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CONTROLLABLE SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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Summary. In the paper definitions of various kinds of stability and boundedness of solutions of linear controllable systems of partial differential equations are introduced and their interconnections are derived. By means of Ljapunov's functions theorems are proved which give necessary and sufficient conditions for particular kinds of stability and boundedness of the solutions.

Keywords. Controllable system, Ljapunov's function, stability, asymptotic stability, uniform stability, uniform asymptotic stability, boundedness, asymptotic boundedness, uniform boundednes, uniform asymptotic boundedness.

Notation. In the paper we study the properties of stability and of boundedness of a controllable system of linear partial differential equations. Before introducing the fundamental definitions, we introduce the notation which will be used throughout the text. By the symbol $\boldsymbol{R}^n$ we denote the real n-dimensional Euclidean space of column vectors with the zero vector o, with the norm $\| \ \|$; by $\boldsymbol{R}^+$ we denote the set of all positive numbers. If $\boldsymbol{I}_1, \boldsymbol{I}_2, \ldots, \boldsymbol{I}_m$ are open and unbounded from above intervals in $\boldsymbol{R}^1$, then an interval in R^m will be denoted by

$$\boldsymbol{I} = \boldsymbol{I}_1 \times \boldsymbol{I}_2 \times \ldots \times \boldsymbol{I}_m \quad \text{and} \quad \boldsymbol{o} = \{(o, a) \in \boldsymbol{R}^n \times \boldsymbol{I} : a \in \boldsymbol{I}\} .$$

For $a = (a_1, a_2, \ldots, a_m) \in \boldsymbol{I}$ let us define a half-closed interval $\langle a, \infty) = \langle a_1, +\infty) \times \langle a_2, +\infty) \times \ldots \times \langle a_m, +\infty)$ and let the symbol $a \to \infty$ denote $a_h \to +\infty$ for each $h = 1, 2, \ldots, m$. For $t = (t_1, t_2, \ldots, t_m) \in \boldsymbol{I}$, $a = (a_1, a_2, \ldots, a_m) \in \boldsymbol{I}$, where $t_1 > a_1$, $t_2 > a_2, \ldots, t_m > a_m$ we introduce the notation $t > a$. By the symbol $t \geqq a$ we denote $t > a$ or $t = a$; the sets $\boldsymbol{D}, \boldsymbol{R}^m_+$ are defined by the rule $\boldsymbol{D} = \{(t, x, a) \in \boldsymbol{I} \times \boldsymbol{R}^m \times \boldsymbol{I} : t \geqq a\}$, $\boldsymbol{R}^m_+ = \{t \in \boldsymbol{R}^m : t \geqq o\}$. The set of all real matrices of the type (n, n) will be denoted by the symbol $\mathscr{R}_n$.

If $A(t) = \{{}^hA(t) : t \in \boldsymbol{I}, h = 1, 2, \ldots, m\}$ is a system of continuous matrix functions from $\mathscr{R}_n$ such that for each $t \in \boldsymbol{I}$ the equality

$$(\mathrm{H}_1) \qquad \frac{\partial\, {}^hA}{\partial t_{h'}} - \frac{\partial\, {}^{h'}A}{\partial t_h} + {}^hA\, {}^{h'}A - {}^{h'}A\, {}^hA = 0 \, ; \quad h, h' = 1, 2, \ldots, m \, ; \quad h \neq h' \, ,$$

holds, the symbol $F_A(\cdot, a, x): \{t \in \boldsymbol{I}: t \geqq a\} \to \boldsymbol{R}^n$ we denote such a solution of the system of partial differential equations

$$\frac{\partial F}{\partial t_h} = {}^hA(t)\,F\,, \quad h = 1, 2, \ldots, m\,,$$

for which $F_A(a, a, x) = x$ (see [1]).

In this paper we shall generalize the concept of the solution of a system of partial differential equations also for the case when the system of matrices $A(t)$ is not continuous and differentiable in the interval $\boldsymbol{I}$. We shall suppose that the system of matrix functions $A(t) = \{{}^hA(t): t \in \boldsymbol{I},\ h = 1, 2, \ldots, m\}$ has the property

(H_2) There exists a sequence of vectors $\{{}^ja: j = 0, 1, 2, \ldots\} \subset \boldsymbol{I}$, ${}^0a < {}^1a < \ldots < {}^ja < \ldots$, such that for each $h = 1, 2, \ldots, m$ the matrix functions ${}^hA(t)$ are continuous on the set $\mathscr{M} = \bigcup_{j=0}^{+\infty} (({}^ja, \infty) - \langle {}^{j+1}a, \infty))$, for each $t \in \mathscr{M}$ they satisfy the relations (H_1) and for each $b \in \langle {}^ja, \infty) - ({}^ja, \infty)$, $j = 0, 1, 2, \ldots$ there are

$$\lim_{t \to b, t \in ({}^ja, \infty)} {}^hA(t) = {}^hA(b) \in \mathscr{R}_n\,, \quad \lim_{t \to b, t \notin ({}^ja, \infty)} {}^hA(t) \in \mathscr{R}_n\,, \quad \lim_{t \to b, t \in ({}^ja, \infty)} \frac{\partial^h A}{\partial t_{h'}} \in \mathscr{R}_n\,,$$

$$\lim_{t \to b, t \notin ({}^ja, \infty)} \frac{\partial^h A}{\partial t_{h'}} \in \mathscr{R}_n\,, \quad h, h' = 1, 2, \ldots, m\,.$$

Definition. The *solution* of the system of partial differential equations

$$\frac{\partial F}{\partial t_h} = {}^hA(t)\,F\,, \quad h = 1, 2, \ldots, m\,, \tag{0}$$

where the system of matrix functions $A(t) = \{{}^hA(t): t \in \boldsymbol{I},\ h = 1, 2, \ldots, m\}$ fulfils (H_2), is a mapping $F_A(\cdot, a, x): \{t \in \boldsymbol{I}: t \geqq a\} \to \boldsymbol{R}^n$ which satisfies the relations (0) in the regions $({}^ja, \infty) - \langle {}^{j+1}a, \infty)$, $j = 0, 1, 2, \ldots$, is continuous in the vectors ${}^ja$ and for each $b \in \langle {}^ja, \infty) - ({}^ja, \infty)$ we have $\lim_{t \to b, t \in ({}^ja, \infty)} F_A(t, a, x) = F_A(b, a, x)$, $F_A(a, a, x) = x$.

From the paper [1] it follows that for each $a \in \boldsymbol{I}$ there exists a matrix function $F_A(t, a) \in \mathscr{R}_n$ defined in the interval $\boldsymbol{I}$ such that $\lim_{t \to b, t \in ({}^ja, \infty)} F_A(t, a) = F_A(b, a)$ for each $b \in \boldsymbol{I} - (a, \infty)$ and for each $t \geqq a$ and for each column vector $x \in \boldsymbol{R}^n$ the relation

$$F_A(t, a, x) = F_A(t, a)\,x \tag{0_1}$$

holds.

