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ON DIFFERENCE LINEAR PERIODIC SYSTEMS II. NON-HOMOGENEOUS CASE

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1. INTRODUCTION

Consider the linear homogeneous system of equations in finite differences

$$x(n+1) = A(n)\,x(n)\,, \quad n \in \mathbb{N}\,, \tag{1.1}$$

where $A(n)$ is an N-periodic matrix sequence of order q with values in $\mathbb{C}$:

$$A\colon \mathbb{N} \to M_q(\mathbb{C})\,, \quad A(n+N) = A(n) \quad \text{for all} \quad n \in \mathbb{N}$$

and $x(n)$ is a vector sequence with complex values.

In [3] we used a method to obtain solutions of (1.1) in closed form based on the reduction of this system to N linear homogeneous systems with constant coefficients.

In this paper we use a similar method for the resolution in closed form of the non-homogeneous system.

Adjoint system

According to Halanay's definition ([2]), the system

$$\xi(n-1) = \xi(n)\,A(n-1)\,, \quad n \in \mathbb{N}^* \tag{1.2}$$

where $\xi(n)$ is a row q-vector, is the adjoint system of the system (1.1).

If $x(n)$ and $\xi(n)$ are solutions of (1.1) and (1.2), respectively, then $\xi(n)\,x(n) = \xi(0)\,x(0)$ for all $n \in \mathbb{N}$, and in consequently ([3])

$$\xi(n)\,\Big[\prod_{j=s-1}^{0} A(j)\Big]\Big[\prod_{j=N-1}^{0} A(j)\Big]^k = \xi(0)\,, \quad k \in \mathbb{N}\,, \quad (s = 0, 1, \dots, N-1)$$

where $\prod_{j=s-1}^{0} A(j) := A(s-1)\,A(s-2)\dots A(0)$ and $\prod_{j=-1}^{0} A(j) := E$, the identity matrix.

Now, as a solution $\xi(n)$ of (1.2) is N-periodic if and only if $\xi(N) = \xi(0)$, $\xi(n)$ is N-periodic if and only if

$$\xi(N)\left(\left[\prod_{j=N-1}^{0} A(j)\right] - E\right) = 0\,. \tag{1.3}$$

Thus, the dimension of the subspace of N-periodic solutions of (1.2) coincides with that of the subspace of N-periodic solutions of (1.1).

2. NON-HOMOGENEOUS SYSTEM

Let

$$x(n + 1) = A(n)\,x(n) + f(n)\,, \quad n \in \mathbb{N}. \tag{2.1}$$

be a non-homogeneous linear difference N-periodic system, where $A(n)$ is as above and $f(n)$ is a N-periodic column q-vector with complex values

$$f\colon \mathbb{N} \to \mathbb{C}^q\,, \quad f(n + N) = f(n) \quad \text{for all} \quad n \in \mathbb{N}\,.$$

Theorem 1. *Every term of each solution sequence $x(n)$ of (2.1) can be expressed in the form*

$$x(n) = \left[\prod_{j=s-1}^{0} A(j)\right]\left[\prod_{j=N-1}^{0} A(j)\right]^k x(0) + \tag{2.2}$$

$$+ \left[\prod_{j=s-1}^{0} A(j)\right]\left(\sum_{r=0}^{k-1}\left[\prod_{j=N-1}^{0} A(j)\right]^r\right)\left(\sum_{m=0}^{N-1}\left[\prod_{j=N-1}^{m+1} A(j)\right] f(m)\right) + \sum_{m=0}^{s-1}\left[\prod_{j=s-1}^{m} A(j)\right] f(m)\,,$$

where k and s are, respectively, the quotient and the rest of the division of n by N, for each $n \in \mathbb{N}$, provided we put

$$\sum_{i=0}^{-1} := 0 \quad \text{and} \quad \prod_{j=i-1}^{i} A(j) := E \quad (i \in \mathbb{N})\,.$$

Proof. We will see that (2.2) is a solution of the system (2.1) that verifies the initial condition $x(0)$; and so it is the only one with that initial condition.

