Josef Matušů; Josef Novák

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CONSTRUCTIONS OF INTERPOLATION CURVES FROM GIVEN SUPPORTING ELEMENTS (II)

JOSEF MATUŠŮ, JOSEF NOVÁK

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Summary. This papers deals with the constructions of interpolation curves which pass through given supporting points (nodes) and touch supporting tangent vectors given at only some of these points or, as the case may be, at all these points. The mathematical kernel of these constructions is based on the Lienhard's interpolation method. Formulae for the curvature of plane and space interpolation curves are derived.

Keywords: Interpolation, curves, curvature.

1. LIENHARD'S INTERPOLATION METHOD

Our approach is based on the papers [1], [2], [3]. In [3] the case of a spatial closed interpolation curve was considered (Example 4) which had a different osculation plane with respect to the arc $P_{i-1}P_i$ than with respect to the arc P_iP_{i+1}. This "deficiency" will be removed now.

First we recall the substance of Lienhard's interpolation method (see [1], [2]). Further in the text this method will be briefly referred to as method I.

Let $n \geqq 3$ be an integer. In the space $\mathbf{R}^m$ ($m > 1$ integer) let n different points $P_i = x_j^{(i)}$ ($i = 1, \ldots, n$; $j = 1, \ldots, m$) be given. The symbol $x_j^{(i)}$ denotes also the corresponding ordered m-tuple of coordinates, or rather the vector which has these coordinates. Thus, the elements of the set $\mathbf{R}^m$ are either points or vectors, according to which of the notions corresponds more to our conception in the given context. As a rule, we use the notion of a point in situations when location in the space $\mathbf{R}^m$ is discussed while the notion of a vector indicates that we are interested in the direction. Also, bold types will be sometimes used to denote vectors.

We shall look for polynomials in the real variable t (of degree at most K, not determined more precisely at the moment)

$$P_{x_j}^{(i)}(t) = \sum_{k=0}^{K} a_{jk}^{(i)} t^k \quad (i = 1, \ldots, n - 1) \tag{1.1}$$

such that

(1.2) $$P_{x_j}^{(i)}(-1) = x_j^{(i)}, \quad P_{x_j}^{(i)}(1) = x_j^{(i+1)},$$

(1.3) $$\frac{d}{dt} P_{x_j}^{(i)}(1) = \frac{d}{dt} P_{x_j}^{(i+1)}(-1),$$

(1.4) $$\frac{d^2}{dt^2} P_{x_j}^{(i)}(1) = \frac{d^2}{dt^2} P_{x_j}^{(i+1)}(-1).$$

Conditions (1.2) guarantee that the interpolation arc parametrized with the aid of the functions $P_{x_j}^{(i)}(t)$ $(j = 1, \ldots, m)$ passes through the points P_i, P_{i+1}. Conditions (1.3) guarantee fluent transition from arc to arc. Finally, conditions (1.4) guarantee, in the planar as well as in the spatial case, that the osculation circle at the point P_i with respect to the arc $P_{i-1}P_i$ is the same as the osculation circle at this point with respect to the arc P_iP_{i+1} (provided P_i is not a point of inflection of the interpolation curve). To satisfy conditions (1.3), (1.4) we have to know the values of the functions $dP_{x_j}^{(i)}(t)/dt$, $d^2P_{x_j}^{(i)}(t)/dt^2$ at the points P_i, P_{i+1}:

(1.5) $$\frac{d}{dt} P_{x_j}^{(i)}(-1) = Dx_j^{(i)}, \quad \frac{d^2}{dt^2} P_{x_j}^{(i)}(-1) = D^2x_j^{(i)},$$

(1.6) $$\frac{d}{dt} P_{x_j}^{(i)}(1) = Dx_j^{(i+1)}, \quad \frac{d^2}{dt^2} P_{x_j}^{(i)}(1) = D^2x_j^{(i+1)};$$

by convention, $Dx_j^{(i)}$, $Dx_j^{(i+1)}$, $D^2x_j^{(i)}$, $D^2x_j^{(i+1)}$ is the notation for these values. The manner of their determination will be discussed later. By (1.2), (1.3), (1.4) six definite conditions are given for every polynomial (1.1). With their aid each polynomial is then uniquely determined as a polynomial of degree at most $K = 5$:

(1.7) $$P_{x_j}^{(i)}(t) = \sum_{k=0}^{5} a_{jk}^{(i)} t^k.$$

We have

(1.8) $$\frac{d}{dt} P_{x_j}^{(i)}(t) = \sum_{k=1}^{5} k a_{jk}^{(i)} t^{k-1},$$

(1.9) $$\frac{d^2}{dt^2} P_{x_j}^{(i)}(t) = \sum_{k=2}^{5} k(k-1) a_{jk}^{(i)} t^{k-2}.$$

If we substitute the values $t = -1, 1$ into (1.8), (1.9), we obtain (taking (1.2), (1.5), (1.6) into account) the following system of six linear equations for the six unknown coefficients $a_{jk}^{(i)}$ of the polynomial (1.7):

(1.10) $$\sum_{k=0}^{5} (-1)^k a_{jk}^{(i)} = x_j^{(i)},$$

$$\sum_{k=1}^{5} (-1)^k k a_{jk}^{(i)} = Dx_j^{(i)},$$

$$\sum_{k=2}^{5} (-1)^k k(k-1) a_{jk}^{(i)} = D^2 x_j^{(i)},$$

$$\sum_{k=0}^{5} a_{jk}^{(i)} = x_j^{(i+1)},$$

$$\sum_{k=1}^{5} k a_{jk}^{(i)} = D x_j^{(i+1)}$$

$$\sum_{k=2}^{5} k(k-1) a_{jk}^{(i)} = D^2 x_j^{(i+1)}.$$

We introduce the matrices

(1.11) $$\mathbf{A}_{ij} = (a_{j0}^{(i)}, a_{j1}^{(i)}, a_{j2}^{(i)}, a_{j3}^{(i)}, a_{j4}^{(i)}, a_{j5}^{(i)}),$$

(1.12) $$\mathbf{X}_{ij} = (x_j^{(i)}, D x_j^{(i)}, D^2 x_j^{(i)}, x_j^{(i+1)}, D x_j^{(i+1)}, D^2 x_j^{(i+1)}).$$

The matrix of the coefficients of system (1.10), which is necessarily regular in view of the uniqueness of the determination of the desired polynomials, is denoted by $\mathbf{A}$. Then the solution of system (1.10) is expressed in matrix notation by the relation

(1.13) $$\mathbf{A}_{ij}^{\mathrm{T}} = \mathbf{A}^{-1} \circ \mathbf{X}_{ij}^{\mathrm{T}},$$

where the superscript T denotes transposed matrices to the matrices (1.11), (1.12), and $\mathbf{A}^{-1}$ denotes the inverse matrix to $\mathbf{A}$.

The values of the first and second derivatives at the points P_i, P_{i+1} (see (1.5), (1.6)) are determined as follows (see [3], Section 1). By Fig. 1 the points $(2h, x_j^{(i+h)})$ $(-1 \leqq h \leqq 1$, h integer) determine uniquely a polynomial of at most second degree

(1.14) $$R_{x_j}^{(i)}(t) = \sum_{k=0}^{2} b_{jk}^{(i)} t^k$$

with the aid of which we put

(1.15) $$D x_j^{(i)} = \frac{d}{dt} R_{x_j}^{(i)}(0) = b_{j1}^{(i)}, \quad D^2 x_j^{(i)} = \frac{d^2}{dt^2} R_{x_j}^{(i)}(0) = 2 b_{j2}^{(i)}.$$

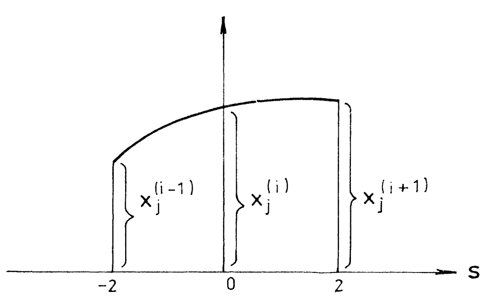

Fig. 1.

Since every coefficient of the polynomial (1.14) is a certain linear combination of the values $x_j^{(i-1)}$, $x_j^{(i)}$, $x_j^{(i+1)}$ the same holds for the derivatives $\mathrm{D}x_j^{(i)}$, $\mathrm{D}^2x_j^{(i)}$. Therefore there exists a matrix $\boldsymbol{B}$ of type (2, 3) such that

$$(1.16)\qquad (\mathrm{D}x_j^{(i)}, \mathrm{D}^2x_j^{(i)}) = (x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}) \circ \boldsymbol{B}^{\mathrm{T}}\,.$$

Then we have

$$(1.17)\qquad (x_j^{(i)}, \mathrm{D}x_j^{(i)}, \mathrm{D}^2x_j^{(i)}) = (x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}, x_j^{(i+2)}) \circ \begin{bmatrix} 0 & \\ 1 & \boldsymbol{B}^{\mathrm{T}} \\ 0 & \\ 0 & 0 \quad 0 \end{bmatrix}.$$

Analogously we obtain

$$(1.18)\quad (x_j^{(i+1)}, \mathrm{D}x_j^{(i+1)}, \mathrm{D}^2x_j^{(i+1)}) = (x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}, x_j^{(i+2)}) \circ \begin{bmatrix} 0 & 0 \quad 0 \\ 0 & \\ 1 & \boldsymbol{B}^{\mathrm{T}} \\ 0 & \end{bmatrix};$$

in this case we replace the number i in Fig. 1 by the number $i + 1$. By (1.17), (1.18) it is then possible to express the matrix (1.12) in the form

$$(1.19)\qquad \boldsymbol{X}_{ij} = (x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}, x_j^{(i+2)}) \circ \begin{bmatrix} 0 & & 0 & 0 \quad 0 \\ 1 & \boldsymbol{B}^{\mathrm{T}} & 0 & \\ 0 & & 1 & \boldsymbol{B}^{\mathrm{T}} \\ 0 & 0 \quad 0 & 0 & \end{bmatrix}.$$

