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MULTISTAGE REGRESSION MODEL

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Summary. Necessary and sufficient conditions are given under which the best linear unbiased estimator (BLUE) $\hat{\beta}_i(\mathbf{Y}_1, \dots, \mathbf{Y}_i)$ is identical with the BLUE $\hat{\beta}_i(\hat{\beta}_1, \dots, \hat{\beta}_{i-1}, \mathbf{Y}_i)$; $\mathbf{Y}_1, \dots, \mathbf{Y}_i$ are subvectors of the random vector \mathbf{Y} in a general regression model $(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma})$, $(\beta_1', \dots, \beta_i')' = \boldsymbol{\beta}$ a vector of unknown parameters; the design matrix \mathbf{X} having a special so called multistage structure and the covariance matrix $\boldsymbol{\Sigma}$ are given.

Keywords: regression model, mixed linear model.

AMS classification: 62J05.

1. INTRODUCTION

The following special structure of the general linear model $(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma})$ is frequently encountered in practice:

$$(*) \left\{ \begin{array}{l} \mathbf{Y} = \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \\ \mathbf{Y}_3 \\ \vdots \\ \mathbf{Y}_p \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} \mathbf{X}_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{C}_{2,1} & \mathbf{X}_2 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{C}_{3,1} & \mathbf{C}_{3,2} & \mathbf{X}_3 & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{C}_{p,1} & \mathbf{C}_{p,2} & \mathbf{C}_{p,3} & \dots & \mathbf{X}_p \end{pmatrix} \\ \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{1,1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_{2,2} & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & \boldsymbol{\Sigma}_{p,p} \end{pmatrix}, \end{array} \right.$$

where \mathbf{Y}_i , $i = 1, 2, \dots, p$, are column vectors of measurements, $\boldsymbol{\beta}_i$ vectors of parameters, \mathbf{X}_i , $\mathbf{C}_{i,j}$, $\boldsymbol{\Sigma}_{ii}$ matrices of the corresponding dimensions.

As an example of a real situation producing this structure we may consider the following problem: Suppose that it is required to determine the value of two sets of etalons, say, $\boldsymbol{\beta}_1 = (\beta_1^{(1)}, \dots, \beta_4^{(1)})'$ and $\boldsymbol{\beta}_2 = (\beta_1^{(2)}, \dots, \beta_4^{(2)})'$ and that the scheme of practically feasible measurements is described by the oriented graph in Fig. 1. In

this graph the vertices represent the etalons and the oriented edges the measurements of differences between the etalons at the end and the beginning of the arrows; only the value of the etalon E is given. The measurement in the framework of the first stage is stochastically independent of the measurement performed at the second stage. If $\mathcal{M}(\mathbf{C}'_{2,1})$ means the column space of the matrix $\mathbf{C}'_{2,1}$, then by inspection of Fig. 1 it can be easily seen that $\mathcal{M}(\mathbf{C}'_{2,1}) \subset \mathcal{M}(\mathbf{X}'_1)$. (On the importance of the

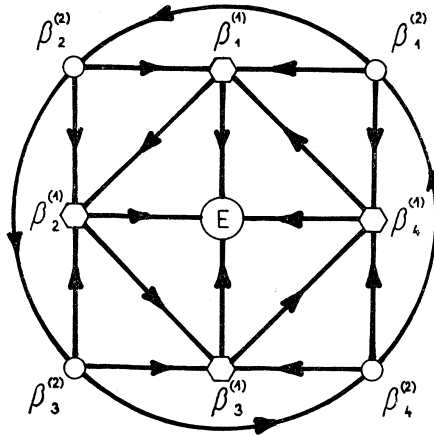


Fig. 1.

condition $\mathcal{M}(\mathbf{C}_{2,1}) \subset \mathcal{M}(\mathbf{X}_2)$ in the two-stage model see [5].) Another example see in [1].