Let us suppose that $x(n)$ is given by (2.2), then $x(n + 1)$ has two different expressions for $s = N - 1$ or $0 \leqq s < N - 1$:

i) $s = N - 1$. Then $n + 1 = (k + 1)\,N$.

In this case,

$$x(n+1) = \left[\prod_{j=N-1}^{0} A(j)\right]^{k+1} x(0) + \left(\sum_{r=0}^{k}\left[\prod_{j=N-1}^{0} A(j)\right]^r\right)\left(\sum_{m=0}^{N-1}\left[\prod_{j=N-1}^{m+1} A(j)\right] f(m)\right), \quad k \in \mathbb{N}$$

and

$$x(n) = \left(\prod_{j=N-2}^{0} A(j)\right]\left[\prod_{j=N-1}^{0} A(j)\right]^k x(0) + \left[\prod_{j=N-2}^{0} A(j)\right]\left(\sum_{r=0}^{k-1}\left[\prod_{j=N-1}^{0} A(j)\right]^r\right)$$

$$(\sum_{m=0}^{N-1} [\prod_{j=N-1}^{m+1} A(j)] f(m)) + \sum_{m=0}^{N-2} [\prod_{j=N-2}^{m+1} A(j)] f(m)\,.$$

Since $A(n) = A(N-1)$ and $f(n) = f(N-1)$, we have

$$A(n)\,x(n) + f(n) = A(N-1)\,x(n) + f(N-1) =$$

$$= [\prod_{j=N-1}^{0} A(j)]^{k+1} x(0) + [\prod_{j=N-1}^{0} A(j)] (\sum_{r=0}^{k-1} [\prod_{j=N-1}^{0} A(j)]^r) (\sum_{m=0}^{N-1} [\prod_{j=N-1}^{m+1} A(j)] f(m)) +$$

$$+ A(N-1) \sum_{m=0}^{N-2} [\prod_{j=N-2}^{m+1} A(j)] f(m) + f(N-1) =$$

$$= [\prod_{j=N-1}^{0} A(j)]^{k+1} x(0) + (\sum_{r=0}^{k} [\prod_{j=N-1}^{0} A(j)]^r) (\sum_{m=0}^{N-1} [\prod_{j=N-1}^{m+1} A(j)] f(m))\,.$$

So in this case $x(n+1) = A(n)\,x(n) + f(n)$.

ii) $0 \leqq s < N-1$. In this case the development is similar, leading to the same conclusion.

Moreover, because of the established conventions, in (2.2) $x(n)|_{n=0} = x(0)$. So, the sequence $x(n)$ given by (2.2) is the only solution of (2.1) that takes the value $x(0)$ when $n = 0$. ■

We can observe that in (2.2) the first addend of the second member of the equality is the solution of the homogeneous system associated with (2.1) and that the two last addends are a particular solution of (2.1).

Theorem 2. *Let us consider the initial value problem*

$$\text{(2.3)} \qquad \begin{cases} x(n+1) = A(n)\,x(n) + f(n)\,, & n \in \mathbb{N}\,, \\ x(0) = x^0\,, \end{cases}$$

where $A(n)$ and $f(n)$ are N-periodic. The sequence $x(n)$ is a solution of (2.3) if and only if any subsequence of $x(n)$ in the form $x(kN+s)$, $(k = 0, 1, 2, \ldots)$ is a solution for $s = 0, 1, 2, \ldots, N-1$, respectively, of the initial value problem with constant coefficients

$$(\mathrm{P}_s)\begin{cases} z^s(k+1) = [\prod_{j=s-1}^{0} A(j)] [\prod_{j=N-1}^{s} A(j)]\, z^s(k) + [\prod_{j=s-1}^{0} A(j)] \sum_{m=s}^{N-1} [\prod_{j=N-1}^{m+1} A(j)] f(m) + \\ \qquad + \sum_{m=0}^{s-1} [\prod_{j=s-1}^{m+1} A(j)] f(m)\,, \quad k \in \mathbb{N}\,; \\ z^s(0) \quad = [\prod_{j=s-1}^{0} A(j)]\, x^0 + \sum_{m=0}^{s-1} [\prod_{j=s-1}^{m+1} A(j)] f(m)\,, \end{cases}$$

where $z^s(k)$ is for each s fixed, by definition, the subsequence $(x(kN+s))_{k=0,1,2,\ldots}$.