Definition. We say that a non-empty class of systems of matrix function $A(t) = \{{}^hA(t): t \in \boldsymbol{I},\ h = 1, 2, \ldots, m\}$ is the *set of regulators of* $\mathscr{A}$, if and only if for each system $A(t) \in \mathscr{A}$ there exists a sequence of vectors $\{{}^ja: j = 0, 1, 2, \ldots\} \subset \boldsymbol{I}$,

${}^0a < {}^1a < \ldots < {}^ja < \ldots$, with the property (H_2) and this class satisfies the condition

(H_3) $(b, x, a) \in \boldsymbol{D}$, $A \in \mathscr{A}$, $(c, F_A(b, a)\, x, b) \in \boldsymbol{D}$, $B \in \mathscr{A} \Rightarrow$ there exists $C \in \mathscr{A}$ such that

$$F_C(t, a)\, x = \begin{cases} F_A(t, a)\, x & \text{for} \quad t \in \langle a, \infty) - \langle b, \infty)\,, \\ F_B(t, b)\, F_A(b, a)\, x & \text{for} \quad t \in \langle b, \infty) - \langle c, \infty)\,, \end{cases}$$

$$F_C(c, a)\, x = F_B(c, b)\, F_A(b, a)\, x\,.$$

Throughout the paper we suppose that a *controllable system* of partial differential equations

$$(1) \qquad \frac{\partial F}{\partial t_h} = {}^hA(t)\, F\,, \quad A(t) = \{{}^hA(t)\colon t \in \boldsymbol{I},\, h = 1, 2, \ldots, m\} \in \mathscr{A}$$

is given, where A is the set of regulators.

Definition. We say that a partial mapping $v\colon \boldsymbol{R}^n \times \boldsymbol{I} \to \boldsymbol{R}^+$ is a *Ljapunov function* of the controllable system of equations (1) if and only if the following holds:

$$(2) \qquad y = F_A(b, a)\, x,\ A \in \mathscr{A},\ (x, a) \in \text{domain } v\,, \quad (y, b) \in \text{domain } v \Rightarrow v(y, b) \leqq v(x, a)\,.$$

Definition. We say that the set $\boldsymbol{o}$ is *stable* with respect to the controllable system of equations (1) if and only if there exists a mapping $z_1\colon \boldsymbol{I} \times \boldsymbol{R}^+ \to \boldsymbol{R}^+$ such that the following implication holds:

$$(3) \qquad (t, x, a) \in \boldsymbol{D}\,, \quad \|x\| \leqq z_1(a, r)\,, \quad A \in \mathscr{A} \Rightarrow \|F_A(t, a)\, x\| \leqq r\,.$$

We say that the controllable system of equations (1) is *bounded* with respect to the set $\boldsymbol{o}$ if and only if there exists a mapping $z_2\colon \boldsymbol{I} \times \boldsymbol{R}^+ \to \boldsymbol{R}^+$ such that the following implication holds:

$$(4) \qquad (t, x, a) \in \boldsymbol{D}\,, \quad \|x\| \leqq r\,, \quad A \in \mathscr{A} \Rightarrow \|F_A(t, a)\, x\| \leqq z_2(a, r)\,.$$

Theorem 1. *The set $\boldsymbol{o}$ is stable with respect to the controllable system of equations* (1) *if and only if the controllable system of equations* (1) *is bounded with respect to the set $\boldsymbol{o}$.*

Proof. Let the set $\boldsymbol{o}$ be stable. Define the mapping z_2 from (4) by putting $z_2(a, r) = r^2/z_1(a, r)$, where z_1 is the mapping from (3). Let $(t, x, a) \in D$, $\|x\| \leqq r$, $A \in \mathscr{A}$ be given. Then $\|z_1(a, r)\, x/r\| \leqq z_1(a, r)$ and (3) implies $\|F_A(t, a)\, z_1(a, r)\, x/r\| \leqq r$ and thus $\|F_A(t, a)\, x\| \leqq r^2/z_1(a, r) = z_2(a, r)$. Hence the controllable system of equations (1) is bounded with respect to the set $\boldsymbol{o}$.

Let the controllable system of equations (1) be bounded with respect to the set $\boldsymbol{o}$. Define the mapping z_1 from (3) by putting $z_1(a, r) = r^2/z_2(a, r)$, where z_2 is the mapping from (4). Let $(t, x, a) \in \boldsymbol{D}$, $\|x\| \leqq z_1(a, r) = r^2/z_2(a, r)$, $A \in \mathscr{A}$ be given. Then $\|z_2(a, r)\, x/r\| \leqq r$ and (4) implies $\|F_A(t, a)\, z_2(a, r)\, x/r\| \leqq z_2(a, r)$ and thus $\|F_A(t, a)\, x\| \leqq r$. Hence the set $\boldsymbol{o}$ is stable with respect to the controllable system of equations (1).

Theorem 2. *The controllable system of equations* (1) *is bounded with respect to the set* $\boldsymbol{o}$ *if and only if there exist a constant* $k \in \boldsymbol{R}^+$, *a partial mapping* $v: \boldsymbol{R}^n \times \boldsymbol{I} \to \boldsymbol{R}^+$, *a partial increasing mapping* $f: \boldsymbol{R}^+ \to \boldsymbol{R}^+$, $f(r) \to +\infty$ *for* $r \to +\infty$, *and a partial mapping* $z_0: \boldsymbol{I} \times \boldsymbol{R}^+ \to \boldsymbol{R}^+$ *with the following properties*:

(i) v *is a Ljapunov function of the controllable system of equations* (1) *with* domain $v = \{(x, a) \in \boldsymbol{R}^n \times \boldsymbol{I}: \|x\| \geqq k\}$,

(ii) $(x, a) \in$ domain $v \Rightarrow f(\|x\|) \leqq v(x, a)$,

(iii) $(x, a) \in$ domain v, $\|x\| \leqq r \Rightarrow v(x, a) \leqq z_0(a, r)$.

