E_2(t^3 - t_1^3) + E_1 t_1^3] ,$$
$$K_\varphi = \tfrac{1}{12}[E_1(t^3 - t_1^3) + E_2 t_1^3] ,$$
$$K_v = \tfrac{1}{12}Et^3 ;$$

(iv) twisting rigidity:

(4d) $$K_{x\varphi} = \tfrac{1}{12}Gt^3 ,$$

in which E_1, E_2, E_v and G are four moduli of elasticity, $t = t_1 + 2t_2$ is the thickness of the shell (Fig. 2) and the shell is simultaneously subject to three simple loads (Fig. 1):

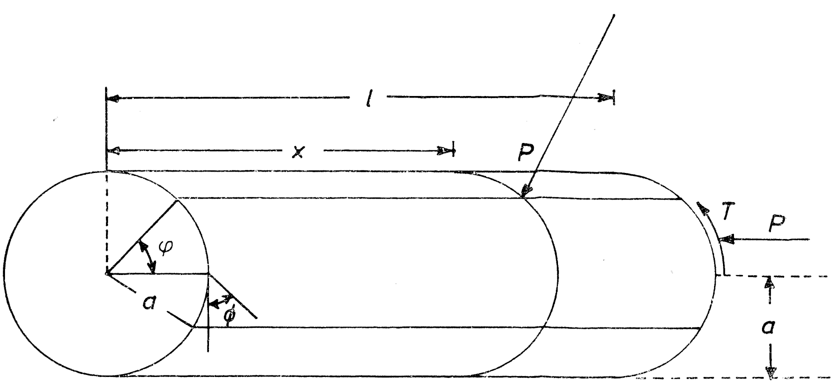

Fig. 1

(i) a uniform normal pressure on its wall, $p_r = -p$;

(ii) an axial compression applied at the edges, the force per unit circumference being P;

(iii) a shear load applied at the edges so as to produce a troque in the cylinder, the shearing force (shear flow) being T.

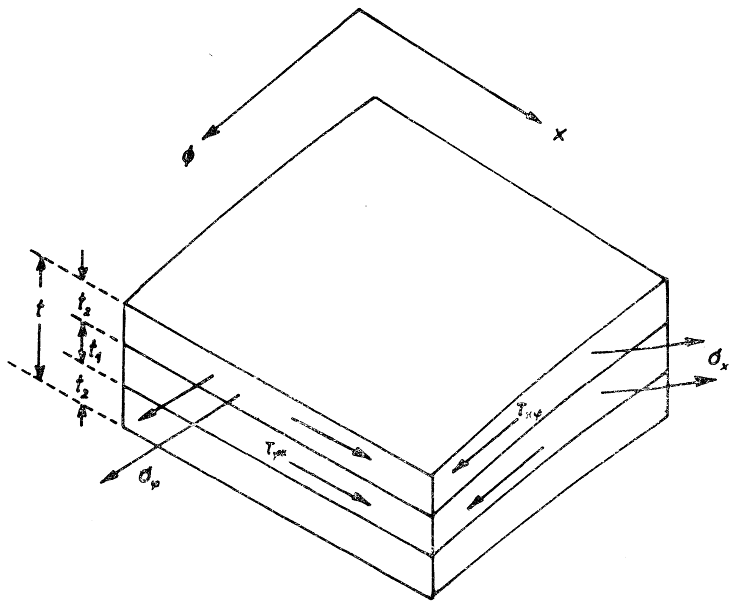

Fig 2

3. THE PROBLEM AND ITS SOLUTION

We consider a cylindrical shell put between the plates of a testing machine (Fig. 3a). When the load P (per unit circumference) is applied, a negative stress $\sigma_x = -P/t$ is produced and the length of the cylinder decreases. As the length decreases, consequently, due to the rigidity of the material, the diameter increases but this increase is prevented at the edges because of the friction between the cylinder and the plates of the press (Fig. 3b). Evidently the bending stresses will appear, and we must find out whether they or a possible instability will determine the strength of the shell.