The following best linear unbiased estimators (BLUES) of the values β_1, \dots, β_p in the multistage model can be considered: BLUE $\hat{\beta}_i(\mathbf{Y}_1, \dots, \mathbf{Y}_i)$ of β_i based on the realization of the vectors $\mathbf{Y}_1, \dots, \mathbf{Y}_i$, $i = 1, \dots, p$; BLUE $\hat{\beta}_i(\hat{\beta}_1, \hat{\beta}_2, \dots, \hat{\beta}_{i-1}, \mathbf{Y}_i)$ of β_i based on the realization of estimators $\hat{\beta}_1, \dots, \hat{\beta}_{i-1}$ and on the realization of \mathbf{Y}_i , $i = 2, \dots, p$, and BLUE $\hat{\beta}_i(\mathbf{Y}_i, \dots, \mathbf{Y}_p)$ of β_i based on the realization of the measurement at all stages. This last estimator, however, is not used in practice because the results of the measurement at the i -th stage (the realization of the vector \mathbf{Y}_i) must not have any influence on the values $\hat{\beta}_1, \hat{\beta}_2, \dots, \hat{\beta}_{i-1}$.

The matrix

$$\text{Var} [\hat{\beta}_i(\hat{\beta}_1, \dots, \hat{\beta}_{i-1}, \mathbf{Y}_i)] - \text{Var} [\hat{\beta}_i(\mathbf{Y}_1, \dots, \mathbf{Y}_i)],$$

where $\text{Var} [\hat{\beta}_i(\hat{\beta}_1, \dots, \hat{\beta}_{i-1}, \mathbf{Y}_i)]$, $\text{Var} [\hat{\beta}_i(\mathbf{Y}_1, \dots, \mathbf{Y}_i)]$ are the covariance matrices of the estimators $\hat{\beta}_i(\hat{\beta}_1, \dots, \hat{\beta}_{i-1}, \mathbf{Y}_i)$ and $\hat{\beta}_i(\mathbf{Y}_1, \dots, \mathbf{Y}_i)$, is obviously positive semidefinite (p.s.d.). However, the estimator $\hat{\beta}_i(\hat{\beta}_1, \dots, \hat{\beta}_{i-1}, \mathbf{Y}_i)$ requires substantially less calculations than the estimator $\hat{\beta}_i(\mathbf{Y}_1, \dots, \mathbf{Y}_i)$ and that is why only the former is used in practice.

If the equality $\hat{\beta}_i(\hat{\beta}_1, \dots, \hat{\beta}_{i-1}, \mathbf{Y}_i) = \hat{\beta}_i(\mathbf{Y}_1, \dots, \mathbf{Y}_i)$ is valid then no information on β_i contained in $\mathbf{Y}_1, \dots, \mathbf{Y}_i$ is lost but a substantial saving of calculation is gained.

The purpose of this paper is to obtain conditions for the matrices $\mathbf{C}_{i,j}$ under which such a situation occurs.

2. DEFINITION AND AUXILIARY STATEMENTS

Definition. A regression model $(\mathbf{Y}, \mathbf{X}\beta, \Sigma)$ is called a p -stage model, when it has the structure (*) and fulfils the condition: $\mathcal{M}(\mathbf{C}_{i,j}) \subset \mathcal{M}(\mathbf{X}'_j)$, $i = 2, \dots, p$, $j = 1, \dots, i - 1$.

Lemma 2.1. Let matrices $\mathbf{X}_1, \mathbf{X}_2, \Sigma_{1,1}, \mathbf{S}_{2,2}$ be of types $n_1 \times k_1, n_2 \times k_2, n_1 \times n_1, n_2 \times n_2$, respectively, and let the matrix

$$\mathbf{S} = \begin{pmatrix} \Sigma_{1,1} & \mathbf{S}_{1,2} \\ \mathbf{S}_{2,1} & \mathbf{S}_{2,2} \end{pmatrix}$$

be symmetric and p.s.d. Then the minimum \mathbf{S} -seminorm g -inversion of the matrix $\begin{pmatrix} \mathbf{X}'_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{X}'_2 \end{pmatrix}$ is given by the relation