Proof. Let the controllable system of equations (1) be bounded with respect to the set $\boldsymbol{o}$. Choose $k \in \boldsymbol{R}^+$ and define the partial mapping $v: \boldsymbol{R}^n \times \boldsymbol{I} \to \boldsymbol{R}^+: v(x, a) = \sup\{\|F_A(t, a)\, x: A \in \mathscr{A}, a \leqq t \in \boldsymbol{I}\}$ for $\|x\| \geqq k$, $f: \boldsymbol{R}^+ \to \boldsymbol{R}^+: f(r) = r$, $z_0: \boldsymbol{I} \times \boldsymbol{R}^+ \to \boldsymbol{R}^+: z_0(a, r) = z_2(a, r)$, where z_2 is the mapping from (4). Now we shall prove that v, f, z_0 have the properties (i), (ii), (iii). Let $(x, a) \in$ domain v be given. From (4) it follows that v is really defined by the described rule and that it has the property (iii). Further, for $y = F_A(b, a)\, x$ and for each $u = F_B(t. b)\, y$ there exists $C \in \mathscr{A}$ such that $u = F_C(t, a)\, x$. This implies

$$v(y, b) = \sup\{\|F_A(t, b)\, y\|: A \in \mathscr{A}, b \leqq t \in \boldsymbol{I}\} \leqq$$
$$\leqq \sup\{\|F_A(t, a)\, x\|: A \in \mathscr{A}, a \leqq t \in \boldsymbol{I}\} = v(x, a)$$

and thus v is a Ljapunov function of the controllable system of equations (1). Evidently $\|x\| \in \{\|F_A(t, a)\, x\|: a \leqq t \in \boldsymbol{I}\}$ and thus $\|x\| \leqq v(x, a)$. Let there exist partial mappings v, f, z_0 and a constant $k \in \boldsymbol{R}^+$ with the properties (i), (ii), (iii). Define the mapping $z_2: \boldsymbol{I} \times \boldsymbol{R}^+ \to \boldsymbol{R}^+$ so that the inequality $f(kz_2(a, r)/r) \geqq z_0(a, k)$ might be fulfilled and show that z_2 satisfies (4). Let $(t, x, a) \in \boldsymbol{D}$, $0 < \|x\| \leqq r$, $A \in \mathscr{A}$ be given. Then for $(F_A(t, a)\, kx/r, t) \in$ domain v we have

$$f(\|F_A(t, a)\, kx/r\|) \leqq f(\|F_A(t, a)\, kx/\|x\|\,\|) \leqq v(F_A(t, a)\, kx/\|x\|, t) \leqq$$
$$\leqq v(kx/\|x\|, a) \leqq z_0(a, k) \leqq f(kz_2(a, r)/r)\,,$$

therefore $\|F_A(t, a)\, x\| \leqq z_2(a, r)$. If $\|F_A(t, a)\, kx/r\| < k$. then $\|F_A(t, a)\, x\| < r \leqq z_2(a, r)$. Hence the controllable system of equations (1) is bounded with respect to the set $\boldsymbol{o}$.

Theorem 3. *The controllable system of equations* (1) *is bounded with respect to the set* $\boldsymbol{o}$ *if and only if there exists a mapping* $z_3: \boldsymbol{I} \to \boldsymbol{R}^+$ *such that*

$$(5) \qquad (t, a, A) \in \boldsymbol{I} \times \boldsymbol{I} \times \mathscr{A}\,, \quad t \geqq a \Rightarrow \|F_A(t, a)\| \leqq z_3(a)\,,$$

where $F_A(t, a)$ is the matrix from (0_1).

Proof. Let the controllable system of equations (1) be bounded with respect to the set $\boldsymbol{o}$. Define the mapping z_3 by putting $z_3(a) = z_2(a, 1)$, where z_2 is the mapping from (4). Then the implication

$$(t, x, a) \in \boldsymbol{D}, \quad \|x\| \leqq 1, \quad A \in \mathscr{A} \Rightarrow \|F_A(t, a)\, x\| \leqq z_2(a, 1) = z_3(a)$$

holds and thus $\|F_A(t, a)\| \leqq z_3(a)$.

For the matrix $F_A(t, a)$ let the implication (5) be fulfilled. Define the mapping z_2 by putting $z_2(a, r) = r z_3(a)$. Then for any $(t, x, a) \in \boldsymbol{D}$, $\|x\| \leqq r$, $A \in \mathscr{A}$ the inequality $\|F_A(t, a)\, x/r\| \leqq z_3(a)$ holds, i.e. $\|F_A(t, a)\, x\| \leqq r z_3(a) = z_2(a, r)$. Thus the theorem is proved.

Theorem 4. *For each $A \in \mathscr{A}$ let the solutions of the system of equations (1) be bounded with respect to the set $\boldsymbol{o}$, i.e. let there exist a mapping z'_2: $\boldsymbol{I} \times \boldsymbol{R}^+ \times \mathscr{A} \to \boldsymbol{R}^+$ such that the following implication holds:*

$$(t, x, a) \in \boldsymbol{D}, \quad \|x\| \leqq r, \quad A \in \mathscr{A} \Rightarrow \|F_A(t, a)\, x\| \leqq z'_2(a, r, A).$$

Then there exists $s \in \boldsymbol{I}$ such that the controllable system of equations (1) is bounded on $\boldsymbol{R}^n \times \langle s, \infty)$ with respect to the set $\boldsymbol{o}$, i.e. the set $\boldsymbol{o}$ is stable on $\boldsymbol{R}^n \times \langle s, \infty)$ with respect to the controllable system of equations (1).

Proof. Define the relation $\boldsymbol{L}: \boldsymbol{I} \to \{\boldsymbol{R}^n\}$ by putting

$$\boldsymbol{L}(a) = \{x \in \boldsymbol{R}^n: \sup\{\|F_A(t, a)\, x\|: A \in \mathscr{A}, a \leqq t\} < +\infty\}$$

and show that this relation has the following properties:

(i) for each $a \in \boldsymbol{I}$ the set $\boldsymbol{L}(a)$ is a vector space,

(ii) $a \in \boldsymbol{I}$, $b \in \boldsymbol{I}$, $a \leqq b \Rightarrow \dim \boldsymbol{L}(a) \leqq \dim \boldsymbol{L}(b)$,

(iii) $c \in \boldsymbol{I}$, $b \in \boldsymbol{I}$, $c \geqq b \in \{a: \dim \boldsymbol{L}(a) = \max\{\dim \boldsymbol{L}(t) : t \in \boldsymbol{I}\}\}$,
$x \notin \boldsymbol{L}(b)$, $A \in \mathscr{A} \Rightarrow F_A(c, b)\, x \notin \boldsymbol{L}(c)$,

(iv) if there exists $s \in \boldsymbol{I}$ such that $\boldsymbol{L}(s) = \boldsymbol{R}^n$, then the controllable system of equations (1) is bounded on $\boldsymbol{R}^n \times \langle s, \infty)$ with respect to the set $\boldsymbol{o}$.

Ad (i). The assertion is evident from the definition of the relation $\boldsymbol{L}$.