The deformation shown in Fig. 3b may be produced in two steps. First we assume that the ends of the cylinder are so well lubricated that the edges can expand without constraint, according to a hoop strain $\varepsilon_\varphi = A_3P/Et$ (Fig. 3c). Then we apply a radial load H (Fig. 3d) of sufficient magnitude to bring the edges back to zero deflection.

The first of this deformation is trivial. In the second part we have normal force $N_{xI} = -P$ as a large basic force and the additional small load H which produces the small displacements u, v, w. To this deformation we may apply equations (1) which were established by De [4] under similar circumstances for the same anisotropic shells. The essential difference is that the solution we seek now is not an incidental deformation which becomes possible when P assumes a certain critical value but a deformation which is produced by the load H.

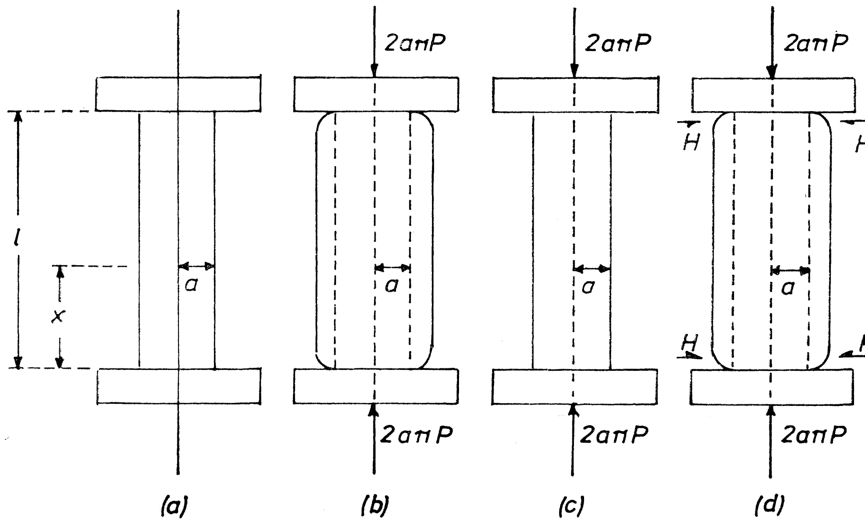

Fig. 3

In this particular case, the equations (1) simplify considerably. Since there is only the axial load P (i.e. $p = T = 0$) we have from (3) $q_1 = q_3 = 0$. Again, since the deformation is expected to have axial symmetry, we must put $v = 0$ and must drop all the dot derivatives. The equation (1b) vanishes altogether, and the other two simplify to

(5a) $$u'' + A_3 w' - k_1 w''' - q_2 u'' = 0\,,$$

(5b) $$A_{10} u' + w + k_1\{-A_9 u''' + A_9 w'''' + A_{12} w\} + A_9 q_2 w'' = 0\,.$$

There are two boundary conditions at each end of the shell, say at $x = 0$ and $x = 1$. First, the radial displacement w must cancel the displacement

$$a\varepsilon_\varphi = aA_3 P/Et$$

of Fig. 3c, i.e. we must have

(6a) $$w = -aA_3 \frac{P}{Et} = -a \frac{D_v}{Et}\, q_2\,.$$

Second, we want to have hinged edges, i.e. $M_x = 0$ and hence from p. 295, Eqns. (93), cf. [1],

(6b)
$$w'' - u' = 0 .$$

Then we have the condition that the load H has no component in the axial direction. In Fig. 3c the axial force per unit of the undeformed circumference is $-P$, after deformation this is cf. [1] p. 417,

$$-P\left(1 + \frac{u'}{a}\right) + N_x ,$$

where N_x is connected with u, w by the relation cf. [1], p. 295, Eqns. (93b):

(7)
$$N_x = \frac{D_x}{a} u' + \frac{D_v}{a} w - \frac{K_x}{a^3} w'' .$$

We have, therefore, the condition that at each edge the equality

(8)
$$N_x = Pu'/a$$

holds. Substituting the value of N_x from (7) in (8) and simplifying we obtain

(6c)
$$u' + A_3 w - k_1 w'' = q_2 u' ,$$

where A_3, k_1, q_2 are given by (2) and (3).