$$\begin{pmatrix} \mathbf{X}'_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{X}'_2 \end{pmatrix}_{m(\mathbf{S})}^- = \begin{pmatrix} \mathbf{B}_{1,1} & \mathbf{B}_{1,2} \\ \mathbf{B}_{2,1} & \mathbf{B}_{2,2} \end{pmatrix}$$

where

$$\mathbf{B}_{1,1} = (\mathbf{X}'_1)_{m(*)}^-; \quad \mathbf{B}_{1,2} = -\{\mathbf{I} - (\mathbf{X}'_1)_{m(\Sigma_{1,1})}^- \mathbf{X}'_1\} \Sigma_{1,1}^- \mathbf{S}_{1,2} (\mathbf{X}'_2)_{m(**)}^-;$$

$$\mathbf{B}_{2,1} = -\{\mathbf{I} - (\mathbf{X}'_2)_{m(\mathbf{S}_{2,2})}^- \mathbf{X}'_2\} \mathbf{S}_{2,2}^- \mathbf{S}_{2,1} (\mathbf{X}'_1)_{m(*)}^-; \quad \mathbf{B}_{2,2} = (\mathbf{X}'_2)_{m(**)}^-;$$

$$(*) = \Sigma_{1,1} - \mathbf{S}_{1,2} \mathbf{S}_{2,2}^- [\mathbf{S}_{2,2} - \mathbf{S}_{2,2} (\mathbf{X}'_2)_{m(\mathbf{S}_{2,2})}^- \mathbf{X}'_2] \mathbf{S}_{2,2}^- \mathbf{S}_{2,1},$$

$$(**) = \mathbf{S}_{2,2} - \mathbf{S}_{2,1} \Sigma_{1,1}^- [\Sigma_{1,1} - \Sigma_{1,1} (\mathbf{X}'_1)_{m(\Sigma_{1,1})}^- \mathbf{X}'_1] \Sigma_{1,1}^- \mathbf{S}_{1,2},$$

and \mathbf{I} is the identity matrix. (The operations with the upper index $-$ and with the lower index $m(\cdot)$ see in [4].)

Proof. See Theorem 3.1 in [2].

Lemma 2.2. In the 2-stage model

$$\left[\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{C}_{2,1} & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_{1,1} & \mathbf{0} \\ \mathbf{0} & \Sigma_{2,2} \end{pmatrix} \right]$$

the minimum Σ -seminorm g -inversion of the matrix

$$\begin{pmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{C}_{2,1} & \mathbf{X}_2 \end{pmatrix}'$$

is

$$\left[\begin{pmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{C}_{2,1} & \mathbf{X}_2 \end{pmatrix}' \right]_{m(\Sigma)}^- = \begin{pmatrix} \mathbf{D}_{1,1} & \mathbf{D}_{1,2} \\ \mathbf{D}_{2,1} & \mathbf{D}_{2,2} \end{pmatrix}',$$