Ad (ii). Let $a \in \boldsymbol{I}$ and let a basis $\{{}^j x: j = 1, 2, \ldots, k\}$ of the vector space $\boldsymbol{L}(a)$ be given. For any $A \in \mathscr{A}$, $b \in \boldsymbol{I}$, $a \leqq b$, the inclusion $\{F_A(b, a)\, {}^j x: j = 1, 2, \ldots, k\} \subset \subset \boldsymbol{L}(b)$ evidently holds. Suppose that the set of vectors $\{F_A(b, a)\, {}^j x: j = 1, 2, \ldots, k\}$ is linearly dependent for some $A \in \mathscr{A}$. Thus there exist constants

$${}^j r \in R^1, \quad j = 1, 2, \ldots, k, \quad \sum_{j=1}^{k} |{}^j r| > 0,$$

such that

$$\sum_{j=1}^{k} {}^j r\, F_A(b, a)\, {}^j x = o$$

and therefore

$$F_A(b, a) \sum_{j=1}^{k} {}^j r\, {}^j x = o.$$

From this and from the unicity of the solution of the system of linear partial differential equations we obtain the equality

$$\sum_{j=1}^{k} {}^j r \, {}^j x = o ,$$

which contradicts the linear independence of the set of vectors $\{{}^j x : j = 1, 2, \ldots, k\}$. Hence the vectors $F_A(b, a)\, {}^j x$, $j = 1, 2, \ldots, k$, are linearly independent, which proves (ii).

Ad (iii). Let the assumptions from (iii) be fulfilled and let the set of vectors $\{{}^j x : j = 1, 2, \ldots, k\}$ be a basis of the vector space $\boldsymbol{L}(b)$. Then the set of vectors $\{F_A(c, b) \,.\, {}^j x : j = 1, 2, \ldots, k\}$ forms a basis of the vector space $\boldsymbol{L}(c)$. Suppose that $F_A(c, b)\, x \in \boldsymbol{L}(c)$; then

$$F_A(c, b)\, x = \sum_{j=1}^{k} {}^j r \, F_A(c, b)\, {}^j x ,$$

where ${}^j r \in \boldsymbol{R}^1$. This implies that for any $b \leqq t \leqq c$ the equality

$$F_A(t, b)\, x = \sum_{j=1}^{k} {}^j r \, F_A(t, b)\, {}^j x$$

holds. In particular

$$x = \sum_{j=1}^{k} {}^j r x ,$$

which is a contradiction with our assumption. Thus the property (iii) is proved.

Ad (iv). Let there exist $s \in \boldsymbol{I}$ such that $\boldsymbol{L}(s) = \boldsymbol{R}^n$ and let $(x, a) \in \boldsymbol{R}^n \times \langle s, \infty)$. Denote by $\{{}^j x : j = 1, 2, \ldots, n\}$ a basis of the vector space $\boldsymbol{R}^n$ and put $\bar{z}_j(a) = \sup \{\|F_A(t, a)\, {}^j x\| : A \in \mathscr{A}, t \geqq a\}$, $j = 1, 2, \ldots, n$. Then

$$x = \sum_{j=1}^{n} {}^j r \, {}^j x, \quad {}^j r \in \boldsymbol{R}^1, \quad |{}^j r| \leqq \varrho \|x\| ,$$

where ϱ is a constant depending only on the basis of the vector space $\boldsymbol{R}^n$. Then for each $A \in \mathscr{A}$ we have

$$\|F_A(t, a)\, x\| = \|F_A(t, a) \sum_{j=1}^{n} {}^j r \, {}^j x\| \leqq \sum_{j=1}^{n} |{}^j r| \, \|F_A(t, a)\, {}^j x\| \leqq \varrho \|x\| \sum_{j=1}^{n} \bar{z}_j(a) .$$

If we define the mapping $z_2 : \langle s, \infty) \times \boldsymbol{R}^+ \to \boldsymbol{R}^+$ by putting

$$z_2(a, r) = \varrho r \sum_{j=1}^{n} \bar{z}_j(a) ,$$

the preceding inequality yields

$$(t, x, a) \in \boldsymbol{D} , \quad (x, a) \in \boldsymbol{R}^n \times \langle s, \infty) , \quad \|x\| \leqq r ,$$
$$A \in \mathscr{A} \Rightarrow \|F_A(t, a)\, x\| \leqq z_2(a, r) ,$$

which proves the property (iv).

Now suppose that there exists no $s \in \boldsymbol{I}$ such that $\boldsymbol{R}^n = \boldsymbol{L}(s)$. Then $\dim \boldsymbol{L}(a) < n$ for each $a \in \boldsymbol{I}$. Denote ${}^0a \in \boldsymbol{I}$, ${}^0a \in \{a: \dim \boldsymbol{L}(a) = \max\{\dim \boldsymbol{L}(t): t \in \boldsymbol{I}\}\}$. Then there exist ${}^0x \notin \boldsymbol{L}({}^0a)$, ${}^0x \in \boldsymbol{R}^n$, ${}^1a$, ${}^0a \in \boldsymbol{I}$, ${}^1a > {}^0a$, $A_1 \in \mathscr{A}$ such that $\|F_{A_1}({}^1a, {}^0a) \,.\, {}^0x\| > 1$. The property (iii) implies that ${}^1x = F_{A_1}({}^1a, {}^0a)\, {}^0x \notin \boldsymbol{L}({}^1a)$. If we define consecutively the sequences ${}^jx \notin L({}^ja)$, $\|{}^jx\| > j$, $A_j \in \mathscr{A}$ so that ${}^jx = F_{A_j}({}^ja, {}^{j-1}a) \,.\, {}^{j-1}x \notin \boldsymbol{L}({}^ja)$, $j = 1, 2, \ldots,$ we obtain that for $A \in \mathscr{A}$, where $F_A(t, {}^{j-1}a)\, {}^{j-1}x = F_{A_j}(t, {}^{j-1}a)\, {}^{j-1}x$ for $t \in \langle {}^{j-1}a, \infty) - \langle {}^ja, \infty)$, the solution F_A is unbounded with respect to the set $\boldsymbol{o}$, which is a contradiction with the assumption of the theorem. Thus the theorem is proved.

Definition. We say that the set $\boldsymbol{o}$ is *uniformly stable* with respect to the controllable system of equations (1) if and only if there exists a mapping $z_1: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ such that the following implication holds:

$$(6) \qquad (t, x, a) \in \boldsymbol{D}, \quad \|x\| \leqq z_1(r), \quad A \in \mathscr{A} \Rightarrow \|F_A(t, a)\, x\| \leqq r\,.$$

We say that the controllable system of equations (1) is *uniformly bounded* with respect to the set $\boldsymbol{o}$ if and only if there exists a mapping $z_2: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ such that the following implication holds:

$$(7) \qquad (t, x, a) \in \boldsymbol{D}, \quad \|x\| \leqq r, \quad A \in \mathscr{A} \Rightarrow \|F_A(t, a)\, x\| \leqq z_2(r)\,.$$

Theorem 5. *The set $\boldsymbol{o}$ is uniformly stable with respect to the controllable system of equations* (1) *if and only if the controllable system of equations* (1) *is uniformly bounded with respect to the set $\boldsymbol{o}$.*

The proof follows from the proof of Theorem 1 and from the fact that the mappings z_1 and z_2 in (6) and (7) do not depend on a.