From (5a) it is evident that it is enough to enforce the condition (6c) at $x = 0$ and that then it will be fulfilled everywhere, including the other edge $x = 1$. Finally, we may exclude or fix a rigid body displacement of the whole shell by prescribing u for one value of the coordinate x.

The differential equations (5) have constant coefficients and may be solved by exponential functions:

(9)
$$u = B\, e^{\lambda x/a} , \quad w = C\, e^{\lambda x/a} .$$

Substituting these values of u and w from (9) in (5) we get two linear equations for B and C:

(10a)
$$B(1 - q_2)\, \lambda^2 + C(A_3 - k_1 \lambda^2) = 0 ,$$

(10b)
$$B(A_{10} - k_1 A_9 \lambda^2) + C(1 + k_1 A_9 \lambda^4 + k_1 A_{12} + q_2 \lambda^2) = 0 .$$

Since these equations are homogeneous their determinant must vanish, and this yields an equation for λ. When the small quantities k_1 and q_2 are neglected compared with unity, we have after simplifications using the relations (2)

(11)
$$A_9 k_1 \lambda^6 + (a A_{10} k_1 + q_2)\, \lambda^4 + (1 - A_3 A_{10})\, \lambda^2 = 0 .$$

The equation (11) has the double root

$$\lambda_5 = \lambda_6 = 0$$

and four non-trivial roots

$$(12)\quad \lambda_{1,2,3,4,} = \pm \sqrt{\left(-\frac{q_2}{2A_9k_1} - \frac{a}{2}A_3 \pm \frac{1}{2A_9k_1}\sqrt{(q_2^2 - 4(1 - A_3 . A_{10})\,A_9k_1)}\right)} .$$

While these four roots lead to true exponential solutions, the fifth and sixth solution degenerate into linear functions of x, and we have

$$(13a)\qquad u = \sum_{n=1}^{4} B_n\, e^{\lambda_n x/a} + B_5 + B_6 \frac{x}{a},$$

$$(13b)\qquad w = \sum_{n=1}^{4} C_n\, e^{\lambda_n x/a} + C_5 + C_6 \frac{x}{a}.$$

For $n = 1, 2, 3, 4,$ the constants B_n and C_n are connected by the equations (10), and since the determinant of these equations is zero, we may use either one to formulate the relation. We choose the relation (10a) obtaining

$$(14a)\qquad B_n = -C_n \frac{A_3 - k_1\lambda_n^2}{\lambda_n}.$$

The degenerate solutions with $n = 5, 6$ must be introduced in the differential equations (5) to make sure that they really are solution and to determine how their constants are interconnected. We find that

$$(14b)\qquad C_6 = 0\,, \quad B_6 = -C_5 \frac{1 + A_{12}k_1}{A_{10}}.$$

This indicates that C_6x/a is no solution at all and that $u = B_6x/a$ and $w = C_5$ together are the fifth independent solution of the equations (5). The sixth solution is $u = B_5$, $w = 0$.

This last solution evidently represents a rigid body motion of the cylinder, and we may simply discard it. The remaining constants $C_1, \ldots, C_5$ can then be determined from the boundary conditions (6a–c). We being with the equation (6c). When we introduce there the solution (13) and make use of the equations (14), we find that the exponential solutions cancel out and we are left with $C_5 = 0$.