where

$$\begin{aligned} \mathbf{D}_{1,1} &= [(\mathbf{X}'_1)_{m(*)}^-]' + [(\mathbf{X}'_1)_{m(*)}^-]' \mathbf{S}_{1,2} \mathbf{S}_{2,2}^- \{ \mathbf{I} - \mathbf{X}_2 [(\mathbf{X}'_2)_{m(\mathbf{S}_{2,2})}^-]' \} \mathbf{C}_{2,1} [(\mathbf{X}'_1)_{m(\Sigma_{1,1})}^-]', \\ \mathbf{D}_{1,2} &= - [(\mathbf{X}'_1)_{m(*)}^-]' \mathbf{S}_{1,2} \mathbf{S}_{2,2}^- \{ \mathbf{I} - \mathbf{X}_2 [(\mathbf{X}'_2)_{m(\mathbf{S}_{2,2})}^-]' \}, \\ \mathbf{D}_{2,1} &= - [(\mathbf{X}'_2)_{m(\mathbf{S}_{2,2})}^-]' \mathbf{C}_{2,1} [(\mathbf{X}'_1)_{m(\Sigma_{1,1})}^-]' - \\ &\quad - [(\mathbf{X}'_2)_{m(\mathbf{S}_{2,2})}^-]' \mathbf{S}_{2,1} \Sigma_{1,1}^- \{ \mathbf{I} - \mathbf{X}_1 [(\mathbf{X}'_1)_{m(\Sigma_{1,1})}^-]' \}, \\ \mathbf{D}_{2,2} &= [(\mathbf{X}'_2)_{m(\mathbf{S}_{2,2})}^-]', \\ (*) &= \Sigma_{1,1} - \mathbf{S}_{1,2} \mathbf{S}_{2,2}^- [\mathbf{S}_{2,2} - \mathbf{S}_{2,2} (\mathbf{X}'_2)_{m(\mathbf{S}_{2,2})}^- \mathbf{X}'_2] \mathbf{S}_{2,2}^- \mathbf{S}_{2,1}, \\ \mathbf{S}_{1,2} &= -\Sigma_{1,1} (\mathbf{X}'_1)_{m(\Sigma_{1,1})}^- \mathbf{C}'_{2,1} = \mathbf{S}'_{2,1}, \\ \mathbf{S}_{2,2} &= \Sigma_{2,2} + \mathbf{C}_{2,1} [(\mathbf{X}'_1)_{m(\Sigma_{1,1})}^-]' \Sigma_{1,1} (\mathbf{X}'_1)_{m(\Sigma_{1,1})}^- \mathbf{C}'_{2,1}. \end{aligned}$$

Proof. For an arbitrary $m \times n$ matrix \mathbf{A} , a p.s.d. $n \times n$ symmetric matrix \mathbf{N} and a regular $n \times n$ matrix \mathbf{R} , the minimum \mathbf{N} -seminorm g -inversion of the matrix \mathbf{A} can be expressed as follows: $\mathbf{A}_{m(\mathbf{N})}^- = \mathbf{R}(\mathbf{A}\mathbf{R})_{m(\mathbf{R}'\mathbf{N}\mathbf{R})}^-$. If

$$\mathbf{A} = \begin{pmatrix} \mathbf{X}'_1 & \mathbf{C}'_{2,1} \\ \mathbf{0} & \mathbf{X}'_2 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} \mathbf{I} & -(\mathbf{X}'_1)_{m(\Sigma_{1,1})}^- \mathbf{C}'_{2,1} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \quad \mathbf{N} = \Sigma,$$

then

$$\begin{aligned} \left(\begin{pmatrix} \mathbf{X}'_1 & \mathbf{C}'_{2,1} \\ \mathbf{0} & \mathbf{X}'_2 \end{pmatrix}' \right)_{m(\Sigma)}^- &= \begin{pmatrix} \mathbf{I} & -(\mathbf{X}'_1)_{m(\Sigma_{1,1})}^- \mathbf{C}'_{2,1} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \left(\begin{pmatrix} \mathbf{X}'_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{X}'_2 \end{pmatrix}' \right)_{m(\mathbf{S})}^- = \\ &= \begin{pmatrix} \mathbf{I} & -(\mathbf{X}'_1)_{m(\Sigma_{1,1})}^- \mathbf{C}'_{2,1} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{B}_{1,1} & \mathbf{B}_{1,2} \\ \mathbf{B}_{2,1} & \mathbf{B}_{2,2} \end{pmatrix}, \end{aligned}$$

where

$$\mathbf{S} = \begin{pmatrix} \Sigma_{1,1} & \mathbf{S}_{1,2} \\ \mathbf{S}_{2,1} & \Sigma_{2,2} \end{pmatrix}$$

and $\mathbf{B}_{i,j}$, $i, j = 1, 2$ are the matrices from Lemma 2.1. The matrix (***) from Lemma 2.1 in this case is

$$\begin{aligned} (***) &= \mathbf{S}_{2,2} - \mathbf{C}_{2,1} [(\mathbf{X}'_1)_{m(\Sigma_{1,1})}^-]' \cdot \\ &\quad \cdot \Sigma_{1,1} \Sigma_{1,1}^- \{ \Sigma_{1,1} - \Sigma_{1,1} (\mathbf{X}'_1)_{m(\Sigma_{1,1})}^- \mathbf{X}'_1 \} \Sigma_{1,1}^- \Sigma_{1,1} (\mathbf{X}'_1)_{m(\Sigma_{1,1})}^- \mathbf{C}'_{2,1}. \end{aligned}$$