After substituting (1.19) into (1.13) we have

$$(1.20)\qquad \boldsymbol{A}_{ij}^{\mathrm{T}} = \boldsymbol{C} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix},$$

where

$$(1.21)\qquad \boldsymbol{C} = \boldsymbol{A}^{-1} \circ \begin{bmatrix} 0 \quad 1 \quad 0 & 0 \\ & 0 \\ \boldsymbol{B} & \\ & 0 \\ 0 \quad 0 \quad 1 & 0 \\ 0 & \\ \quad \boldsymbol{B} & \\ 0 & \end{bmatrix}.$$

By simple computation we find that

$$(1.22)\qquad \mathbf{A}^{-1} = \frac{1}{16}\begin{bmatrix} 8 & 5 & 1 & 8 & -5 & 1 \\ -15 & -7 & -1 & 15 & -7 & 1 \\ 0 & -6 & -2 & 0 & 6 & -2 \\ 10 & 10 & 2 & -10 & 10 & -2 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ -3 & -3 & -1 & 3 & -3 & 1 \end{bmatrix},$$

$$(1.23)\qquad \mathbf{B} = \frac{1}{4}\begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}.$$

With the aid of (1.21), (1.22), (1.23) it is then possible to express (1.13) in the following form (see [2], formula (29)):

$$(1.24)\qquad 32\mathbf{A}_{ij}^{\mathrm{T}} = \begin{bmatrix} -2 & 18 & 18 & -2 \\ 3 & -25 & 25 & -3 \\ 2 & -2 & -2 & 2 \\ -4 & 12 & -12 & 4 \\ 0 & 0 & 0 & 0 \\ 1 & -3 & 3 & -1 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}.$$

Grouping of nodes will be carried out in the same manner as in [3], Section 2.

Example 1. In the plane $\mathbf{R}^2$ let us consider the points $P_1 = (0, 0)$, $P_2 = (2, 3)$, $P_3 = (15, -6)$, $P_4 = (2, -10)$, $P_5 = (10,5)$ (see Example 1 in [3]). By (1.24) we then have the following parametric equations for the individual arcs of the unclosed planar interpolation curve $P_1P_2P_3P_4P_5$:

(1.25)

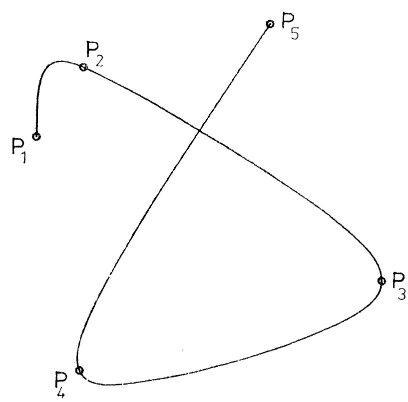

Fig. 2.

$$P_1P_2 \ldots \begin{aligned} P_{x_1}^{(1)}(t) &= 0{\cdot}0625 + 0{\cdot}34375t + 0{\cdot}9375t^2 + 0{\cdot}875t^3 - 0{\cdot}21875t^5\,, \\ P_{x_2}^{(1)}(t) &= 1{\cdot}875 + 3{\cdot}1875t - 0{\cdot}375t^2 - 2{\cdot}25t^3 + 0{\cdot}5625t^5\,, \end{aligned}$$

$$P_2P_3 \ldots \begin{aligned} P_{x_1}^{(2)}(t) &= 9{\cdot}4375 + 9{\cdot}96875t - 0{\cdot}9375t^2 - 4{\cdot}625t^3 + 1{\cdot}15625t^5\,, \\ P_{x_2}^{(2)}(t) &= -1{\cdot}0625 - 6{\cdot}09375t - 0{\cdot}4375t^2 + 2{\cdot}125t^3 - 0{\cdot}53125t^5\,, \end{aligned}$$

$$P_3P_4 \ldots \begin{aligned} P_{x_1}^{(3)}(t) &= 8{\cdot}8125 - 10{\cdot}90625t - 0{\cdot}3125t^2 + 5{\cdot}875t^3 - 1{\cdot}46875t^5\,, \\ P_{x_2}^{(3)}(t) &= -9{\cdot}5 - 3{\cdot}3125t + 1{\cdot}5t^2 + 1{\cdot}75t^3 - 0{\cdot}4375t^5\,, \end{aligned}$$

$$P_4P_5 \ldots \begin{aligned} P_{x_1}^{(4)}(t) &= 5{\cdot}6875 + 7{\cdot}46875t + 0{\cdot}3125t^2 - 4{\cdot}625t^3 + 1{\cdot}15625t^5\,, \\ P_{x_2}^{(4)}(t) &= -1{\cdot}8125 + 12{\cdot}09375t - 0{\cdot}6875t^2 - 6{\cdot}125t^3 + 1{\cdot}53125t^5\,. \end{aligned}$$

The interpolation curve is drawn in Fig. 2.

2. COMPUTATION OF THE VECTORS OF THE FIRST AND SECOND DERIVATIVES (*under method I*)

Consider the nodes P_i, P_{i+1}, P_{i+2} and look for the tangent vector (vector of the first derivative) at the point P_{i+1} with respect to the interpolation arc P_iP_{i+1}. Since it is required that condition (1.3) be valid, this vector is equal to the tangent vector at the point P_{i+1} with respect to the interpolation arc $P_{i+1}P_{i+2}$. By (1.24), from the relation $32P_{x_j}^{(i)}(t) = (1, t, t^2, t^3, t^4, t^5) \circ 32\boldsymbol{A}_{ij}^{\mathrm{T}}$ we obtain, by differentiation,

$$(2.1) \qquad 32P_{x_j}^{\prime(i)}(t) = (1, t, t^2, t^3, t^4, t^5) \circ \begin{bmatrix} 3 & -25 & 25 & -3 \\ 4 & -4 & -4 & 4 \\ -12 & 36 & -36 & 12 \\ 0 & 0 & 0 & 0 \\ 5 & -15 & 15 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}.$$

For $t = 1$, (2.1) yields

$$(2.2) \qquad 4P_{x_j}^{\prime(i)}(1) = (0, -1, 0, 1) \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix} = x_j^{(i+2)} - x_j^{(i)} = P_{i+2} - P_i\,,$$

i.e., the tangent vector at the point P_{i+1} with respect to the interpolation arcs P_iP_{i+1} and $P_{i+1}P_{i+2}$ is collinear with the vector $\overrightarrow{P_iP_{i+2}}$, and its length is four times smaller than that of the vector $\overrightarrow{P_iP_{i+2}}$.

Let us consider the points P_i, P_{i+1}, P_{i+2} once more, and let us look for the vector of the second derivative at the point P_{i+1} with respect to the interpolation arc P_iP_{i+1}. Since we require that condition (1.4) be valid, this vector is equal to the vector of the second derivative at the point P_{i+1} with respect to the interpolation arc $P_{i+1}P_{i+2}$.

By differentiation of relation (2.1) we obtain

$$(2.3)\qquad 32P''^{(i)}_{x_j}(t) = (1, t, t^2, t^3, t^4, t^5) \circ \begin{bmatrix} 4 & -4 & -4 & 4 \\ -24 & 72 & -72 & 24 \\ 0 & 0 & 0 & 0 \\ 20 & -60 & 60 & -20 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}.$$

For $t = 1$, (2.3) yields

$$(2.4)\qquad 4P''^{(i)}_{x_j}(1) = (0, 1, -2, 1) \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix} = x_j^{(i)} - 2x_j^{(i+1)} + x_j^{(i+2)} =$$

$$= (P_{i+2} - P_{i+1}) + (P_i - P_{i+1}).$$

3. MODIFICATION OF LIENHARD'S INTERPOLATION METHOD

When constructing interpolation curves in Section 1 the mutual distances of the nodes were not taken into account. Now we shall consider these distances and introduce a modification of Lienhard's interpolation method which differs somewhat from the modification given in [2]. In the text below we shall briefly refer to the present modification as to method II.