Proof. *Necessary condition.* If $x(n)$ is a solution of (2.3), it verifies (2.2). For fixed s and k, $z^s(k)$ and $z^s(k+1)$ are two terms of $x(n)$ and thus verify also (2.2) for k and $k+1$, respectively.

Then

$$z^s(k) = [\prod_{j=s-1}^{0} A(j)] [\prod_{j=N-1}^{0} A(j)]^k x^0 + [\prod_{j=s-1}^{0} A(j)] (\sum_{r=0}^{k-1} [\prod_{j=N-1}^{0} A(j)]^r)$$
$$(\sum_{m=0}^{N-1} [\prod_{j=N-1}^{m+1} A(j)] f(m)) + \sum_{m=0}^{s-1} [\prod_{j=s-1}^{m+1} A(j)] f(m).$$

We multiply both sides of the equality by $[\prod_{j=s-1}^{0} A(j)] [\prod_{j=N-1}^{s} A(j)]$; in the second addend we add and we subtract $[\prod_{j=s-1}^{0} A(j)] \sum_{m=0}^{N-1} [\prod_{j=N-1}^{m+1} A(j)] f(m)$. Then we add to both sides the expression $\sum_{m=0}^{s-1} [\prod_{j=s-1}^{m+1} A(j)] f(m)$ and after rearranging the equality the convenient terms, we obtain

$$[\prod_{j=s-1}^{0} A(j)] [\prod_{j=N-1}^{s} A(j)] z^s(k) + [\prod_{j=s-1}^{0} A(j)] \sum_{m=0}^{N-1} [\prod_{j=N-1}^{m+1} A(j)] f(m) -$$
$$- [\prod_{j=s-1}^{0} A(j)] \sum_{m=0}^{s-1} [\prod_{j=N-1}^{m+1} A(j)] f(m) + \sum_{m=0}^{s-1} [\prod_{j=s-1}^{m+1} A(j)] f(m) z^s(k+1).$$

Taking out the common factor in the second and third addend on the left we conclude that

$$z^s(k+1) = [\prod_{j=s-1}^{0} A(j)] [\prod_{j=N-1}^{s} A(j)] z^s(k) +$$
$$+ [\prod_{j=s-1}^{0} A(j)] \sum_{m=s}^{N-1} [\prod_{j=N-1}^{m+1} A(j)] f(m) + \sum_{m=0}^{s-1} [\prod_{j=s-1}^{m+1} A(j)] f(m).$$

Moreover, by definition, $z^s(0) = x(s)$ $(s = 0, 1, \ldots, N-1)$. Hence according to Theorem 1

$$z^s(0) = [\prod_{j=s-1}^{0} A(j)] x^0 + \sum_{m=0}^{s-1} [\prod_{j=s-1}^{m+1} A(j)] f(m).$$

Sufficient condition. Each of the problems (P_s) $(s = 0, 1, \ldots, N-1)$ is a initial value problem with constant coefficients and constant independent term. A formula of Bellman says that if $X(n)$ is a fundamental matrix of the homogeneous system, every solution $x(n)$ of the non-homogeneous system has the representation

$$x(n) = X(n) x(0) + \sum_{m=0}^{n-1} X(n) X^{-1}(m+1) f(m).$$

This formula gives the solution of the problems (P_s)

$$z^s(k) = ([\prod_{j=s-1}^{0} A(j)] [\prod_{j=N-1}^{s} A(j)])^k z^s(0) + \sum_{r=0}^{k-1} ([\prod_{j=s-1}^{0} A(j)] [\prod_{j=N-1}^{s} A(j)])^{k-r-1}$$
$$([\prod_{j=s-1}^{0} A(j)] \sum_{m=s}^{N-1} [\prod_{j=N-1}^{m+1} A(j)] f(m) + \sum_{m=0}^{s-1} [\prod_{j=s-1}^{m+1} A(j)] f(m)).$$