As $\mathcal{M}(\mathbf{C}'_{2,1}) \subset \mathcal{M}(\mathbf{X}'_1) \Leftrightarrow \exists \{ \mathbf{E}: \mathbf{X}'_1 \mathbf{E} = \mathbf{C}'_{2,1} \}$ and $\Sigma_{1,1} (\mathbf{X}'_1)_{m(\Sigma_{1,1})}^- \mathbf{X}'_1 = \mathbf{X}'_1 [(\mathbf{X}'_1)_{m(\Sigma_{1,1})}^-]' \Sigma_{1,1}$ we obtain (***) $\approx \mathbf{S}_{2,2}$. If this relation is considered in the expression

$$\begin{pmatrix} \mathbf{I} & -(\mathbf{X}'_1)_{m(\Sigma_{1,1})}^- \mathbf{C}'_{2,1} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{B}_{1,1} & \mathbf{B}_{1,2} \\ \mathbf{B}_{2,1} & \mathbf{B}_{2,2} \end{pmatrix}$$

the lemma is immediately verified.

3. BLUE IN THE p -STAGE REGRESSION MODEL

For the sake of simplicity it is assumed in the following that the vectors β_1, \dots, β_p are unbiasedly estimable. In the general case the following theorems are valid for the unbiasedly estimable linear functions of vectors β_1, \dots, β_p (for details see Chpt. 4 in [3]).

Theorem 3.1. *In the 2-stage model, $\hat{\beta}_2(\mathbf{Y}_1, \mathbf{Y}_2) = \hat{\beta}_2(\hat{\beta}_1, \mathbf{Y}_2)$.*

Proof. In the model

$$\left[\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{C}_{2,1}, & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_{1,1}, & \mathbf{0} \\ \mathbf{0}, & \Sigma_{2,2} \end{pmatrix} \right],$$

according to Lemma 2.2 the BLUE of β_2 is

$$\begin{aligned} \hat{\beta}_2(\mathbf{Y}_1, \mathbf{Y}_2) &= (\mathbf{0}, \mathbf{I}) \left\{ \left[\begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{C}_{2,1}, & \mathbf{X}_2 \end{pmatrix}' \right]_{m(\Sigma)}^{-1} \right\}' \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} = \\ &= [(\mathbf{X}'_2)_{m(\Sigma_{2,2})}^{-1}]' \{ \mathbf{Y}_2 - \mathbf{C}_{2,1} [(\mathbf{X}'_1)_{m(\Sigma_{1,1})}^{-1}]' \mathbf{Y}_1 \}. \end{aligned}$$

If in the regression model $(\mathbf{Y}_2, \mathbf{C}_{2,1}\beta_1 + \mathbf{X}_2\beta_2, \Sigma_{2,2})$ (2nd stage) the BLUE $\hat{\beta}_1(\mathbf{Y}_1) = [(\mathbf{X}'_1)_{m(\Sigma_{1,1})}^{-1}]' \mathbf{Y}_1$ from the model $(\mathbf{Y}_1, \mathbf{X}_1\beta_1, \Sigma_{1,1})$ (1st stage) is used instead of β_1 then the model

$$\begin{aligned} &(\mathbf{Y}_2 - \mathbf{C}_{2,1}\hat{\beta}_1(\mathbf{Y}_1), \mathbf{X}_2\beta_2, \Sigma_{2,2} + \mathbf{C}_{2,1}[(\mathbf{X}'_1)_{m(\Sigma_{1,1})}^{-1}]' \Sigma_{1,1} \cdot \\ &\cdot (\mathbf{X}'_1)_{m(\Sigma_{1,1})}^{-1} \mathbf{C}'_{2,1} = \mathbf{S}_{2,2}) \end{aligned}$$