Theorem 6. *The controllable system of equations* (1) *is uniformly bounded with respect to the set $\boldsymbol{o}$ if and only if there exist a constant $k \in \boldsymbol{R}^+$, a partial mapping $v: \boldsymbol{R}^n \times \boldsymbol{I} \to \boldsymbol{R}^+$, a partial increasing mapping $f: \boldsymbol{R}^+ \to \boldsymbol{R}^+$, $f(r) \to +\infty$ for $r \to +\infty$ and a partial mapping $z_0: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ with the following properties:*

(i) *v is a Ljapunov function of the controllable system of equations* (1) *with* domain $v = \{(x, a) \in \boldsymbol{R}^n \times \boldsymbol{I}: \|x\| \geqq k\}$,

(ii) $(x, a) \in$ domain $v \Rightarrow f(\|x\|) \leqq v(x, a) \leqq z_0(\|x\|)$.

The proof follows from the proof of Theorem 2 and from the fact that the mapping z_2 in (7) and the partial mapping z_0 in (ii) do not depend on a.

Theorem 7. *The controllable system of equations* (1) *is uniformly bounded with respect to the set $\boldsymbol{o}$ if and only if there exists a constant $z_3 \in \boldsymbol{R}^+$ such that the following implication holds:*

$$(8) \qquad (t, a, A) \in \boldsymbol{I} \times \boldsymbol{I} \times \mathscr{A}, \quad t \geqq a \Rightarrow \|F_A(t, a)\| \leqq z_3\,,$$

where $F_A(t, a)$ is the matrix from (0_1).

The proof is straightforward, because Theorem 7 is the uniform modification of Theorem 3.

Definition. We say that the set $\boldsymbol{o}$ is *asymptotically stable* with respect to the controllable system of equations (1) if and only if it is stable with respect to the system (1) and there exists a constant $k_1 \in \boldsymbol{R}^+$ and a mapping $z_4: \boldsymbol{I} \times \boldsymbol{R}^+ \to \boldsymbol{R}^m_+$ such that the following implication holds:

$$(9) \quad (t, x, a) \in \boldsymbol{D}\,, \quad \|x\| \leqq k_1\,, \quad t \geqq a + z_4(a, r)\,, \quad A \in \mathscr{A} \Rightarrow \|F_A(t, a)\, x\| \leqq r\,.$$

We say that the controllable system of equations (1) is *asymptotically bounded* with respect to the set $\boldsymbol{o}$ if and only if the system (1) is bounded with respect to the set $\boldsymbol{o}$ and there exist a constant $k_2 \in \boldsymbol{R}^+$ and a mapping $z_5: \boldsymbol{I} \times \boldsymbol{R}^+ \to \boldsymbol{R}^m_+$ such that the following implication holds:

$$(10) \qquad (t, x, a) \in \boldsymbol{D}\,, \quad \|x\| \leqq r\,, \quad t \geqq a + z_5(a, r)\,,$$
$$A \in \mathscr{A} \Rightarrow \|F_A(t, a)\, x\| \leqq k_2\,.$$

Remark. If there exist a constant k_2 and a mapping z_5 such that (10) holds, then for each $k_2 \in \boldsymbol{R}^+$ there exists z_5 depending on k_2 such that (10) holds.

Theorem 8. *The set $\boldsymbol{o}$ is asymptotically stable with respect to the controllable system of equations* (1) *if and only if the controllable system of equations* (1) *is asymptotically bounded with respect to the set $\boldsymbol{o}$.*

Proof. Let the set $\boldsymbol{o}$ be asymptotically stable with respect to the controllable system of equations (1). By Theorem 1, the system (1) is bounded with respect to the set $\boldsymbol{o}$. Define the mapping z_5 and the constant $k_2 \in \boldsymbol{R}^+$ by putting $z_5(a, r) = z_4(a, k_1^2/r)$, $k_2 = k_1$, where z_4 and k_1 are from (9). Let $(t, x, a) \in \boldsymbol{D}$, $\|x\| \leqq r$, $A \in \mathscr{A}$, $t \geqq a + z_4(a, k_1^2/r)$ be given. Then $\|k_1 x/r\| \leqq k_1$ and (9) implies $\|F_A(t, a)\, k_1 x/r\| \leqq k_1^2/r$, therefore $\|F_A(t, a)\, x\| \leqq k_1 = k_2$. Hence the controllable system of equations (1) is asymptotically bounded with respect to the set $\boldsymbol{o}$. By Theorem 1, the set $\boldsymbol{o}$ is stable with respect to the system (1). If $(t, x, a) \in \boldsymbol{D}$, $\|x\| \leqq r$, $t \geqq a + z_5(a, r)$, $A \in \mathscr{A}$, then (10) implies $\|F_A(t, a)\, x\| \leqq k_2$ and therefore the linear mapping $F_A(t, a, \cdot) = F_A(t, a): \boldsymbol{R}^n \to \boldsymbol{R}^n$ is bounded. If we denote its norm by the symbol $\|F_A(t, a)\|$, then for all $t \geqq a + z_5(a, r)$ and for each $A \in \mathscr{A}$ the following inequality holds:

$$(11) \qquad \|F_A(t, a)\| \leqq k_2/r\,.$$

Put $k_2/r = r_1$, $k_1 = 1$ and define the mapping z_4 by putting $z_4(a, r_1) = z_5(a, k_2/r_1)$. If $(t, x, a) \in \boldsymbol{D}$, $\|x\| \leqq 1$, $A \in \mathscr{A}$, $t \geqq a + z_5(a, k_2/r_1)$ are given, then (11) implies $\|F_A(t, a)\, x\| \leqq \|F_A(t, a)\| \leqq r_1$. This proves the theorem.