We shall proceed in the same way as in Section 1 with the only difference that the values $-2, 0, 2$ (see Fig. 1) of the variable t will be replaced by the values $-2q_{i,-1}/q_{i,0}$, 0, $2q_{i,1}/q_{i,0}$; here $q_{i,-1} = |P_{i-1}P_i|$, $q_{i,1} = |P_iP_{i+1}|$ (the distances of the respective points), $q_{i,0} = (q_{i,-1} + q_{i,1})/2$. Applying the corresponding polynomial (1.14) and conditions (1.15) we then have (cf. (1.16))

$$(3.1)\qquad (Dx_j^{(i)}, D^2x_j^{(i)}) = (x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}) \circ \mathbf{B}_i^{\mathrm{T}},$$

where

$$(3.2)\qquad \mathbf{B}_i = \tfrac{1}{8}\begin{bmatrix} -2r_i & 2r_i - 2r_i^{-1} & 2r_i^{-1} \\ 1 + r_i & -(2 + r_i + r_i^{-1}) & 1 + r_i^{-1} \end{bmatrix},$$

$r_i = q_{i,1}/q_{i,-1}$. Thus we have (cf. (1.17))

$$(3.3)\qquad (x_j^{(i)}, Dx_j^{(i)}, D^2x_j^{(i)}) = (x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}, x_j^{(i+2)}) \circ \begin{bmatrix} 0 & & \\ 1 & \mathbf{B}_i^{\mathrm{T}} & \\ 0 & & \\ 0 & 0 & 0 \end{bmatrix}$$

and analogously (cf. (1.18))

$$(3.4)\quad (x_j^{(i+1)}, \mathrm{D}x_j^{(i+1)}, \mathrm{D}^2x_j^{(i+1)}) = (x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}, x_j^{(i+2)}) \circ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{} \\ 1 & \multicolumn{2}{c}{\boldsymbol{B}_{i+1}^{\mathrm{T}}} \\ 0 & \multicolumn{2}{c}{} \end{bmatrix}.$$

By (3.3), (3.4) it is then possible to express the matrix (1.12) in the form

$$(3.5)\qquad \boldsymbol{X}_{ij} = (x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}, x_j^{(i+2)}) \circ \begin{bmatrix} 0 & & & 0 & 0 & 0 \\ 1 & \multicolumn{2}{c}{\boldsymbol{B}_i^{\mathrm{T}}} & 0 & & \\ 0 & & & 1 & \multicolumn{2}{c}{\boldsymbol{B}_{i+1}^{\mathrm{T}}} \\ 0 & 0 & 0 & 0 & & \end{bmatrix}.$$

After substituting (3.5) into (1.13) we have

$$(3.6)\qquad \boldsymbol{A}_{ij}^{\mathrm{T}} = \boldsymbol{C}_i \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix},$$

where

$$(3.7)\qquad \boldsymbol{C}_i = \boldsymbol{A}^{-1} \circ \begin{bmatrix} 0 & 1 & 0 & 0 \\ & & & 0 \\ \multicolumn{3}{c}{\boldsymbol{B}_i} & \\ & & & 0 \\ 0 & 0 & 1 & 0 \\ 0 & & & \\ & \multicolumn{3}{c}{\boldsymbol{B}_{i+1}} \\ 0 & & & \end{bmatrix}.$$

Thus it is possible to summarize: If in the unmodified case, i.e., when not respecting the mutual distances of the nodes, the matrix $\boldsymbol{B}$ (cf. (1.23)) of formula (1.21) is constant for all arcs P_iP_{i+1}, then this matrix changes from arc to arc in the modified case, i.e., when the mutual distances of the nodes are taken into account. Simultaneously, the matrix $\boldsymbol{C}$ of formula (1.21) also changes and passes into matrix (3.7). With the aid of (3.7), (1.22), (3.2) it is then possible to write (1.24) in the following form:

$$(3.8)\qquad 128\boldsymbol{A}_{ij}^{\mathrm{T}} = \left[\begin{array}{ll} 1 - 9r_i & 63 + 9r_i - 11r_i^{-1} + 11r_{i+1} \\ -1 + 13r_i & -117 - 13r_i + 15r_i^{-1} + 15r_{i+1} \\ -2 + 10r_i & 2 - 10r_i + 14r_i^{-1} - 14r_{i+1} \\ 2 - 18r_i & 74 + 18r_i - 22r_i^{-1} - 22r_{i+1} \\ 1 - r_i & -1 + r_i - 3r_i^{-1} + 3r_{i+1} \\ -1 + 5r_i & -21 - 5r_i + 7r_i^{-1} + 7r_{i+1} \end{array}\right.$$

$$\begin{bmatrix} 63 + 11r_i^{-1} - 11r_{i+1} + 9r_{i+1}^{-1} & 1 - 9r_{i+1}^{-1} \\ 117 - 15r_i^{-1} - 15r_{i+1} + 13_{i+1}^{-1} & 1 - 13r_{i+1}^{-1} \\ 2 - 14r_i^{-1} + 14r_{i+1} - 10r_{i+1}^{-1} & -2 + 10r_{i+1}^{-1} \\ -74 + 22r_i^{-1} + 22r_{i+1} - 18r_{i+1}^{-1} & -2 + 18r_{i+1}^{-1} \\ -1 + 3r_i^{-1} - 3r_{i+1} + r_{i+1}^{-1} & 1 - r_{i+1}^{-1} \\ 21 - 7r_i^{-1} - 7r_{i+1} + 5r_{i+1}^{-1} & 1 - 5r_{i+1}^{-1} \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}.$$

Example 2. In the plane $\mathbf{R}^2$ let us consider the same nodes P_1, P_2, P_3, P_4, P_5 as in Example 1. For the individual arcs of the unclosed planar interpolation curve $P_1P_2P_3P_4P_5$ we have, by (3.8), the following parametric equations:

$$P_1P_2 \ldots \quad \begin{aligned} P_{x_1}^{(1)}(t) &= 0{\cdot}18627 + 0{\cdot}56955t + 0{\cdot}89401t^2 + 0{\cdot}62745t^3 - \\ &\quad - 0{\cdot}08028t^4 - 0{\cdot}197t^5 , \\ P_{x_2}^{(1)}(t) &= 0{\cdot}51373 + 1{\cdot}29174t + 1{\cdot}27858t^2 + 0{\cdot}47255t^3 - \\ &\quad - 0{\cdot}29231t^4 - 0{\cdot}26429t^5 , \end{aligned}$$

$$P_2P_3 \ldots \quad \begin{aligned} P_{x_1}^{(2)}(t) &= 9{\cdot}3558 + 10{\cdot}8846t - 0{\cdot}73258t^2 - 6{\cdot}00707t^3 - \\ &\quad - 0{\cdot}12322t^4 + 1{\cdot}62247t^5 , \\ P_{x_2}^{(2)}(t) &= 0{\cdot}18624 - 8{\cdot}02079t - 1{\cdot}90401t^2 + 4{\cdot}87211t^3 + \\ &\quad + 0{\cdot}21777t^4 - 1{\cdot}35132t^5 , \end{aligned}$$

$$P_3P_4 \ldots \quad \begin{aligned} P_{x_1}^{(3)}(t) &= 8{\cdot}89481 - 9{\cdot}93111t - 0{\cdot}41973t^2 + 4{\cdot}47326t^3 + \\ &\quad + 0{\cdot}02492t^4 - 1{\cdot}04215t^5 , \\ P_{x_2}^{(3)}(t) &= -9{\cdot}17068 - 2{\cdot}94243t + 1{\cdot}12923t^2 + 1{\cdot}22173t^3 + \\ &\quad + 0{\cdot}04145t^4 - 0{\cdot}2793t^5 , \end{aligned}$$

$$P_4P_5 \ldots \quad \begin{aligned} P_{x_1}^{(4)}(t) &= 5{\cdot}32167 + 7{\cdot}98606t + 0{\cdot}74119t^2 - 5{\cdot}35665t^3 - \\ &\quad - 0{\cdot}06286t^4 + 1{\cdot}37059t^5 , \\ P_{x_2}^{(4)}(t) &= -2{\cdot}14047 + 12{\cdot}54666t - 0{\cdot}28144t^2 - 6{\cdot}75094t^3 - \\ &\quad - 0{\cdot}07809t^4 + 1{\cdot}73428t^5 . \end{aligned}$$

The interpolation curve is shown in Fig. 3.

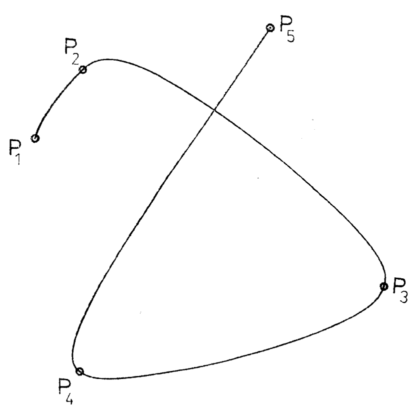

Fig. 3.

4. COMPUTATION OF THE VECTORS OF THE FIRST AND SECOND DERIVATIVES (*under method II*)

According to (3.8) we have, by simple computation,

$$(4.1)\qquad 4P'^{(i)}_{x_j}(1) = (0, -r_{i+1}, r_{i+1} - r_{i+1}^{-1}, r_{i+1}^{-1}) \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix} =$$

$$= r_{i+1}^{-1}(P_{i+2} - P_{i+1}) - r_{i+1}(P_i - P_{i+1}),$$

i.e., the tangent vector at the point P_{i+1} with respect to the interpolation arcs P_iP_{i+1}, $P_{i+1}P_{i+2}$ is a linear combination of the vectors $\overrightarrow{P_iP_{i+1}}$, $\overrightarrow{P_{i+1}P_{i+2}}$. For the vector of the second derivatives at the point P_{i+1}, with respect to these arcs, we have

$$(4.2)\qquad 8P''^{(i)}_{x_j}(1) = (0, 1 + r_{i+1}, -(2 + r_{i+1} + r_{i+1}^{-1}), 1 + r_{i+1}^{-1}) \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix} =$$

$$= (1 + r_{i+1}^{-1})(P_{i+2} - P_{i+1}) + (1 + r_{i+1})(P_i - P_{i+1}).$$

5. CASES WITH PRESCRIBED SUPPORTING TANGENT VECTORS (*method $\mathscr{T}$*)

Below we shall investigate cases for which supporting tangent vectors are prescribed at some supporting points (or at all supporting points as the case may be).