Now, let $x(n)$ be a sequence determinated by its N "interlaced" subsequences $x(kN + s) := z^s(k)$, $k \in \mathbb{N}$ $(s = 0, 1, \ldots, N - 1)$.

Let us verify that $x(n)$ has the form (2.2).

In the first place, taking into account the expression of the initial conditions given in (P_s),

$$[[\prod_{j=s-1}^{0} A(j)]\,[\prod_{j=N-1}^{s} A(j)]]^k\, z^s(0) \quad \text{gives rise to two addendes}$$

$$(\mathrm{s}_1)\quad [[\prod_{j=s-1}^{0} A(j)]\,[\prod_{j=N-1}^{s} A(j)]]^k\,[\prod_{j=s-1}^{0} A(j)]\, x^0 = [\prod_{j=s-1}^{0} A(j)]\,[\prod_{j=N-1}^{0} A(j)]^k\, x^0\,,$$

$$(\mathrm{s}_2)\quad [[\prod_{j=s-1}^{0} A(j)]\,[\prod_{j=N-1}^{s} A(j)]]^k \sum_{m=0}^{s-1}[\prod_{j=s-1}^{m+1} A(j)]\, f(m) =$$

$$= [\prod_{j=s-1}^{0} A(j)]\,[\prod_{j=N-1}^{0} A(j)]^{k-1}\,[\prod_{j=N-1}^{s} A(j)] \sum_{m=0}^{s-1}[\prod_{j=s-1}^{m+1} A(j)]\, f(m) =$$

$$= [\prod_{j=s-1}^{0} A(j)]\,[\prod_{j=N-1}^{0} A(j)]^{k-1} \sum_{m=0}^{s-1}[\prod_{j=N-1}^{m+1} A(j)]\, f(m)\,.$$

On the other hand,

$$\sum_{r=0}^{k-1}[[\prod_{j=s-1}^{0} A(j)]\,[\prod_{j=N-1}^{s} A(j)]]^{k-r-1} =$$

$$= [\prod_{j=s-1}^{0} A(j)]\,(\sum_{r=0}^{k-2}[\prod_{j=N-1}^{0} A(j)]^r)\,[\prod_{j=N-1}^{s} A(j)] + E\,,$$

which multiplied by

$$[\prod_{j=s-1}^{0} A(j)] \sum_{m=s}^{N-1}[\prod_{j=N-1}^{m+1} A(j)]\, f(m) + \sum_{m=0}^{s-1}[\prod_{j=s-1}^{m+1} A(j)]\, f(m)$$

gives rise to other four addends

$$(\mathrm{s}_3)\quad [\prod_{j=s-1}^{0} A(j)] \sum_{m=s}^{N-1}[\prod_{j=N-1}^{m+1} A(j)]\, f(m)\,,$$

$$(\mathrm{s}_4)\quad \sum_{m=0}^{s-1}[\prod_{j=s-1}^{m+1} A(j)]\, f(m)\,,$$

$$(\mathrm{s}_5)\quad [\prod_{j=s-1}^{0} A(j)]\,(\sum_{r=0}^{k-2}[\prod_{j=N-1}^{0} A(j)]^r)\,[\prod_{j=N-1}^{s} A(j)]\,[\prod_{j=s-1}^{0} A(j)] \sum_{m=s}^{N-1}[\prod_{j=N-1}^{m+1} A(j)]\, f(m) =$$