Theorem 9. *Let there exist a constant $k_2 \in \mathbf{R}^+$ and a mapping $z_5: \mathbf{I} \times \mathbf{R}^+ \to \mathbf{R}^m_+$ such that for the controllable system of equations* (1) *the implication* (10) *holds and for any $(a, A) \in \mathbf{I} \times \mathscr{A}$, $\sup \{\|F_A(t, a)\| : t \in \langle a, \infty) - (a + z_5(a, 1), \infty)\} \in \mathbf{R}^+$. Then there is $s \in \mathbf{I}$ such that the controllable system of equations* (1) *is asymptotically bounded on $\mathbf{R}^n \times \langle s, \infty)$ with respect to the set $\boldsymbol{o}$.*

Proof. The implication (10) implies that for any $(t, x, a) \in \mathbf{D}$, $\|x\| \leqq 1$, $t \geqq a + z_5(a, 1)$, $A \in \mathscr{A}$, $\|F_A(t, a)\, x\| \leqq k_2$ holds and hence $\|F_A(t, a)\| \leqq k_2$. If we define $z_3(a, A) = \max \{\sup \{\|F_A(t, a)\| : t \in \langle a, \infty) - (a + z_5(a, 1), \infty), k_2\}$ and a mapping $z'_2: \mathbf{I} \times \mathbf{R}^+ \times \mathscr{A} \to \mathbf{R}^+$ by $z'_2(a, r, A) = r z_3(a, A)$, then for any $(t, x, a) \in \mathbf{D}$, $\|x\| \leqq r$, $A \in \mathscr{A}$ the inequality $\|F_A(t, a)\, x/r\| \leqq z_3(a, A)$, i.e. $\|F_A(t, a)\, x\| \leqq r z_3(a, A) = z'_2(a, r, A)$ holds. This and Theorem 4 imply theorem 9.

Theorems 1, 8 and 9 yield

Theorem 10. *Let there exist a constant $k_1 \in \mathbf{R}^+$ and a mapping $z_4: \mathbf{I} \times \mathbf{R}^+ \to \mathbf{R}^m_+$ such that for the controllable system of equations* (1) *the implication* (9) *holds and for any $(a, A) \in \mathbf{I} \times \mathscr{A}$, $\sup \{\|F_A(t, a)\| : t \in \langle a, \infty) - (a + z_4(a, k_1^2), \infty)\} \in \mathbf{R}^+$. Then there is $s \in \mathbf{I}$ such that the set $\boldsymbol{o}$ is asymptotically stable on $\mathbf{R}^n \times \langle s, \infty)$ with respect to the controllable system of equations* (1).

Theorem 11. *The controllable system of equations* (1) *is asymptotically bounded with respect to the set $\boldsymbol{o}$ if and only if there exist a constant $k \in \mathbf{R}^+$, a partial mapping $v: \mathbf{R}^n \times \mathbf{I} \to \mathbf{R}^+$, a partial increasing mapping $f: \mathbf{R}^+ \to \mathbf{R}^+$, $f(r) \to +\infty$ for $r \to +\infty$, and a partial mapping $z_0: \mathbf{I} \times \mathbf{R}^+ \to \mathbf{R}^+$ with the properties* (i), (ii), (iii) *from Theorem* 2, *a constant $k_0 \in \mathbf{R}^+$ and a mapping $z_6: \mathbf{I} \times \mathbf{R}^+ \to \mathbf{R}^m_+$ with the property*

$$\text{(iv)} \qquad \|x\| \leqq r\,, \quad t \geqq a + z_6(a, r)\,, \quad A \in \mathscr{A}\,,$$
$$(F_A(t, a)\, x, t) \in \text{domain } v \Rightarrow v(F_A(t, a)\, x, t) \leqq k_0\,.$$

Proof. Let the controllable system of equations (1) be asymptotically bounded with respect to the set $\boldsymbol{o}$. Then Theorem 2 implies that there exist a constant $k \in \mathbf{R}^+$ and partial mappings v, f, z_0 with the properties (i), (ii), (iii) from Theorem 2. Put $k_0 = k_2$ and define the mapping $z_6: \mathbf{I} \times \mathbf{R}^+ \to \mathbf{R}^m_+$ by putting $z_6(a, r) = z_5(a, r)$, where k_2 and z_5 are from (10). Then for any $(t, x, a) \in \mathbf{D}$, $\|x\| \leqq r$, $t \geqq a + z_6(a, r)$, $A \in \mathscr{A}$ we have $\|F_A(t, a)\, x\| \leqq k_0$ and from the definition of the mapping v in the proof of Theorem 2 the inequality $v(F_A(t, a)\, x, t) \leqq k_0$ follows. Hence the mapping v has the property (iv).

Let there exist constants $k \in \mathbf{R}^+$, $k_0 \in \mathbf{R}^+$ and partial mappings v, f, z_0, z_6 with the properties (i), (ii), (iii), (iv). According to Theorem 2 the properties (i), (ii), (iii) imply that the controllable system of equations (1) is bounded with respect to the set $\boldsymbol{o}$. Define the constant $k_2 \geqq k$ in such a way that the inequality $f(k_2) \geqq k_0$ might be fulfilled and choose z_6 for z_5. We shall show that k_2 and z_5 fulfil (10). Let $(t, x, a) \in \mathbf{D}$,

$\|x\| \leqq r$, $t \geqq a + z_5(a, r)$, $A \in \mathscr{A}$ be given. Then for $(F_A(t, a)\, x, t) \in$ domain v we have

$$f(\|F_A(t, a)\, x\|) \leqq v(F_A(t, a)\, x, t) \leqq k_0 \leqq f(k_2)$$

and thus $\|F_A(t, a)\, x\| \leqq k_2$. Hence the controllable system of equations (1) is asymptotically bounded with respect to the set $\boldsymbol{o}$.

Theorem 12. *If the controllable system of equations* (1) *is uniformly bounded with respect to the set* $\boldsymbol{o}$ *and if* $\lim_{t \to \infty} \|F_A(t, a)\, x\| = 0$ *for any* $(x, a, A) \in \boldsymbol{R}^n + \boldsymbol{I} \times \mathscr{A}$, *then the controllable system of equations* (1) *is asymptotically bounded with respect to the set* $\boldsymbol{o}$, *i.e. the set* $\boldsymbol{o}$ *is asymptotically stable with respect to the controllable system of equations* (1).

Proof. Choose $r \in \boldsymbol{R}^+$ and define the relation $\boldsymbol{L}: \boldsymbol{I} \to \{\boldsymbol{R}^n\}$ by putting

$$\boldsymbol{L}(a) = \{x \in \boldsymbol{R}^n: \limsup_{t \to \infty} \sup \{\|F_A(t, a)\, x\|: A \in \mathscr{A}\} \leqq r/2\}$$

and show that this relation has the following properties:

(i) $a \in \boldsymbol{I}$, $b \in \boldsymbol{I}$, $a \leqq b$, $x \in \boldsymbol{L}(a)$, $A \in \mathscr{A} \Rightarrow F_A(b, a)\, x \in \boldsymbol{L}(b)$;

(ii) for each $a \in \boldsymbol{I}$ the set $\boldsymbol{L}(a)$ is closed;

(iii) if $K \subset \boldsymbol{R}^n$ is a compact set, then there exists a mapping $\bar{z}_4: \boldsymbol{I} \times \boldsymbol{R}^+ \to \boldsymbol{R}^m_+$ such that the following implication holds:

$$x \in \boldsymbol{K} \cap \boldsymbol{L}(a)\,, \quad t \geqq a + \bar{z}_4(a, \eta)\,, \quad A \in \mathscr{A} \Rightarrow \|F_A(t, a)\, x\| < r/2 + \eta\;;$$

(iv) if $F_A(b, a)\, x \in \boldsymbol{L}(b)$ for each $A \in \mathscr{A}$, then $x \in \boldsymbol{L}(a)$;

(v) $a \in \boldsymbol{I} \Rightarrow \boldsymbol{L}(a) = \boldsymbol{R}^n$.