is obtained. In this model the BLUE $\hat{\beta}_2(\hat{\beta}_1, \mathbf{Y}_2)$ of β_2 is $\hat{\beta}_2(\hat{\beta}_1, \mathbf{Y}_2) = [(\mathbf{X}'_2)_{m(\Sigma_{2,2})}^{-1}]' \cdot (\mathbf{Y}_2 - \mathbf{C}_{2,1}\hat{\beta}_1(\mathbf{Y}_1))$. Thus $\hat{\beta}_2(\hat{\beta}_1, \mathbf{Y}_1) = \hat{\beta}_2(\mathbf{Y}_1, \mathbf{Y}_2)$.

Remark 3.1. If $\Sigma_{1,1} = \sigma_1^2 \mathbf{V}_1$, $\Sigma_{2,2} = \sigma_2^2 \mathbf{V}_2$, where the values σ_1^2, σ_2^2 are not known but the value $\varrho = \sigma_2^2/\sigma_1^2$ is known, then the situation is not changed essentially. In Theorem 3.1 the matrix $\begin{pmatrix} \mathbf{V}_1, & \mathbf{0} \\ \mathbf{0}, & \varrho \mathbf{V}_2 \end{pmatrix}$ can be used instead of the matrix Σ . The case of ϱ unknown is investigated among other things in [5].

Theorem 3.2. *If the condition $\mathbf{C}_{i,j} = \mathbf{0} \Leftrightarrow i - j \geq 2$, $i = 3, \dots, p$, $j = 1, \dots, i - 1$ is fulfilled in the p -stage model then $\hat{\beta}_i(\hat{\beta}_{i-1}, \mathbf{Y}_i) = \hat{\beta}_i(\mathbf{Y}_1, \dots, \mathbf{Y}_i)$, $i = 2, \dots, p$.*

Proof. The case $i = 2$ is proved in Theorem 3.1. Let $p = 3$. Models

$$\left[\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \\ \mathbf{Y}_3 \end{pmatrix}, \begin{pmatrix} \mathbf{X}_1, & \mathbf{0}, & \mathbf{0} \\ \mathbf{C}_{2,1}, & \mathbf{X}_2, & \mathbf{0} \\ \mathbf{0}, & \mathbf{C}_{3,2}, & \mathbf{X}_3 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \Sigma_{1,1}, & \mathbf{0}, & \mathbf{0} \\ \mathbf{0}, & \Sigma_{2,2}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{0}, & \Sigma_{3,3} \end{pmatrix} \right]$$

and

$$\left[\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \\ \mathbf{Y}_3^* \end{pmatrix}, \begin{pmatrix} \mathbf{X}_1, & \mathbf{0} & \mathbf{0} \\ \mathbf{C}_{2,1}, & \mathbf{X}_2 & \mathbf{0} \\ \mathbf{0}, & \mathbf{0} & \mathbf{X}_3 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \begin{pmatrix} \Sigma & \mathbf{S}_{1,3} \\ \mathbf{S}_{3,1} & \mathbf{S}_{3,3} \end{pmatrix} \right],$$

where

$$\mathbf{Y}_3^* = \mathbf{Y}_3 - (\mathbf{0}, \mathbf{C}_{3,2}) \left\{ \left[\begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{C}_{2,1}, & \mathbf{X}_2 \end{pmatrix}' \right]_{m(\Sigma)}^- \right\}' \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix},$$

$$\Sigma = \begin{pmatrix} \Sigma_{1,1}, & \mathbf{0} \\ \mathbf{0}, & \Sigma_{2,2} \end{pmatrix},$$

$$\mathbf{S}_{3,1} = -(\mathbf{0}, \mathbf{C}_{3,2}) \left\{ \left[\begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{C}_{2,1}, & \mathbf{X}_2 \end{pmatrix}' \right]_{m(\Sigma)}^- \right\}' \Sigma = \mathbf{S}'_{1,3},$$