a) To the node P_i with supporting tangent vector $\mathbf{v}_i = v_j^{(i)}$ (or without any supporting tangent vector) we assign the number $K_i = 1$ (or $K_i = 0$, respectively). Let $K_i = 1$, $K_{i+1} = 0$. According to Fig. 4 the points $(2h, x_j^{(i+h)})$ $(-1 \leqq h \leqq 1$, h integer) and the number $v_j^{(i)}$ determine uniquely a polynomial of at most third degree $S_{x_j}^{(i)}(t)$ with the aid of which we put (see (1.15)) $\mathrm{D}x_j^{(i)} = S'^{(i)}_{x_j}(0)$, $\mathrm{D}^2x_j^{(i)} = S''^{(i)}_{x_j}(0)$. By simple computation we find out that

$$(5.1)\qquad (x_j^{(i)}, \mathrm{D}x_j^{(i)}, \mathrm{D}^2x_j^{(i)}) = (x_j^{(i-1)}, x_j^{(i)}, v_j^{(i)}, x_j^{(i+1)}, x_j^{(i+2)}) \circ \tfrac{1}{4} \begin{bmatrix} 0 & 0 & 1 \\ 4 & 0 & -2 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Further, by (1.18), (1.23) we have

$$(5.2)\qquad (x_j^{(i+1)}, \mathrm{D}x_j^{(i+1)}, \mathrm{D}^2x_j^{(i+1)}) = (x_j^{(i-1)}, x_j^{(i)}, v_j^{(i)}, x_j^{(i+1)}, x_j^{(i+2)}) \circ \tfrac{1}{4} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 4 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}.$$

With the aid of (5.1), (5.2) it is possible to express the transposed matrix to the matrix (1.12) in the form

$$(5.3)\qquad \mathbf{X}_{ij}^{\mathrm{T}} = \tfrac{1}{4}\begin{bmatrix} 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ v_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}.$$

Substituting (5.3) into (1.13), where $\mathbf{A}^{-1}$ is the matrix (1.22), we obtain

$$(5.4)\qquad 64\mathbf{A}_{ij}^{\mathrm{T}} = \begin{bmatrix} 1 & 36 & 20 & 31 & -4 \\ -1 & -50 & -28 & 57 & -6 \\ -2 & -4 & -24 & 2 & 4 \\ 2 & 24 & 40 & -34 & 8 \\ 1 & 0 & 4 & -1 & 0 \\ -1 & -6 & -12 & 9 & -2 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ v_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}.$$

b) Let $K_i = 0$, $K_{i+1} = 1$. By (1.17), (1.23) we have

$$(5.5)\qquad (x_j^{(i)}, \mathrm{D}x_j^{(i)}, \mathrm{D}^2x_j^{(i)}) = (x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}, v_j^{(i+1)}, x_j^{(i+2)}) \circ \tfrac{1}{4}\begin{bmatrix} 0 & -1 & 1 \\ 4 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By Fig. 5 the points $(2h, x_j^{(i+1+h)})$ $(-1 \leqq h \leqq 1$, h integer) and the number $v_j^{(i+1)}$ determine uniquely a polynomial of at most third degree $V_{x_j}^{(i+1)}(t)$ with the aid of which we put $\mathrm{D}x_j^{(i+1)} = V_{x_j}'^{(i+1)}(0)$, $\mathrm{D}^2x_j^{(i+1)} = V_{x_j}''^{(i+1)}(0)$. By simple computation we find that

$$(5.6)\qquad (x_j^{(i+1)}, \mathrm{D}x_j^{(i+1)}, \mathrm{D}^2x_j^{(i+1)}) = (x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}, v_j^{(i+1)}, x_j^{(i+2)}) \circ \tfrac{1}{4}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & -2 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

With the aid of (5.5), (5.6) it is possible to express the transposed matrix to the matrix (1.12) in the form

$$(5.7)\qquad \mathbf{X}_{ij}^{\mathrm{T}} = \tfrac{1}{4}\begin{bmatrix} 0 & 4 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 1 & -2 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ v_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}.$$

Substituting (5.7) into (1.13), where $\boldsymbol{A}^{-1}$ is the matrix (1.22), we obtain

$$(5.8)\qquad 64\boldsymbol{A}_{ij}^{\mathrm{T}} = \begin{bmatrix} -4 & 31 & 36 & -20 & 1 \\ 6 & -57 & 50 & -28 & 1 \\ 4 & 2 & -4 & 24 & -2 \\ -8 & 34 & -24 & 40 & -2 \\ 0 & -1 & 0 & -4 & 1 \\ 2 & -9 & 6 & -12 & 1 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ x_j^{(i+1)} \\ v_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}.$$

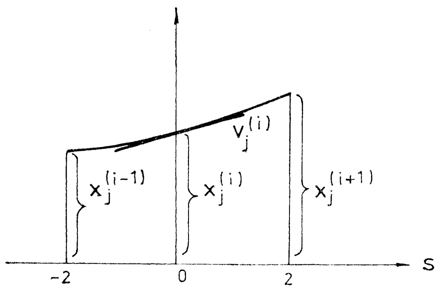

Fig. 4.

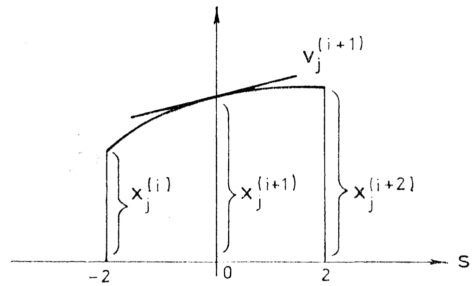

Fig. 5.

c) Let $K_i = K_{i+1} = 1$. By (5.1), (5.6) it is possible to express the transposed matrix to the matrix (1.12) in the form

$$(5.9)\qquad \boldsymbol{X}_{ij}^{\mathrm{T}} = \tfrac{1}{4} \begin{bmatrix} 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ v_j^{(i)} \\ x_j^{(i+1)} \\ v_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}.$$

Substituting (5.9) into (1.13), where $\boldsymbol{A}^{-1}$ is the matrix (1.22), we obtain

$$(5.10)\qquad 64\boldsymbol{A}_{ij}^{\mathrm{T}} = \begin{bmatrix} 1 & 31 & 20 & 31 & -20 & 1 \\ -1 & -57 & -28 & 57 & -28 & 1 \\ -2 & 2 & -24 & 2 & 24 & -2 \\ 2 & 34 & 40 & -34 & 40 & -2 \\ 1 & -1 & 4 & -1 & -4 & 1 \\ -1 & -9 & -12 & 9 & -12 & 1 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ v_j^{(i)} \\ x_j^{(i+1)} \\ v_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}.$$

The individual arcs of the desired interpolation curve which takes into account the given supporting elements are then constructed with the aid of the formulas (1.24) ($K_i = K_{i+1} = 0$), (5.4) ($K_i = 1$, $K_{i+1} = 0$), (5.8) ($K_i = 0$, $K_{i+1} = 1$), and (5.10) ($K_i = K_{i+1} = 1$).

Example 3. Consider the same nodes P_1, P_2, P_3, P_4, P_5 as in Example 1. At the nodes P_3, P_4 let us choose the tangent vectors $\boldsymbol{v}_3 = (1, -2)$, $\boldsymbol{v}_4 = (0, 3)$. Let us construct the unclosed planar interpolation curve $P_1P_2P_3P_4P_5$ (see Example 3 in [3]).

We start with the interpolation arc P_1P_2. Since $K_1 = K_2 = 0$, we apply formula (1.24) to the nodes P_2, P_1, P_2, P_3 and obtain the following parametric equations of the arc P_1P_2 (cf. (1.25)):

$$P_1P_2 \ldots \quad \begin{aligned} P^{(1)}_{x_1}(t) &= 0{,}0625 + 0{\cdot}34375t + 0{\cdot}9375t^2 + 0{\cdot}875t^3 - 0{\cdot}21875t^5 \,, \\ P^{(1)}_{x_2}(t) &= 1{,}875 + 3{,}1875t - 0{\cdot}375t^2 - 2{\cdot}25t^3 + 0{\cdot}5625t^5 \,. \end{aligned}$$

We continue with the arc P_2P_3. Since $K_2 = 0$, $K_3 = 1$, we apply formula (5.8) to the supporting elements P_1, P_2, P_3, $\mathbf{v}_3$, P_4, which leads to the following parametric equations of the arc P_2P_3:

$$P_2P_3 \ldots \quad \begin{aligned} P^{(2)}_{x_1}(t) &= 9{,}\,125 + 9{\cdot}53125t - 0{\cdot}5625t^2 - 4t^3 - 0{\cdot}0625t^4 + \\ &\quad + 0{\cdot}96875t^5 \,, \\ P^{(2)}_{x_2}(t) &= -1{\cdot}45312 - 6{\cdot}64062t + 0{\cdot}03125t^2 + 2{\cdot}90625t^3 - \\ &\quad - 0{\cdot}07813t^4 - 0{\cdot}76563t^5 \,. \end{aligned}$$

We continue with the arc P_3P_4. Since $K_2 = K_3 = 1$, we apply formula (5.10) to the supporting elements P_2, P_3, $\mathbf{v}_3$, P_4, $\mathbf{v}_4$, P_5 and obtain the following parametric equations of the arc P_3P_4:

$$P_3P_4 \ldots \quad \begin{aligned} P^{(3)}_{x_1}(t) &= 8{\cdot}73438 - 11{\cdot}89062t - 0{\cdot}21875t^2 + 7{\cdot}28125t^3 - \\ &\quad - 0{\cdot}01563t^4 - 1{\cdot}89063t^5 \,, \\ P^{(3)}_{x_2}(t) &= -9{\cdot}1875 - 3{\cdot}96875t + 1{\cdot}125t^2 + 2{\cdot}6875t^3 + \\ &\quad + 0{\cdot}0625t^4 - 0{\cdot}71875t^5 \,. \end{aligned}$$

The interpolation arc P_4P_5 remains last. Since $K_4 = 1$, $K_5 = 0$, we apply formula (5.4) to the supporting elements P_3, P_4, $\mathbf{v}_4$, P_5, P_4 and obtain the following parametric equations of the arc P_4P_5:

$$P_4P_5 \ldots \quad \begin{aligned} P^{(4)}_{x_1}(t) &= 6{\cdot}07813 + 6{\cdot}92188t - 0{\cdot}15625t^2 - 3{\cdot}84375t^3 + \\ &\quad + 0{\cdot}07813t^4 + 0{\cdot}92188t^5 \,, \\ P^{(4)}_{x_2}(t) &= -1{\cdot}73438 + 11{\cdot}98437t - 0{\cdot}78125t^2 - 5{\cdot}96875t^2 + \\ &\quad + 0{\cdot}01563t^4 + 1{\cdot}48438t^5 \,. \end{aligned}$$

The interpolation curve is shown in Fig. 6.