$$= [\prod_{j=s-1}^{0} A(j)]\,(\sum_{r=1}^{k-1}[\prod_{j=N-1}^{0} A(j)]^r) \sum_{m=s}^{N-1}[\prod_{j=N-1}^{m+1} A(j)]\, f(m)\,,$$

$$(\mathrm{s}_6)\quad [\prod_{j=s-1}^{0} A(j)]\,(\sum_{r=0}^{k-2}[\prod_{j=N-1}^{0} A(j)]^r)\,[\prod_{j=N-1}^{s} A(j)] \sum_{m=0}^{s-1}[\prod_{j=s-1}^{m+1} A(j)]\, f(m) =$$

$$= [\prod_{j=s-1}^{0} A(j)]\,(\sum_{r=0}^{k-2}[\prod_{j=N-1}^{0} A(j)]^r) \sum_{m=0}^{s-1}[\prod_{j=N-1}^{m+1} A(j)]\, f(m)\,.$$

Adding (s_6) and (s_2) we obtain

$$(s_7) \quad [\prod_{j=s-1}^{0} A(j)]\,(\sum_{r=0}^{k-1}[\prod_{j=N-1}^{0} A(j)]^r)\sum_{m=0}^{s-1}[\prod_{j=N-1}^{m+1} A(j)]\,f(m)\,.$$

Adding (s_5) and (s_3), we have

$$(s_8) \quad [\prod_{j=s-1}^{0} A(j)]\,(\sum_{r=0}^{k-1}[\prod_{j=N-1}^{0} A(j)]^r)\sum_{m=s}^{N-1}[\prod_{j=N-1}^{m+1} A(j)]\,f(m)\,.$$

Adding (s_7) and (s_8) we obtain

$$(s_9) \quad [\prod_{j=s-1}^{0} A(j)]\,(\sum_{r=0}^{k-1}[\prod_{j=N-1}^{0} A(j)]^r)\sum_{m=0}^{N-1}[\prod_{j=N-1}^{m+1} A(j)]\,f(m)\,.$$

Finally, as $z^s(k)$ is the sum of (s_1), (s_4) and (s_9) we arrive to

$$z^s(k) = [\prod_{j=s-1}^{0} A(j)]\,[\prod_{j=N-1}^{0} A(j)]^k\, x^0 + [\prod_{j=s-1}^{0} A(j)]\,(\sum_{r=0}^{k-1}[\prod_{j=N-1}^{0} A(j)]^r)$$
$$(\sum_{m=0}^{N-1}[\prod_{j=N-1}^{m+1} A(j)]\,f(m)) + \sum_{m=0}^{s-1}[\prod_{j=s-1}^{m+1} A(j)]\,f(m)\,.$$

Moreover, this sequence $x(n)$ verifies

$$x(0) = z^0(0) = [\prod_{j=-1}^{0} A(j)]x^0 + \sum_{m=0}^{-1}[\prod_{j=-1}^{0} A(j)]\,f(m) = x^0\,;$$

so, it is a solution of the problem (2.3). ■

3. COMMENTS TO THEOREM 2

The importance of Theorem 2 is similar to that of Theorem 1 of [3] for the homogeneous case: both of them guarantee that it is possible to obtain explicitly the solution of an N-periodic finite difference system.

It is not necessary to resolve the N problems (P_s) given in Theorem 2; it is enough to solve (P_0). Once having resolved this problem and obtained $z^0(k)$, we calculate the remaining $N-1$ "interlaced" subsequences, which conform the solution $x(n)$, by the formula

$$(3.1) \quad z^s(k) = [\prod_{j=s-1}^{0} A(j)]\, z^0(k) + \sum_{m=0}^{s-1}[\prod_{j=s-1}^{m+1} A(j)]\,f(m)\,, \quad k \in \mathbb{N} \quad (s = 1, \ldots, N-1)\,.$$

Moreover, since the solution of (P_0), $z^0(k)$, is

$$z^0(k) = [\prod_{j=N-1}^{0} A(j)]^k\, x^0 + \sum_{r=0}^{k-1}[\prod_{j=N-1}^{0} A(j)]^r \sum_{m=0}^{N-1}[\prod_{j=N-1}^{m+1} A(j)]\,f(m)\,,$$

it is not necessary to require that the N-periodic matrix $A(n)$ be regular for all $n \in \mathbb{N}$.