Ad (i). The property (i) is evident from the definition of the relation $\boldsymbol{L}$.

Ad (ii). Let a sequence $\{{}^j x: j = 1, 2, \ldots\} \subset \boldsymbol{L}(a)$ having $\lim_{j \to \infty} {}^j x = x$ and a number $\psi \in \boldsymbol{R}^+$ be given. Then there exists a positive integer $j_0(\psi)$ such that for all positive integers $j > j_0(\psi)$ we have $\|{}^j x - x\| \leqq z_1(\psi)$, where z_1 is the mapping from (6). Then

$$\limsup_{t \to \infty} \sup \{\|F_A(t, a)\, x\|: A \in \mathscr{A}\} \leqq \limsup_{t \to \infty} \sup \{\|F_A(t, a)\, x - F_A(t, a)\, {}^j x\|: A \in \mathscr{A}\} + \limsup_{t \to \infty} \sup \{\|F_A(t, a)\, {}^j x\|: A \in \mathscr{A}\} \leqq \psi + r/2\,.$$

As $\psi \in \boldsymbol{R}^+$ is arbitrary, this implies the inequality

$$\limsup_{t \to \infty} \sup \{\|F_A(t, a)\, x\|: A \in \mathscr{A}\} \leqq r/2$$

and thus $x \in \boldsymbol{L}(a)$. Thus the property (ii) is proved.

Ad (iii). Let $a \in \boldsymbol{I}$, $t \in \boldsymbol{I}$, $t \geqq a$, $A \in \mathscr{A}$ be given. If we denote the norm of the linear mapping $F_A(t, a, \cdot): \boldsymbol{R}^n \to \boldsymbol{R}^n$ by the symbol $\|F_A(t, a)\|$, then $\|F_A(t, a)\| \leqq z_2(1)$, where z_2 is the mapping from (7). Choose $\eta \in \boldsymbol{R}^+$ and denote

$$\boldsymbol{S}(a, \eta) = \{s \in \boldsymbol{R}^n: \inf \{\|s - x\|: x \in \boldsymbol{L}(a)\} < \eta/(2z_2(1))\} .$$

Then for each $s \in \boldsymbol{S}(a, \eta)$, $A \in \mathscr{A}$ there exists $x \in \boldsymbol{L}(a)$ such that

$$\|F_A(t, a)\, s\| \leqq \|F_A(t, a)\,(s - x)\| + \|F_A(t, a)\, x\| \leqq z_2(1)\, \eta/(2z_2(1)) + \|F_A(t, a)\, x\| ,$$

and thus

$$\limsup_{t \to \infty} \sup \{\|F_A(t, a)\, s\|: A \in \mathscr{A}\} \leqq \eta/2 + r/2 . \tag{12}$$

Let $\{\boldsymbol{G}_u: u \in \boldsymbol{B}\}$ be such a system of open $\boldsymbol{G}_u \subset \boldsymbol{R}^n$ that $\boldsymbol{K} \cap \boldsymbol{L}(a) \subset \bigcup_{u \in \boldsymbol{B}} \boldsymbol{G}_u$ and for each $u \in \boldsymbol{B}$ there exists a set $\{{}^j s: j = 0, 1. 2, \ldots, n\} \subset \boldsymbol{S}(a, \eta)$ such that

$$\boldsymbol{G}_u = \{x \in \boldsymbol{R}^n: x = \sum_{j=0}^{n} {}^j r\, {}^j s, \sum_{j=0}^{n} {}^j r = \sum_{j=0}^{n} |{}^j r| < 1\} .$$

As $\boldsymbol{K} \cap \boldsymbol{L}(a)$ is a compact set, there exists a finite subset $\boldsymbol{B}_0 \subset \boldsymbol{B}$ such that

$$\boldsymbol{K} \cap \boldsymbol{L}(a) \subset \bigcup_{u \in \boldsymbol{B}_0} \boldsymbol{G}_u .$$

Let $u \in \boldsymbol{B}_0$ and let the corresponding sequences be $\{{}^j s: j = 0, 1, 2, \ldots, n\} \subset \boldsymbol{S}(a, \eta)$. Then for each ${}^j s$ according to (12) there exists a mapping ${}^u\bar{z}_j: \boldsymbol{I} \times \boldsymbol{R}^+ \to \boldsymbol{R}^+$ such that for all $t_h \geqq a_h + {}^u\bar{z}_j(a, \eta)$, $h = 1, 2, \ldots, m$ we have $\sup \{\|F_A(t, a)\, {}^j s\|: A \in \mathscr{A}\} < r/2 + \eta$. Put ${}^u\bar{z}(a, \eta) = \max \{{}^u\bar{z}_j(a, \eta): j = 0, 1, 2, \ldots, n\}$. Then for $t_h \geqq a_h + {}^u\bar{z}(a, \eta)$, $h = 1, 2, \ldots, m$, $x \in \boldsymbol{G}_u$, $A \in \mathscr{A}$, the inequality

$$\|F_A(t, a)\, x\| = \|F_A(t, a) \sum_{j=0}^{n} {}^j r\, {}^j s\| \leqq \sum_{j=0}^{n} {}^j r \|F_A(t, a)\, {}^j s\| < r/2 + \eta \text{ holds} .$$

If we define the mapping $\bar{z}_4: \boldsymbol{I} \times \boldsymbol{R}^+ \to \boldsymbol{R}^m_+$ by putting

$$\bar{z}_{4h}(a, \eta) = \max \{{}^u\bar{z}(a, \eta): u \in \boldsymbol{B}_0\} , \quad h = 1, 2, \ldots, m ,$$

we obtain that the property (iii) is fulfilled.

Ad (iv). Let the assumption of (iv) be fulfilled. By $\boldsymbol{K}$ denote the closure of the set $\{y \in \boldsymbol{R}^n: y = F_A(b, a)\, x$ for some $A \in \mathscr{A}\}$. According to the property (ii) evidently $\boldsymbol{K} \subset \boldsymbol{L}(b)$ and $\sup \{\|y\|: y \in \boldsymbol{K}\} \leqq z_2(1)\, \|x\|$, where z_2 is the mapping from (7). Therefore the set $\boldsymbol{K}$ is compact and from the property (iii) we obtain

$$\sup \{\|F_A(t, b)\, y\|: A \in \mathscr{A}, y \in \boldsymbol{K}, t \geqq b + \bar{z}_4(b, \eta)\} < r/2 + \eta .$$

Evidently, for each $\eta \in \boldsymbol{R}^+$ we have

$$\sup \{\|F_A(t, a)\, x\|: A \in \mathscr{A}, t \geqq b + \bar{z}_4(b, \eta)\} =$$
$$= \sup \{\|F_A(t, b)\, F_A(b, a)\, x\|: A \in \mathscr{A}, t \geqq b + \bar{z}_4(b, \eta)\} < r/2 + \eta ,$$

therefore $x \in \boldsymbol{L}(a)$. Thus the property (iv) is proved.