Thus we got rid of all the linear terms in the equations (13). The boundary conditions (6a,b) written for $x = 0$ and $x = b$ yield four equations for the remaining four unknown coefficients. They may easily be brought into the following form:

$$(15)\qquad \begin{aligned} C_1 + \quad C_2 + \quad C_3 + \quad C_4 &= -\frac{D_v}{Et} q_2 a\,, \\ \lambda_1^2 C_1 + \quad \lambda_2^2 C_2 + \quad \lambda_3^2 C_3 + \quad \lambda_4^2 C_4 &= A_3 \frac{D_v}{Et} q_2 a\,, \end{aligned}$$

$$e^{\lambda_1 l/a}C_1 + e^{\lambda_2 l/a}C_2 + e^{\lambda_3 l/a}C_3 + e^{\lambda_4 l/a}C_4 = -\frac{D_v}{Et} q_2 a ,$$

$$\lambda_1^2 e^{\lambda_1 l/a}C_1 + \lambda_2^2 e^{\lambda_2 l/a}C_2 + \lambda_3^2 e^{\lambda_3 l/a}C_3 + \lambda_4^2 e^{\lambda_4 l/a}C_4 = A_3 \frac{D_v}{Et} q_2 a .$$

We shall not go into the details of solving these equations, but we shall discuss the solution as obtained when the results are introduced into the equations (13).

Assume that $A_3 A_{10} < 1$. We see from (12) that for small values of q_2 all four values λ_n^2 are complex but they are real and negative if q_2 grows beyond the limit

(16) $$q_2^2 = 4(1 - A_3 A_{10}) A_9 k_1 .$$

If q_2 is smaller than this limit we may write

$$\lambda_1 = -\lambda_3 = -\alpha + i\beta , \quad \lambda_2 = -\lambda_4 = -\alpha - i\beta$$

with real positive quantities α, β. When q_2 is small and the cylinder is long then $e^{-\alpha l/a}$ is a very small quantity. In this case it turns out that $C_1, C_2 \gg C_3, C_4$ so that C_3, C_4 may be neglected in the equations (15a,b) and C_1, C_2 in the equations (15c, d). The solution then becomes extremely simple. For small values of x/a only the terms C_1 and C_2 make appreciable contributions and yield

(17) $$\omega = -\frac{D_v}{Et} q_2 a\, e^{-\alpha x/a}\left(\cos\frac{\beta x}{a} + \frac{A_3 + \alpha^2 - \beta^2}{2\alpha\beta}\sin\frac{\beta x}{a}\right),$$

while near $x = 1$ only C_3 and C_4 are essential and we have

(18) $$\omega = -\frac{D_v}{Et} q_2 a\, e^{-\alpha(l-x)/a}\left(\cos\frac{\beta(l-x)}{a} + \frac{A_3 + \alpha^2 - \beta^2}{2\alpha\beta}\sin\frac{\beta(l-x)}{a}\right).$$

These formulas represent two identical end disturbances produced by the constraint imposed upon the circumferential expansion of the shell, and these disturbances affect only two narrow border zones.

When q_2 is increased these disturbances penetrate deeper into the shell. Increasing q_2 gradually as it is nearer to the limiting value the two disturbance zones reach each other and finally they overlap completely.

At the loading of this stage, α decreases so far that it is no longer possible to neglect anything in equation (5). The solution in that case can be written in terms of hyperbolic and trigonometric functions and we leave out those bulky formulae.

The critical value of q_2 is given by (16). Beyond this value all four values λ_n are purely imaginary, say

$$\lambda_1 = -\lambda_3 = i\mu_1 , \quad \lambda_2 = -\lambda_4 = i\mu_2 ,$$

and the solution assumes the following form:

$$(19)\quad \omega = -\frac{D_v}{Et}\frac{q_2 a}{\mu_2^2 - \mu_1^2}\left[\frac{\mu_2^2 - A_3}{\cos\frac{\mu_1 l}{2a}}\cos\frac{\mu_1(l-2x)}{2a} - \frac{\mu_1^2 - A_3}{\cos\frac{\mu_2 l}{2a}}\cos\frac{\mu_2(l-2x)}{2a}\right].$$

As q_2 increases, one of the cosine denominators very rapidly approaches zero. Consequently, one of the terms in the brackets outgrouws the other one, and the deflection approaches a pure sinusoidal shape, but at the same time its amplitude increases beyond bounds.