4. PERIODIC SOLUTIONS

A solution $x(n)$ of (2.1) is N-periodic if and only if $x(N) = x(0)$, but as $x(N) = z^0(1)$ and $x(0) = z^0(0)$, a solution $x(n)$ of (2.1) is N-periodic if and only if its subsequence $z^0(k)$ is constant.

By extension, a solution $x(n)$ of (2.1) is Np-periodic if and only if its subsequence $z^0(k)$ is p-periodic.

Theorem 3. *The non-homogeneous and N-periodic system* (2.1) *has, at least, an N-periodic solution if and only if for all N-periodic solution* $\xi(n)$ *of the adjoint system associated with the homogeneous one of* (2.1), *we have*

$$\sum_{m=0}^{N-1} \xi(m + 1) f(m) = 0 .$$

Proof. To begin with,

$$x(N) = [\prod_{j=N-1}^{0} A(j)]\, x(0) + \sum_{m=0}^{N-1} [\prod_{j=N-1}^{m+1} A(j)]\, f(m) .$$

A solution $x(n)$ of (2.1) is N-periodic if and only if

$$([\prod_{j=N-1}^{0} A(j)] - E)\, x(0) = - \sum_{m=0}^{N-1} [\prod_{j=N-1}^{m+1} A(j)]\, f(m) . \tag{4.1}$$

For the sake of convenience, we denote

$$B := [\prod_{j=N-1}^{0} A(j)] - E ,$$

$$b := \sum_{m=0}^{N-1} [\prod_{j=N-1}^{m+1} A(j)]\, f(m) .$$

Then (4.1) becomes

$$B\, x(0) = -b .$$

Now, this system has a solution if and only if

$$eb = 0$$

for each row vector e such that $eB = 0$.

But, according to (1.3), the row vectors e which satisfy this condition are the q-vector $\xi(N)$ that gives rise to the N-periodic solutions of the adjoint system of the homogeneous one of (2.1).

We write the condition $eb = 0$ in the form

$$e \sum_{m=0}^{N-1} [\prod_{j=N-1}^{m+1} A(j)]\, f(m) = 0 ;$$

that is,

$$\sum_{m=0}^{N-1} e[\prod_{j=N-1}^{m+1} A(j)] f(m) = 0 .$$

As

$$e[\prod_{j=N-1}^{m+1} A(j)] = \xi(m + 1) ,$$

we conclude that

$$\sum_{m=0}^{N-1} \xi(m + 1) f(m) = 0$$

for all N-periodic solution $\xi(n)$ of the adjoint system. ∎

The next theorem expresses an existence condition for N-periodic solutions of a non-homogeneous system. This theorem could be also called the "resonance phenomenon".

Theorem 4. *If the non-homogeneous and N-periodic system (2.1) does not admit N-periodic solutions, then none of its solutions are bounded.*

Proof. If the system (2.1) does not admit N-periodic solutions, the system

$$z^0(k + 1) = [\prod_{j=N-1}^{0} A(j)] z^0(k) + \sum_{m=0}^{N-1} [\prod_{j=N-1}^{m+1} A(j)] f(m) \tag{4.2}$$

does not admit constant solutions.

We will see that in these circumstances all the solutions of (4.2) are unbounded. We will adopt the notation of Theorem 3.