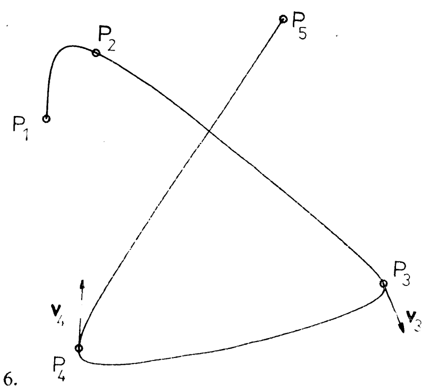

Fig. 6.

Example 4. In the space $\mathbf{R}^3$ let the points $P_1 = (0, 0, 0)$, $P_2 = (10, 5, 5)$, $P_3 = (0, 10, 15)$, $P_4 = (-5, 3, 8)$ be given. At the points P_1, P_3 let us consider the supporting tangent vectors $\mathbf{v}_1 = (4, 0, 0)$, $\mathbf{v}_3 = (-2, -2, 2)$. Let us construct the spatial closed interpolation curve $P_1P_2P_3P_4P_1$ which takes into account the given supporting elements (see Example 4 in [3]).

We start with the arc P_1P_2. Since $K_1 = 1$, $K_2 = 0$, we apply formula (5.4) to the supporting elements P_4, P_1, $\mathbf{v}_1$, P_2, P_3, which leads to the following parametric equations of the arc P_1P_2:

$$
\begin{aligned}
P^{(1)}_{x_1}(t) &= 6{\cdot}01563 + 7{\cdot}23438t - 1{\cdot}03125t^2 - 2{\cdot}96875t^3 + \\
&\quad + 0{\cdot}01562t^4 + 0{\cdot}73437t^5\,, \\
P_1P_2 \ldots P^{(1)}_{x_2}(t) &= 1{\cdot}84375 + 3{\cdot}46875t + 0{\cdot}6875t^2 - 1{\cdot}3125t^3 - \\
&\quad - 0{\cdot}03125t^4 + 0{\cdot}34375t^5\,, \\
P^{(1)}_{x_3}(t) &= 1{\cdot}60937 + 2{\cdot}92188t + 0{\cdot}84375t^2 - 0{\cdot}53125t^3 + \\
&\quad + 0{\cdot}04688t^4 + 0{\cdot}10938t^5\,.
\end{aligned}
\tag{5.11}
$$

We continue with the arc P_2P_3. Since $K_2 = 0$, $K_3 = 1$, we apply formula (5.8) to the supporting elements P_1, P_2, P_3, $\mathbf{v}_3$, P_4 and obtain the following parametric equations of the arc P_2P_3:

$$
\begin{aligned}
P^{(2)}_{x_1}(t) &= 5{\cdot}39062 - 8{\cdot}10938t - 0{\cdot}28125t^2 + 4{\cdot}21876t^3 - \\
&\quad - 0{\cdot}10937t^4 - 1{\cdot}10938t^5\,, \\
P_2P_3 \ldots P^{(2)}_{x_2}(t) &= 8{\cdot}71875 + 4{\cdot}28125t - 1{\cdot}3125t^2 - 2{\cdot}4375t^3 + \\
&\quad + 0{\cdot}09375t^4 + 0{\cdot}65625t^5\,, \\
P^{(2)}_{x_3}(t) &= 10{\cdot}35938 + 6{\cdot}51562t - 0{\cdot}28125t^2 - 1{\cdot}96875t^3 - \\
&\quad - 0{\cdot}07813t^4 + 0{\cdot}45313t^5\,.
\end{aligned}
$$

We continue with the arc P_3P_4. Since $K_3 = 1$, $K_4 = 0$, we apply formula (5.4) to the supporting elements P_2, P_3, $\mathbf{v}_3$, P_4, P_1, which leads to the following parametric equations of the arc P_3P_4:

$$
\begin{aligned}
P^{(3)}_{x_1}(t) &= -2{\cdot}89063 - 3{\cdot}73438t + 0{\cdot}28125t^2 + 1{\cdot}71876t^3 + \\
&\quad + 0{\cdot}10938t^4 - 0{\cdot}48438t^5\,, \\
P_3P_4 \ldots P^{(3)}_{x_2}(t) &= 6{\cdot}53125 - 4{\cdot}34375t + 0{\cdot}0625t^2 + 1{\cdot}0625t^3 - \\
&\quad - 0{\cdot}09375t^4 - 0{\cdot}21875t^5\,, \\
P^{(3)}_{x_3}(t) &= 13{\cdot}01562 - 5{\cdot}54688t - 1{\cdot}59375t^2 + 2{\cdot}78126t^3 + \\
&\quad + 0{\cdot}07813t^4 - 0{\cdot}73438t^5\,.
\end{aligned}
$$

The arc P_4P_1 remains. Since $K_4 = 0$, $K_1 = 1$, we apply formula (5.8) to the supporting elements P_3, P_4, P_1, $\mathbf{v}_1$, P_2 and obtain the following parametric equations of the arc P_4P_1:

$$
\begin{aligned}
P^{(4)}_{x_1}(t) &= -3{\cdot}51562 + 2{\cdot}85937t + 1{\cdot}03125t^2 - 0{\cdot}46875t^3 - \\
&\quad - 0{\cdot}001563t^4 + 0{\cdot}10938t^5\,,
\end{aligned}
$$

$$P_4P_1 \ldots P^{(4)}_{x_2}(t) = \quad 0{\cdot}90625 - 1{\cdot}65625t + 0{\cdot}5625t^2 + 0{\cdot}1875t^3 + \\ + 0{\cdot}03125t^4 - 0{\cdot}03125t^5 ,$$

$$P^{(4)}_{x_3}(t) = \quad 3{\cdot}01563 - 5{\cdot}64063t + 1{\cdot}03125t^2 + 2{\cdot}21876t^3 - \\ - 0{\cdot}04688t^4 - 0{\cdot}57813t^5 .$$

The interpolation curve is shown in axonometric projection in Fig. 7. For the sake of simplicity, the symbol P_i is also used here to denote the axonometric projection of a node while the symbol P'_i denotes its axonometric first projection. The similar holds for supporting vectors.

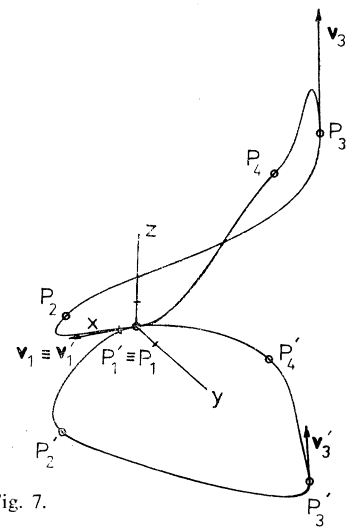

Fig. 7.

Applying formulas (5.4), (5.8) we easily verify that we have

$$(5.13) \qquad 4P''^{(1)}_{x_j}(-1) = 4P''^{(4)}_{x_j}(1) = (x_j^{(2)} - x_j^{(1)}) + (x_j^{(4)} - x_j^{(1)})$$

($j = 1, 2, 3$). Consequently, these values are equal respectively to the numbers 5, 18, 13. They may be obtained also by applying the polynomials (5.11), (5.12). Then the osculation plane of the interpolation curve at the node P_1 has the equation $13x_2 - - 8x_3 = 0$.

It is easily verified that cases a), b) can computed also according to formula (5.10) if, in case a), we prescribe the supporting tangent vector $v_j^{(i+1)} = (P_{i+2} - P_i)/4$ at the node P_{i+1} and if, in case b), we prescribe the supporting tangent vector $v_j^{(i)} = = (P_{i+1} - P_{i-1})/4$ at the node P_i (see (2.2)). For instance, in case a) we then have

$$(5.14) \qquad \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ v_j^{(i)} \\ x_j^{(i+1)} \\ v_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ v_j^{(i)} \\ x_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}$$

and substitution of (5.14) into (5.10) yields (5.4).

By (5.10) we have $64P_{x_j}^{(i)}(t) = (1, t, t^2, t^3, t^4, t^5) \circ 64\mathbf{A}_{ij}^{\mathrm{T}}$. Hence differentiation yields

$$P_{x_j}^{\prime(i)}(-1) = v_j^{(i)}, \quad P_{x_j}^{\prime(i)}(1) = v_j^{(i+1)}, \tag{5.15}$$

$$4P_{x_j}^{\prime\prime(i)}(1) = (P_{i+2} - P_{i+1}) + (P_i - P_{i+1}) \tag{5.16}$$

(cf. (2.4)). For $i = 4$ and the nodes P_4, $P_5 \equiv P_1$, $P_6 \equiv P_2$ we obtain, for $j = 1, 2, 3$, from (5.16) the result $4P_{x_j}^{\prime\prime(4)}(1) = (P_2 - P_1) + (P_4 - P_1)$, which agrees with (5.13).