$$\mathbf{S}_{3,3} = \Sigma_{3,3} + \mathbf{C}_{3,2} [(\mathbf{X}'_2)_{m(\mathbf{S}_{2,2})}^-] \mathbf{S}_{2,2} (\mathbf{X}'_2)_{m(\mathbf{S}_{2,2})}^- \mathbf{C}'_{3,2},$$

$$\mathbf{S}_{2,2} = \Sigma_{2,2} + \mathbf{C}_{2,1} [(\mathbf{X}'_1)_{m(\Sigma_{1,1})}^-] \Sigma_{1,1} (\mathbf{X}'_1)_{m(\Sigma_{1,1})}^- \mathbf{C}'_{2,1},$$

are equivalent (each model is the result of a transformation of the other by a regular matrix). That is why the BLUE $\hat{\beta}_3(\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3)$ from the first model and the BLUE $\hat{\beta}_3(\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3^*)$ from the second model are identical. Due to Theorem 3.1 we have

$$\hat{\beta}_3(\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3^*) = [(\mathbf{X}'_3)_{m(**)}^-] \left\langle \mathbf{Y}_3 - (\mathbf{0}, \mathbf{C}_{3,2}) \left\{ \left[\begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{C}_{2,1}, & \mathbf{X}_2 \end{pmatrix}' \right]_{m(\Sigma)}^- \right\}' \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \right\rangle$$

where

$$(\mathbf{0}, \mathbf{C}_{3,2}) \left\{ \left[\begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{C}_{2,1}, & \mathbf{X}_2 \end{pmatrix}' \right]_{m(\Sigma)}^- \right\}' \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} = \mathbf{C}_{3,2} [(\mathbf{X}'_2)_{m(\mathbf{S}_{2,2})}^-]'$$

$$\cdot \{ \mathbf{Y}_2 - \mathbf{C}_{2,1} [(\mathbf{X}'_1)_{m(\Sigma_{1,1})}^-] \mathbf{Y}_1 \} = \mathbf{C}_{3,2} \hat{\beta}_2(\mathbf{Y}_1, \mathbf{Y}_2) = \mathbf{C}_{3,2} \hat{\beta}_2(\hat{\beta}_1, \mathbf{Y}_2).$$

The matrix (**) can be expressed as follows:

$$\begin{aligned} (**) &= \mathbf{S}_{3,3} - \mathbf{S}_{3,1} \Sigma^- \left\{ \Sigma - \Sigma \left[\begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{C}_{2,1}, & \mathbf{X}_2 \end{pmatrix}' \right]_{m(\Sigma)}^- \begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{C}_{2,1}, & \mathbf{X}_2 \end{pmatrix}' \right\} \Sigma^- \mathbf{S}_{1,3} = \\ &= \mathbf{S}_{3,3} - (\mathbf{0}, \mathbf{C}_{3,2}) \left\{ \left[\begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{C}_{2,1}, & \mathbf{X}_2 \end{pmatrix}' \right]_{m(\Sigma)}^- \right\}' \Sigma \Sigma^-. \end{aligned}$$

$$\cdot \left\{ \Sigma - \Sigma \left[\begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{C}_{2,1}, & \mathbf{X}_2 \end{pmatrix}' \right]_{m(\Sigma)}^- \begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{C}_{2,1}, & \mathbf{X}_2 \end{pmatrix}' \right\} \Sigma^- \Sigma \left[\begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{C}_{2,1}, & \mathbf{X}_2 \end{pmatrix}' \right]_{m(\Sigma)}^- \begin{pmatrix} \mathbf{0} \\ \mathbf{C}_{3,2} \end{pmatrix}.$$