The adjoint system to the homogeneous one of (4.2) is

$$\xi^0(k - 1) = \xi^0(k) \prod_{j=N-1}^{0} A(j) . \tag{4.3}$$

If (4.2) does not admit constant solutions then, by Theorem 3, there exists a constant solution $\xi(k)$ of (4.3) such that

$$\xi(1)\, b \neq 0 .$$

Let $z^0(k)$ be a solution of (4.2), we have

$$z^0(1) = [\prod_{j=N-1}^{0} A(j)] z^0(0) + b .$$

Multipling by the row q-vector $\xi(1)$ yields

$$\xi(1)\, z^0(1) = \xi(1) [\prod_{j=N-1}^{0} A(j)] z^0(0) + \xi(1)\, b ,$$

that is

$$\xi(1)\, z^0(1) = \xi(0)\, z^0(0) + \xi(1)\, b . \tag{4.4}$$

On the other hand,

$$z^0(2) = [\prod_{j=N-1}^{0} A(j)]\, z^0(1) + b\,.$$

Hence

$$\xi(2)\, z^0(2) = \xi(2) [\prod_{j=N-1}^{0} A(j)]\, z^0(1) + \xi(2)\, b\,.$$

As $\xi(k)$ is constant, we conclude that

$$\xi(1)\, z^0(2) = \xi(1)\, z^0(1) + \xi(1)\, b$$

and, by (4.4),

$$\xi(1)\, z^0(2) = \xi(0)\, z^0(0) + 2\, \xi(1)\, b\,.$$

By induction we demonstrate that

$$\xi(1)\, z^0(k) = \xi(0)\, z^0(0) + k\, \xi(1)\, b$$

and as $\xi(1)\, b \neq 0$, it follows that $z^0(k)$ is unbounded for every be the initial condition $z^0(0)$.

Finally, as $z^0(k)$ is the subsequence $x(kN)$ of each solution sequence $x(n)$ of (4.3), no one solution sequence of (2.1) is bounded. ∎

5. COMMENTS TO THEOREM 4

(i) The solution of the system (4.2) can be written in the form

$$z^0(k) = [\prod_{j=N-1}^{0} A(j)]^k\, x(0) + \sum_{m=0}^{k-1} [\prod_{j=N-1}^{0} A(j)]^m\, b\,.$$

Now, if all the eigenvalues λ of $\prod_{j=N-1}^{0} A(j)$ are inside the unity circle, $|\lambda| < 1$, we know that $[\prod_{j=N-1}^{0} A(j)]^k$ converges to zero when k tends to infinity and that

$$\sum_{m=0}^{\infty} [\prod_{j=N-1}^{0} A(j)]^m\, b = -([\prod_{j=N-1}^{0} A(j)] - E)^{-1}\, b\,.$$

Hence

$$\lim_{k\to\infty} z^0(k) = (E - [\prod_{j=N-1}^{0} A(j)])^{-1}\, b\,.$$

Further, if $z^0(k)$ is convergent, then by (3.1), each subsequence $z^s(k)$ is convergent; this brings about two results:

a) The sequence $x(n)$, which is a solution of the N-periodic non-homogeneous system, approachs "interlacely" to the N points of convergence of its N subsequences $z^s(k)$, $s = 0, 1, \ldots, N - 1$, which are its only cluster points (i.e. subsequential limits).

Likewise, we observe that these N cluster points do not depend on the initial conditions $x(0)$. That is to say, they would be the same for all the solution sequences $x(n)$ of the system.

b) The solution sequence $x(n)$ of the N-periodic system is bounded, hence so, the N-periodic non-homogeneous system has an N-periodic solution, for every N-periodic vector sequence $f(n)$.

ii) From Theorem 4, we deduce that if A is a $q \times q$ constant matrix and b is a constant q-vector such that

$$x = Ax + b$$

has no solution for x, then all the solutions of the vector difference equation

$$x(n + 1) = A\, x(n) + b$$

are unbounded.

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Souhrn

DIFERENČNÍ LINEÁRNÍ PERIODICKÉ SYSTÉMY II. NEHOMOGENNÍ PŘÍPAD

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V práci je lineární nehomogenní periodický systém v diferencích převeden na obdobný systém s konstantními koeficienty a absolutním členem. To umožňuje studovat existenci a vlastnosti periodických řešení a odvodit asymptotické chování a uzavřený tvar všech řešení.

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