Concluding this section we determine the curvature k at a node P_{i+1} which is not a point of inflection of a planar or spatial interpolation curve and at which we have the supporting tangent vector $P_{x_j}^{\prime(i)}(1) = v_j^{(i+1)}$ (see (5.15)).

In the planar case we have $(j = 1, 2)$

$$k^2 = \frac{\begin{vmatrix} P_{x_1}^{\prime(i)}(1) & P_{x_2}^{\prime(i)}(1) \\ P_{x_1}^{\prime\prime(i)}(1) & P_{x_2}^{\prime\prime(i)}(1) \end{vmatrix}^2}{\{[P_{x_1}^{\prime(i)}(1)]^2 + [P_{x_2}^{\prime(i)}(1)]^2\}^3} = \frac{\begin{vmatrix} v_1^{(i+1)} & v_2^{(i+1)} \\ 4P_{x_1}^{\prime\prime(i)}(1) & 4P_{x_2}^{\prime\prime(i)}(1) \end{vmatrix}^2}{16|\mathbf{v}_{i+1}|^6}. \tag{5.17}$$

By (5.16) we have

$$4P_{x_j}^{\prime\prime(i)}(1) = (P_{i+2} - P_{i+1}) + (P_i - P_{i+1}) = \mathbf{w}_{i+1} = w_j^{(i+1)}. \tag{5.18}$$

For the vector $\mathbf{w}_{i+1}$ we construct the perpendicular vector $\mathbf{w}_{i+1}^{\perp} = (w_2^{(i+1)}, -w_1^{(i+1)})$; since we have

$$\begin{vmatrix} w_1^{(i+1)} & w_2^{(i+1)} \\ w_2^{(i+1)} & -w_1^{(i+1)} \end{vmatrix} = -|\mathbf{w}_{i+1}|^2 < 0,$$

the orientation of the ordered pair of vectors $\mathbf{w}_{i+1}$, $\mathbf{w}_{i+1}^{\perp}$ is negative. Then the determinant from relation (5.17) is equal to $v_1^{(i+1)}(4P_{x_2}^{\prime\prime(i)}(1)) - v_2^{(i+1)}(4P_{x_1}^{\prime\prime(i)}(1)) = v_1^{(i+1)}w_2^{(i+1)} - v_2^{(i+1)}w_1^{(i+1)} = \mathbf{v}_{i+1} \cdot \mathbf{w}_{i+1}^{\perp} = |\mathbf{v}_{i+1}| \operatorname{proj}_{\mathbf{v}_{i+1}} \mathbf{w}_{i+1}^{\perp}$, where $\operatorname{proj}_{\mathbf{v}_{i+1}} \cdot \mathbf{w}_{i+1}^{\perp}$ stands for the projection of the vector $\mathbf{w}_{i+1}^{\perp}$ onto the vector $\mathbf{v}_{i+1}$. Thus we have

$$k^2 = \frac{(\operatorname{proj}_{\mathbf{v}_{i+1}} \mathbf{w}_{i+1}^{\perp})^2}{16|\mathbf{v}_{i+1}|^4},$$

i.e.

$$k = \frac{|\operatorname{proj}_{\mathbf{v}_{i+1}} \mathbf{w}_{i+1}^{\perp}|}{4|\mathbf{v}_{i+1}|^2} = \frac{\text{const}}{|\mathbf{v}_{i+1}|^2}. \tag{5.19}$$

From relation (5.19) it is evident that if the length $|\mathbf{v}_{i+1}|$ of the vector $\mathbf{v}_{i+1}$ increases, then the curvature k decreases as $C_1|\mathbf{v}_{i+1}|^{-2}$, $C_1 = \text{const}$. In Fig. 8 (for $i = 1$) we have $P_1 = (0, 0)$, $P_2 = (3, 2)$, $P_3 = (8, 0)$, $\mathbf{v}_2 = (2, 0)$, $\mathbf{w}_2 = (P_3 - P_2) + (P_1 - P_2) = (2, -4)$, $\mathbf{w}_2^{\perp} = (-4, -2)$. Further, $|\mathbf{v}_2| = 2$, $\operatorname{proj}_{\mathbf{v}_2} \mathbf{w}_2^{\perp} = -4$. By (5.19) we thus have $k = 1/4$, i.e. the radius of curvature at the node P_2 of the interpolation curve $P_1P_2P_3$ is $r = 4$. The parametric equations of the individual arcs are:

$$P_1P_2 \ldots \begin{array}{l} P^{(1)}_{x_1}(t) = 1 + 1{\cdot}875t + 0{\cdot}5t^2 - 0{\cdot}5t^3 + 0{\cdot}125t^5\,, \\ P^{(1)}_{x_2}(t) = 1 + 1{\cdot}75t - t^3 + 0{\cdot}25t^5\,, \end{array}$$

$$P_2P_3 \ldots \begin{array}{l} P^{(2)}_{x_1}(t) = 6 + 3{\cdot}625t - 0{\cdot}5t^2 - 1{\cdot}5t^3 + 0{\cdot}375t^5\,, \\ P^{(2)}_{x_2}(t) = 1 - 1{\cdot}75t + t^3 - 0{\cdot}25t^5\,. \end{array}$$

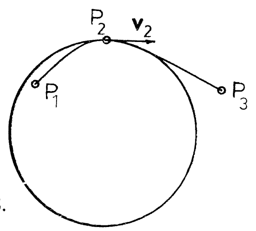

Fig. 8.

In the spatial case ($j = 1, 2, 3$) we have, keeping the notation (see (5.18))

$$k^2 = \frac{\begin{vmatrix} \mathbf{v}_{i+1} \cdot \mathbf{v}_{i+1} & \mathbf{v}_{i+1} \cdot \mathbf{w}_{i+1} \\ \mathbf{v}_{i+1} \cdot \mathbf{w}_{i+1} & \mathbf{w}_{i+1} \cdot \mathbf{w}_{i+1} \end{vmatrix}}{16|\mathbf{v}_{i+1}|^6} = \frac{|\mathbf{v}_{i+1}|^2\,|\mathbf{w}_{i+1}|^2 - \left(|\mathbf{v}_{i+1}|\,\mathrm{proj}_{\mathbf{v}_{i+1}}\,\mathbf{w}_{i+1}\right)^2}{16|\mathbf{v}_{i+1}|^6} =$$

$$= \frac{|\mathbf{w}_{i+1}|^2 - (\mathrm{proj}_{\mathbf{v}_{i+1}}\,\mathbf{w}_{i+1})^2}{16|\mathbf{v}_{i+1}|^4} = \frac{|\mathbf{w}_{i+1}|^2 \sin^2(\mathbf{v}_{i+1}, \mathbf{w}_{i+1})}{16|\mathbf{v}_{i+1}|^4}\,,$$

i.e.

$$k = \frac{|\mathbf{w}_{i+1}| \sin(\mathbf{v}_{i+1}, \mathbf{w}_{i+1})}{4|\mathbf{v}_{i+1}|^2} = \frac{\text{const}}{|\mathbf{v}_{i+1}|^2}\,. \tag{5.20}$$

From (5.20) it is obvious that if the length $|\mathbf{v}_{i+1}|$ of the vector $\mathbf{v}_{i+1}$ increases, then the curvature k is decreasing as $C_2|\mathbf{v}_{i+1}|^{-2}$, $C_2 = \text{const}$.

Substituting into formula (5.20) we verify that the interpolation curve of Example 4 has at the node P_1 radius of curvature $r = 4{\cdot}19278$.

6. CASES WITH PRESCRIBED TANGENT VECTORS (*method* $\tilde{\mathscr{T}}$)

We shall again examine the cases in which supporting tangent vectors are prescribed at some nodes (or at all nodes as the case may be). While under method $\mathscr{T}$ the mutual distances of the nodes are not taken into account, under method $\tilde{\mathscr{T}}$ these distances have their significance.

a) Let $K_1 = 1, K_{i+1} = 0$. In Fig. 4 the values $-2, 0, 2$ of the variable t are replaced by the values $-2q_{i,-1}/q_{i,0}, 0, 2q_{i,1}/q_{i,0}$, where the quantities $q_{i,-1}, q_{i,0}, q_{i,1}$ have the same meaning as in Section 3. Analogously as in Section 5a) we construct a polynomial of at most third degree $\tilde{S}_{x_j}^{(i)}(t)$ with the aid of which we put $Dx_j^{(i)} = \tilde{S}_{x_j}'^{(i)}(0)$, $D^2x_j^{(i)} = \tilde{S}_{x_j}''^{(i)}(0)$. The detailed computation of matrix (1.12), which can be expressed as the product of the matrix $(x_j^{(i-1)}, x_j^{(i)}, v_j^{(i)}, x_i^{(i+1)}, x_j^{(i+2)})$ and a certain matrix of type (5, 6), will not be performed here.

b) Let $K_i = 0, K_{i+1} = 1$. In Fig .5 the values $-2, 0, 2$ of the variable t are replaced by the values $-2q_{i+1,-1}/q_{i+1,0}, 0, 2q_{i+1,1}/q_{i+1,0}$. Analogously as in Section 5b) we construct a polynomial of at most third degree $\tilde{V}_{x_j}^{(i+1)}(t)$ with the aid of which we put $Dx_j^{(i+1)} = \tilde{V}_{x_j}'^{(i+1)}(0)$, $D^2x_j^{(i+1)} = \tilde{V}_{x_j}''^{(i+1)}(0)$. The detailed computation of the matrix (1.12), which can be expressed as the product of the matrix $(x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}, v_j^{(i+1)}, x_j^{(i+2)})$ and a certain matrix of type (5, 6), will again be omitted.