The equivalence

$$\mathcal{M}(\mathbf{C}'_{3,2}) \subset \mathcal{M}(\mathbf{X}'_2) \Leftrightarrow \mathcal{M} \begin{pmatrix} \mathbf{0} \\ \mathbf{C}'_{3,2} \end{pmatrix} \subset \mathcal{M} \begin{pmatrix} \mathbf{X}'_1, & \mathbf{C}'_{2,1} \\ \mathbf{0}, & \mathbf{X}'_2 \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \exists \left\{ \mathbf{E}: (\mathbf{0}, \mathbf{C}_{3,2}) = \mathbf{E}' \begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{C}_{2,1}, & \mathbf{X}_2 \end{pmatrix} \right\}$$

implies the identity (**) = $\mathbf{S}_{3,3}$. Thus we obtain $\hat{\beta}_3(\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3) = \hat{\beta}_3(\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3^*) =$
 $= [(\mathbf{X}'_3)_{m(\mathbf{s}_{3,3})}]' \{ \mathbf{Y}_3 - \mathbf{C}_{3,2} [(\mathbf{X}'_2)_{m(\mathbf{s}_{2,2})}]' \langle \mathbf{Y}_2 - \mathbf{C}_{2,1} [(\mathbf{X}'_1)_{m(\mathbf{s}_{1,1})}]' \mathbf{Y}_1 \rangle \} =$
 $= [(\mathbf{X}'_3)_{m(\mathbf{s}_{3,3})}]' (\mathbf{Y}_3 - \mathbf{C}_{3,2} \hat{\beta}_2(\hat{\beta}_1, \mathbf{Y}_2)).$

If in the model $(\mathbf{Y}_3, \mathbf{C}_{3,2}\beta_2 + \mathbf{X}_3\beta_3, \Sigma_{3,3})$ (3rd stage) the BLUE $\hat{\beta}_1(\mathbf{Y}_1, \mathbf{Y}_2) =$
 $= \hat{\beta}_2(\hat{\beta}_1, \mathbf{Y}_2)$ from the model

$$\left[\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{C}_{2,1}, & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{1,1}, & \mathbf{0} \\ \mathbf{0}, & \Sigma_{2,2} \end{pmatrix} \right]$$

is used instead of β_2 then the model obtained is

$$\{ \mathbf{Y}_2 - \mathbf{C}_{3,2} \hat{\beta}_2(\hat{\beta}_1, \mathbf{Y}_2), \mathbf{X}_3\beta_3, \Sigma_{3,3} + \mathbf{C}_{3,2} [(\mathbf{X}'_2)_{m(\mathbf{s}_{2,2})}]' \mathbf{S}_{2,2} (\mathbf{X}'_2)_{m(\mathbf{s}_{2,2})} \mathbf{C}'_{3,2} = \mathbf{S}_{3,3} \},$$

where

$$\mathbf{S}_{2,2} = \Sigma_{2,2} + \mathbf{C}_{2,1} [(\mathbf{X}'_1)_{m(\mathbf{s}_{1,1})}]' \Sigma_{1,1} (\mathbf{X}'_1)_{m(\mathbf{s}_{1,1})} \mathbf{C}'_{2,1}.$$

In the last model the BLUE of β_3 is

$$\hat{\beta}_3[\hat{\beta}_2(\hat{\beta}_1, \mathbf{Y}_2), \mathbf{Y}_3] = [(\mathbf{X}'_3)_{m(\mathbf{s}_{3,3})}]' [\mathbf{Y}_3 - \mathbf{C}_{3,2} \hat{\beta}_2(\hat{\beta}_1, \mathbf{Y}_2)].$$

Thus $\hat{\beta}_3[\hat{\beta}_2(\hat{\beta}_1, \mathbf{Y}_2), \mathbf{Y}_3] = \hat{\beta}_3(\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3)$.

For $p > 3$ the procedure is analogous.

Remark 3.2. If the condition $\mathbf{C}_{i,j} = \mathbf{0} \Leftrightarrow i - j \geq 2, i = 3, \dots, p, j = 1, \dots, \dots, i - 1$ is not fulfilled then $\text{Var} [\hat{\beta}_i(\hat{\beta}_1, \dots, \hat{\beta}_{i-1}), \mathbf{Y}_i] - \text{Var} [\hat{\beta}_i(\mathbf{Y}_1, \dots, \mathbf{Y}_i)] \neq \mathbf{0}$. To see this, it is sufficient to investigate the case $p = 3$ with $\mathbf{C}_{3,1} \neq \mathbf{0}$.

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