Ad (v). Suppose that there exists ${}^0a \in \boldsymbol{I}$ such that $\boldsymbol{L}({}^0a) \neq \boldsymbol{R}^n$. Choose ${}^0x \in \boldsymbol{R}^n - \boldsymbol{L}({}^0a)$. The property (iv) implies that there exists an increasing sequence $\{{}^{j-1}a: j = 1, 2, \ldots\} \subset \boldsymbol{I}$, a sequence $\{A_j: j = 1, 2, \ldots\} \subset \mathscr{A}$ and a sequence $\{{}^jx: j = 1, 2, \ldots\} \subset \boldsymbol{R}^n$ such that ${}^jx = F_{A_j}({}^ja, {}^{j-1}a)\, {}^{j-1}x \notin \boldsymbol{L}({}^ja)$ for each $j = 1, 2, \ldots$. If we define $A \in \mathscr{A}$ in such a way that $F_A(t, {}^{j-1}a)\, {}^{j-1}x = F_{A_j}(t, {}^{j-1}a)\, {}^{j-1}x$ for $t \in \langle {}^{j-1}a, \infty) - \langle {}^ja, \infty)$, then $\limsup_{t \to \infty} \|F_A(t, {}^0a)\, {}^0x\| \geqq z_1(r/2)$, where z_1 is the mapping from (6). But this is a contradiction with the assumption of the theorem. Thus the property (v) is proved.

Now choose $k_1 \in \boldsymbol{R}^+$ and denote $\boldsymbol{K} = \{x \in \boldsymbol{R}^n: \|x\| \leqq k_1\}$. The set $\boldsymbol{K}$ is evidently compact in $\boldsymbol{R}^n$. Define the mapping $z_4: \boldsymbol{I} \times \boldsymbol{R}^+ \to \boldsymbol{R}^m_+$ by putting $z_4(a, r) = \bar{z}_4(a, r/2)$, where $\bar{z}_4$ is the mapping from (iii). Then according to (iii) and (v), we have $\|F_A(t, a) \,.\, x\| \leqq r$ for each $(t, x, a) \in \boldsymbol{D}$, $\|x\| \leqq k_1$, $t \geqq a + z_4(a, r)$, $A \in \mathscr{A}$. Thus the theorem is proved.

Definition. We say that the set $\boldsymbol{o}$ is *uniformly asymptotically stable* with respect to the controllable system of equations (1) if and only if it is uniformly stable with respect to the controllable system of equations (1) and there exists a constant $k_1 \in \boldsymbol{R}^+$ and a mapping $z_4: \boldsymbol{R}^+ \to \boldsymbol{R}^m_+$ such that the following implication holds:

$$(13) \quad (t, x, a) \in \boldsymbol{D}\,, \quad \|x\| \leqq k_1\,, \quad t \geqq a + z_4(r)\,, \quad A \in \mathscr{A} \Rightarrow \|F_A(t, a)\, x\| \leqq r\,.$$

We say that the controllable system of equations (1) is *uniformly asymptotically bounded* with respect to the set $\boldsymbol{o}$ if and only if the controllable system of equations (1) is uniformly bounded with respect to the set $\boldsymbol{o}$ and there exists a constant $k_2 \in \boldsymbol{R}^+$ and a mapping $z_5: \boldsymbol{R}^+ \to \boldsymbol{R}^m_+$ such that the following implication holds:

$$(14) \quad (t, x, a) \in \boldsymbol{D}\,, \quad \|x\| \leqq r, t \geqq a + z_5(r)\,, \quad A \in \mathscr{A} \Rightarrow \|F_A(t, a)\, x\| \leqq k_2\,.$$

Theorem 13. *The set* $\boldsymbol{o}$ *uniformly asymptotically stable with respect to the controllable system of equations* (1) *if and only if the controllable system of equations* (1) *is uniformly asymptotically bounded with respect to the set* $\boldsymbol{o}$.

The proof follows from the proof of Theorem 8 and from the fact that the mappings z_4 and z_5 in (13) and (14) do not depend on a.

Theorem 14. *The controllable system of equations* (1) *is uniformly asymptotically bounded with respect to the set* $\boldsymbol{o}$ *if and only* if *there exist a constant* $k \in \boldsymbol{R}^+$, *a partial mapping* $v: \boldsymbol{R}^n \times \boldsymbol{I} \to \boldsymbol{R}^+$, *a partial increasing mapping* $f: \boldsymbol{R}^+ \to \boldsymbol{R}^+$, $f(r) \to +\infty$ *for* $r \to +\infty$ *and a partial mapping* $z_0: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ *with the properties* (i), (ii) *from Theorem* 6, *a constant* $k_0 \in \boldsymbol{R}^+$ *and a mapping* $z_6: \boldsymbol{R}^+ \to \boldsymbol{R}^m_+$ *with the property*:

$$\text{(iii)} \qquad \|x\| \leqq r, t \geqq a + z_6(r)\,, \quad A \in \mathscr{A},\,, \quad (F_A(t, a)\, x, t) \in \text{domain } v \Rightarrow$$
$$\Rightarrow v(F_A(t, a)\, x, t) \leqq k_0\,.$$

The proof is evident, because Theorem 14 is a uniform modification of Theorem 11.

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Souhrn

REGULOVATELNÉ SOUSTAVY PARCIÁLNÍCH DIFERENCIÁLNÍCH ROVNIC

František Tumajer

V článku jsou definovány různé druhy stability a omezenosti řešení lineárních regulovatelných soustav parciálních diferenciálních rovnic a jsou odvozeny vztahy mezi nimi. Pomocí ljapunovských funkcí jsou formulovány nutné a postačující podmínky pro speciální druhy stability a omezenosti řešení.

Резюме

РЕГУЛИРУЕМЫЕ СИСТЕМЫ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ В ЧАСТНЫХ ПРОИЗВОДНЫХ

František Tumajer

В статье даны определения различных типов устойчивости и ограниченности решений линейных регулируемых систем дифференциальных уравнений в частных производных и найдены соотношения между ними. При помощи ляпуновских функций доказаны теоремы, которые дают необходимые и достаточные условия для этих типов устойчивости и ограниченности решений.

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