c) Let $K_i = K_{i+1} = 1$. On the basis of the results of the preceding two cases it is possible to derive that for the transposed matrix to the matrix (1.12) we have

$$(6.1)\quad \boldsymbol{X}_{ij}^{T} = \frac{1}{8}\begin{bmatrix} 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 \\ a_i & -(a_i + b_i) & 4c_i & b_i & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & a_{i+1} & 0 & -(a_{i+1} + b_{i+1}) & 4c_{i+1} & b_{i+1} \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ v_j^{(i)} \\ x_j^{(i+1)} \\ v_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix},$$

where $a_i = r_i + r_i^2$, $b_i = r_i^{-1} + r_i^{-2}$, $c_i = r_i - r_i^{-1}$, $r_i = q_{i,1}/q_{i,-1}$. If we substitute (6.1) into (1.13), where $\boldsymbol{A}^{-1}$ is the matrix (1.22), we obtain

$$(6.2)\quad 128\boldsymbol{A}_{ij}^{T} = \begin{bmatrix} a_i & 64 - a_i - b_i + a_{i+1} & 40 + 4c_i & 64 + b_i - a_{i+1} - b_{i+1} & -40 + 4c_{i+1} & b_{i+1} \\ -a_i & -120 + a_i + b_i + a_{i+1} & -56 - 4c_i & 120 - b_i - a_{i+1} - b_{i+1} & -56 + 4c_{i+1} & b_{i+1} \\ -2a_i & 2a_i + 2b_i - 2a_{i+1} & -48 - 8c_i & -2b_i + 2a_{i+1} + 2b_{i+1} & 48 - 8c_{i+1} & -2b_{i+1} \\ 2a_i & 80 - 2a_i - 2b_i - 2a_{i+1} & 80 + 8c_i & -80 + 2b_i + 2a_{i+1} + 2b_{i+1} & 80 - 8c_{i+1} & -2b_{i+1} \\ a_i & -a_i - b_i + a_{i+1} & 8 + 4c_i & b_i - a_{i+1} - b_{i+1} & -8 + 4c_{i+1} & b_{i+1} \\ -a_i & -24 + a_i + b_i + a_{i+1} & -24 - 4c_i & 24 - b_i - a_{i+1} - b_{i+1} & -24 + 4c_{i+1} & b_{i+1} \end{bmatrix} \circ \begin{bmatrix} x_j^{(i-1)} \\ x_j^{(i)} \\ v_j^{(i)} \\ x_j^{(i+1)} \\ v_j^{(i+1)} \\ x_j^{(i+2)} \end{bmatrix}.$$

Similarly as in Section 5 it is possible to show that cases a), b) can be computed also with the aid of formula (6.2) if, in case a), we prescribe the supporting tangent vector

$$(6.3)\qquad v_j^{(i+1)} = \{r_{i+1}^{-1}(P_{i+2} - P_{i+1}) - r_{i+1}(P_i - P_{i-1})\}/4$$

at the node P_{i+1} (see (4.1)) and if, in case b), we prescribe the supporting tangent vector $v_j^{(i)} = \{r_i^{-1}(P_{i+1} - P_i) - r_i(P_{i-1} - P_i)\}/4$ at the node P_i.

Example 5. Applying method $\tilde{\mathscr{T}}$ we now construct the spatial closed interpolation curve $P_1P_2P_3P_4P_1$ of Example 4 which takes into account the tangent vectors $\mathbf{v}_1, \mathbf{v}_3$.

We start with the arc P_1P_2. Since $K_2 = 0$, we prescribe the supporting tangent vector $\mathbf{v}_2 = (1{\cdot}0206;\ 2{\cdot}55155;\ 3{\cdot}57218)$ at the node P_2 and apply formula (6.2) to the supporting elements $P_4, P_1, \mathbf{v}_1, P_2, \mathbf{v}_2, P_3$:

$$(6.4)\qquad \begin{aligned} P^{(1)}_{x_1}(t) &= 5{\cdot}67502 + 6{\cdot}80309t - 0{\cdot}60519t^2 - 2{\cdot}36132t^3 - \\ &\quad - 0{\cdot}06983t^4 + 0{\cdot}55823t^5\,, \\ P_1P_2 \ldots P^{(1)}_{x_2}(t) &= 1{\cdot}80866 + 3{\cdot}43329t + 0{\cdot}7448t^2 - 1{\cdot}25446t^3 - \\ &\quad - 0{\cdot}05346t^4 + 0{\cdot}32117t^5\,, \\ P^{(1)}_{x_3}(t) &= 1{\cdot}66878 + 2{\cdot}94959t + 0{\cdot}76939t^2 - 0{\cdot}54225t^3 + \\ &\quad + 0{\cdot}06183t^4 + 0{\cdot}09266t^5\,. \end{aligned}$$

We continue with the arc P_2P_3, i.e. we apply formula (6.2) to the elements P_1, $P_2, \mathbf{v}_2, P_3, \mathbf{v}_3, P_4$:

$$\begin{aligned} P^{(2)}_{x_1}(t) &= 5{\cdot}64272 - 8{\cdot}61629t - 0{\cdot}53029t^2 + 4{\cdot}97743t^3 - \\ &\quad - 0{\cdot}11243t^4 - 1{\cdot}36114t^5\,, \\ P_2P_3 \ldots P^{(2)}_{x_2}(t) &= 8{\cdot}72049 + 4{\cdot}27623t - 1{\cdot}3031t^2 - 2{\cdot}44034t^3 + \\ &\quad + 0{\cdot}08261t^4 + 0{\cdot}66411t^5\,, \\ P^{(2)}_{x_3}(t) &= 10{\cdot}23352 + 6{\cdot}56937t - 0{\cdot}074t^2 - 2{\cdot}03178t^3 - \\ &\quad - 0{\cdot}15952t^4 + 0{\cdot}46241t^5\,. \end{aligned}$$

We continue with the arc P_3P_4. Since $K_4 = 0$, we prescribe the supporting tangent vector $\mathbf{v}_4 = (0{\cdot}28463;\ -2{\cdot}4023;\ -3{\cdot}80269)$ at the node P_4 and apply formula (6.2) to the supporting elements $P_2, P_3, \mathbf{v}_3, P_4, \mathbf{v}_4, P_1$:

$$\begin{aligned} P^{(3)}_{x_1}(t) &= -3{\cdot}04269 - 3{\cdot}79478t + 0{\cdot}51422t^2 + 1{\cdot}7684t^3 + \\ &\quad + 0{\cdot}02847t^4 - 0{\cdot}47362t^5\,, \\ P_3P_4 \ldots P^{(3)}_{x_2}(t) &= 6{\cdot}49361 - 4{\cdot}39677t + 0{\cdot}11336t^2 + 1{\cdot}14411t^3 - \\ &\quad - 0{\cdot}10697t^4 - 0{\cdot}24734t^5\,, \\ P^{(3)}_{x_3}(t) &= 12{\cdot}97153 - 5{\cdot}49005t - 1{\cdot}49238t^2 + 2{\cdot}68077t^3 + \\ &\quad + 0{\cdot}02085t^4 - 0{\cdot}69072t^5\,. \end{aligned}$$

The arc P_4P_1 remains, thus we apply formula (6.2) to the supporting elements $P_3, P_4, \mathbf{v}_4, P_1, \mathbf{v}_1, P_2$:

$$(6.5)\qquad \begin{aligned} P^{(4)}_{x_1}(t) &= -3{\cdot}44461 + 2{\cdot}7159t + 0{\cdot}96039t^2 - 0{\cdot}25296t^3 - \\ &\quad - 0{\cdot}01578t^4 + 0{\cdot}03706t^5\,, \\ P_4P_1 \ldots P^{(4)}_{x_2}(t) &= 0{\cdot}92505 - 1{\cdot}69334t + 0{\cdot}54932t^2 + 0{\cdot}23725t^3 + \\ &\quad + 0{\cdot}02563t^4 - 0{\cdot}04391t^5\,, \\ P^{(4)}_{x_3}(t) &= 3{\cdot}01272 - 5{\cdot}57722t + 1{\cdot}02388t^2 + 2{\cdot}10512t^3 - \\ &\quad - 0{\cdot}0366t^4 - 0{\cdot}5279t^5\,. \end{aligned}$$

The interpolation curve is drawn in axonometric projection in Fig. 9.

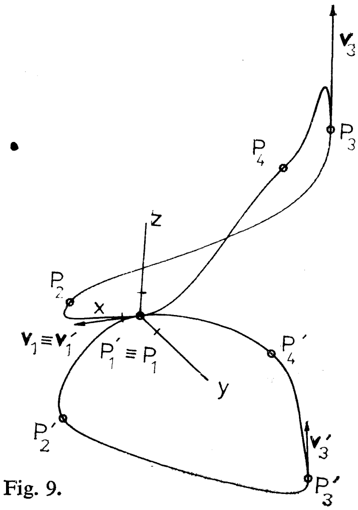

Fig. 9.

By (6.2) we have $128P_{x_j}^{(i)}(t) = (1, t, t^2, t^3, t^4, t^5) \circ 128\mathbf{A}_{ij}^{\mathrm{T}}$. Hence differentiation yields

$$P_{x_j}^{\prime(i)}(-1) = v_j^{(i)}, \quad P_{x_j}^{\prime(i)}(1) = v_j^{(i+1)}, \tag{6.6}$$

$$8P_{x_j}^{\prime\prime(i)}(1) = b_{i+1}(P_{i+2} - P_{i+1}) + a_{i+1}(P_i - P_{i+1}) + 4c_{i+1}v_j^{(i+1)}. \tag{6.7}$$

If we substitute the vector (6.3) into relation (6.7), then a simple computation yields (4.2). For $i = 4$ and the nodes $P_4, P_5 \equiv P_1, P_6 \equiv P_2$ (see Example 5) we obtain from (6.7), for $j = 1, 2, 3$, the result $8P_{x_j}^{\prime\prime(4)}(1) = b_1(P_2 - P_1) + a_1(P_4 - P_1) + 4c_1v_j^{(1)} = 8P_{x_j}^{\prime\prime(1)}(-1)$. With accuracy to two decimal places these values are equal to the numbers 7·64; 15·61; 29·45, respectively. These values can be obtained also by the application of the polynomial (6.4), (6.5). Then the osculation plane of the interpolation curve $P_1P_2P_3P_4P_1$ at node P_1 has the equation $2945x_2 - 1561x_3 = 0$.