This is a strict analogy to the well-known phenomenon that in an ordinary beam column the deflection tends to infinity as the axial load approaches the Euler load. Also the mechanical interpretation is the same in both cases. For the anisotropic shell, the infinite deflection occurs when

$$(20)\quad \mu_{1,2}^2 = \frac{q_2}{2A_9k_1} - \frac{a}{2}A_3 \mp \frac{1}{2A_9k_1}\sqrt{[q_2^2 - 4(1 - A_3A_{10})A_9k_1]} = \left(\frac{n\pi a}{l}\right)^2$$

where n is an odd integer. When this equation is solved for q_2, we arrive at the equation (11) of [6] with $q_1 = 0$, $m = 0$ and such that n is an odd integer (except that in that equation two small terms have been neglected). We conclude that the bending stresses in our grow beyond bounds when the load P approaches a critical value connected with a buckling mode which is axisymmetric ($m = 0$) and also symmetric to the plane $x = l/2$ (n = odd).

On this way to infinity, the bending stress will sooner or later pass the yield limit. As soon as this happens, our theory cases to be valid, and the first and the largest bulge of the cylinder will be squeezed flat. Except for this, the elastic theory is still applicable, and if the test is continued the next bolds will grow until they also start to yield and are squeezed flat.

The equations (5) which govern the bending collapse just described are a special case of the general buckling equations (4) from [6]. We derived the former from the latter essentially by putting $m = 0$. This seems reasonable but is by no means necessary. Our solutions (17) and (19) are just as well solutions of the general equations (4), see [6]. When the load P (or the dimensionless parameter q_2) reaches the buckling load given by the equation (11) of [6] with $q_1 = 0$, then the equations (4) of [6] permit a certain deflection of arbitrary amplitude, and since this deflection satisfies the homogeneous boundary conditions it may be superposed on the solutions of the differential equation (4) of [6] with the boundary conditions (6a), and hence of the beam-column problem. The existence of a solution in which an amplitude coefficient can be varied at discretion, indicates a natural equilibrium, which always stands at the threshold to instability.

The developing bending collapse may, therefore, be interrupted by a true and sudden buckling if the shell reaches a buckling load, either for a mode with $m > 0$ or for one with $m = 0$ and an even value of n.

PARTICULAR CASE

To get the corresponding result for isotropic case we put

$$t_2 = 0\,,\quad t_1 = t\,,\quad E_1 = E_2 = \frac{E}{1 - \nu^3}\,,\quad E_\nu = \frac{E\nu}{1 - \nu^2} \tag{21}$$

$$G = \frac{E}{2(1 + \nu)}\,,\quad (\nu = \text{Poisson's ratio}).$$

Substituting (21) in the equations (5), (6a), (6b), (6c), (10), (12), (13), (14a), (14b), (15), (16), (17), (19) we get the corresponding result for the isotropic shells which are identical with the respective equations (51), (52a), (52b), (52c), (53), (54), (55), (56a), (56b), (57), (58), (59), (60) of [1], pp. 458–461.

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Souhrn

ZTRÁTA STABILITY VÁLCOVÉ SKOŘEPINY POD OSOVÝM A RADIÁLNÍM ZATÍŽENÍM

ANUKUL DE

Článek se zabývá otázkou ztráty stability osovým tlakem, je-li skořepina upnuta do čelistí zkušebního stroje — lisu tak, že se její čela nemohou během stlačování roztahovat, tedy působí silné tření mezi čelistmi a skořepinou. Je nalezeno řešení s nutnými okrajovými podmínkami a jsou odvozeny odpovídající výsledky pro isotropní případ.

Резюме

ПОТЕРА УСТОЙЧИВОСТИ ЦИЛИНДРИЧЕСКОЙ ОБОЛОЧКИ, НАХОДЯЩЕЙСЯ ПОД ДЕЙСТВИЕМ ОСЕВОЙ И РАДИАЛЬНОЙ НАГРУЗОК

В статье рассматривается проблема потери устойчивости под действием осевого давления при условии, что оболочка закреплена в челюсти испытательной машины — пресса таким образом, что ее лобовые поверхности не могут при сжимании растягиваться. Найдено решение с необходимыми краевыми условиями и выведены соответствующие результаты для изотропного случая.

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