Concluding this section we determine the curvature k at a node P_{i+1} which is not an inflection point of a planar or spatial interpolation curve and at which we have the supporting tangent vector $P_{x_j}^{\prime(i)}(1) = v_j^{(i+1)}$ (see (6.6)).

In the planar case ($j = 1, 2$) we have (cf. (5.17))

$$k^2 = \frac{\begin{vmatrix} v_1^{(i+1)} & v_2^{(i+1)} \\ 8P_{x_1}^{\prime\prime(i)}(1) & 8P_{x_2}^{\prime\prime(i)}(1) \end{vmatrix}^2}{64|\mathbf{v}_{i+1}|^6}. \tag{6.8}$$

By (6.7) we have

$$8P_{x_j}^{\prime\prime(i)}(1) = {}^{\sim}\mathbf{w}_{i+1} + 4c_{i+1}\mathbf{v}_{i+1}, \tag{6.9}$$

where $^{\sim}\mathbf{w}_{i+1} = b_{i+1}(P_{i+2} - P_{i+1}) + a_{i+1}(P_i - P_{i+1})$. For the vectors $^{\sim}\mathbf{w}_{i+1} = \tilde{w}_j^{(i+1)}$, $\mathbf{v}_{i+1} = v_j^{(i+1)}$ we construct the perpendicular vectors $^{\sim}\mathbf{w}_{i+1}^{\perp} = (\tilde{w}_2^{(i+1)}, -\tilde{w}_1^{(i+1)})$, $\mathbf{v}_{i+1}^{\perp} = (v_2^{(i+1)}, -v_1^{(i+1)})$. Then the determinant of relation (6.8) is equal to $v_1^{(i+1)}(\tilde{w}_2^{(i+1)} + 4c_{i+1}v_2^{(i+1)}) - v_2^{(i+1)}(\tilde{w}_1^{(i+1)} + 4c_{i+1}v_1^{(i+1)}) = \mathbf{v}_{i+1} \,.\, (^{\sim}\mathbf{w}_{i+1}^{\perp} + 4c_{i+1}\mathbf{v}_{i+1}^{\perp}) = \mathbf{v}_{i+1} \,.\, ^{\sim}\mathbf{w}_{i+1}^{\perp} = |\mathbf{v}_{i+1}| \operatorname{proj}_{\mathbf{v}_{i+1}} {^{\sim}\mathbf{w}_{i+1}^{\perp}}$, where $\operatorname{proj}_{\mathbf{v}_{i+1}} {^{\sim}\mathbf{w}_{i+1}^{\perp}}$ denotes the projection of the vector $^{\sim}\mathbf{w}_{i+1}^{\perp}$ onto the vector $\mathbf{v}_{i+1}$. Thus

$$k^2 = \frac{(\operatorname{proj}_{\mathbf{v}_{i+1}} {^{\sim}\mathbf{w}_{i+1}^{\perp}})^2}{64|\mathbf{v}_{i+1}|^4},$$

i.e.

$$(6.10) \qquad k = \frac{|\operatorname{proj}_{\mathbf{v}_{i+1}} {^{\sim}\mathbf{w}_{i+1}^{\perp}}|}{8|\mathbf{v}_{i+1}|^2} = \frac{\text{const}}{|\mathbf{v}_{i+1}|^2}.$$

From relation (6.10) it is obvious that if the length $|\mathbf{v}_{i+1}|$ of the vector $\mathbf{v}_{i+1}$ increases, then the curvature k decreases as $C_3|\mathbf{v}_{i+1}|^{-2}$, $C_3 = \text{const}$. For the interpolation curve $P_1P_2P_3$, where P_1, P_2, P_3 are the nodes from Fig. 8, we have for $i = 1$: $^{\sim}\mathbf{w}_2 = b_2(P_3 - P_2) + a_2(P_1 - P_2) = 1{\cdot}1178 \,.\, (5, -2) + 3{\cdot}72436 \,.\, (-3, -2) = (-5{\cdot}58408;\ -9{\cdot}68432)$, $^{\sim}\mathbf{w}_2^{\perp} = (-9{\cdot}68432;\ 5{\cdot}58408)$. Further, $|\mathbf{v}_2| = 2$, $\operatorname{proj}_{\mathbf{v}_2} \,.\, {^{\sim}\mathbf{w}_2^{\perp}} = -9{\cdot}68432$. Thus, by (6.9), we have $k = 0{\cdot}30264$, i.e. the radius of curvature at the node P_2 of the interpolation curve $P_1P_2P_3$ is $r = 3{\cdot}30431$. The parametric equations of the individual arcs are as follows:

$$P_1P_2 \ldots \quad \begin{aligned} P_{x_1}^{(1)}(t) &= 0{\cdot}97663 + 1{\cdot}85163t + 0{\cdot}54674t^2 - 0{\cdot}45326t^3 - \\ &\quad - 0{\cdot}02337t^4 + 0{\cdot}10163t^5, \\ P_{x_2}^{(1)}(t) &= 0{\cdot}98684 + 1{\cdot}73684t + 0{\cdot}02632t^2 - 0{\cdot}97368t^3 - \\ &\quad - 0{\cdot}01316t^4 + 0{\cdot}23684t^5, \end{aligned}$$

$$P_2P_3 \ldots \quad \begin{aligned} P_{x_1}^{(2)}(t) &= 5{\cdot}97663 + 3{\cdot}64837t - 0{\cdot}45326t^2 - 1{\cdot}54675t^3 - \\ &\quad - 0{\cdot}02337t^4 + 0{\cdot}39838t^5, \\ P_{x_2}^{(2)}(t) &= 0{\cdot}98684 - 1{\cdot}73684t + 0{\cdot}02632t^2 + 0{\cdot}97368t^3 - \\ &\quad - 0{\cdot}01316t^4 - 0{\cdot}23684t^5. \end{aligned}$$

In the spatial case ($j = 1, 2, 3$) we have, under the same notation (see (6.9))

$$k^2 = \frac{\begin{vmatrix} \mathbf{v}_{i+1} \,.\, \mathbf{v}_{i+1} & \mathbf{v}_{i+1} \,.\, (^{\sim}\mathbf{w}_{i+1} + 4c_{i+1}\mathbf{v}_{i+1}) \\ \mathbf{v}_{i+1} \,.\, (^{\sim}\mathbf{w}_{i+1} + 4c_{i+1}\mathbf{v}_{i+1}) & (^{\sim}\mathbf{w}_{i+1} + 4c_{i+1}\mathbf{v}_{i+1}) \,.\, (^{\sim}\mathbf{w}_{i+1} + 4c_{i+1}\mathbf{v}_{i+1}) \end{vmatrix}}{64|\mathbf{v}_{i+1}|^6},$$

i.e. (proceeding more rapidly now),

$$(6.11) \qquad k = \frac{|^{\sim}\mathbf{w}_{i+1}| \sin(\mathbf{v}_{i+1}, {^{\sim}\mathbf{w}_{i+1}})}{8|\mathbf{v}_{i+1}|^2} = \frac{\text{const}}{|\mathbf{v}_{i+1}|^2}.$$

From formula (6.11) it follows similarly as in the preceding cases (see (5.19), (5.20), (6.10)) that if the length $|\mathbf{v}_{i+1}|$ of the vector $\mathbf{v}_{i+1}$ increases, then the curvature k decreases as $C_4|\mathbf{v}_{i+1}|^{-2}$, $C_4 = \text{const.}$

Substituting into formula (6.11) we verify that the interpolation curve of Example 5 has radius of curvature $r = 3{\cdot}84015$ at the node P_1.

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Souhrn

KONSTRUKCE INTERPOLAČNÍCH KŘIVEK Z DANÝCH OPĚRNÝCH ELEMENTŮ (II)

JOSEF MATUŠŮ, JOSEF NOVÁK

Předmětem článku jsou konstrukce interpolačních křivek procházejících danými opěrnými body a dotýkajících se opěrných tečných vektorů v některých z těchto bodů, popř. ve všech opěrných bodech. Matematickým jádrem těchto konstrukcí je Lienhardova interpolační metoda. Jsou odvozeny vzorce pro křivost rovinných a prostorových interpolačních křivek.

Резюме

КОНСТРУКЦИИ ИНТЕРПОЛЯЦИОННЫХ КРИВЫХ ИЗ ДАННЫХ ОПОРНЫХ ЭЛЕМЕНТОВ (II)

Темой статьи являются конструкции интерполяционных кривых, проходящих через данные опорные точки и соприкасающихся опорных касательных векторов в некоторых из этих точек или во всех точках. Математической сущностью этих конструкций является интерполяционный метод Лингарда. Выведены также формулы для кривизны плоских и пространственных кривых.

Author's addresses: Prof. RNDr. *Josef Matušů*, DrSc., ČVUT, Karlovo nám. 13, 121 35 Praha 2, Doc. RNDr. *Josef Novák*, CSc., ČVUT, Horská 3, 128 03 Praha 2.