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CONVERGENCE OF L_p-NORMS OF A MATRIX

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I. INTRODUCTION

In [8] it was proved that, under certain assumptions, the L_p-norms of a linear operator converge to its spectral norm, and the rate of this convergence was also estimated. (The definition of the L_p-norm of a linear operator can be found, e.g., in [5] and [6].) In the case of matrices, with which this paper deals the general theorem of convergence of the L_p-norms in [8] will be reduced to the simple and well-known results which form also the basis of the well-known computational procedure (see, e.g., [2]) for determination of the spectral radius of an Hermitian matrix with the aid of traces of matrix powers. A simple application of the theorem on estimating the rate of convergence from [8] then yields an a priori bound of the error for the computational procedure mentioned above. With a view to the possibility of practical utilization of the computation of L_p-norms for the approximation of the spectral radius, we shall propose a suitable normalization of the matrix powers and, with the aid of the normalized powers, we shall determine a recurrence relation for computing the L_p-norms of an Hermitian matrix. As a by-product we shall obtain an expression which gives approximately the number of eigenvalues which are equal to the spectral radius in absolute value. We shall then use the theorem on estimating the rate convergence, given in [8], to derive an a posteriori bound of the error for the above-mentioned computational procedure, and we shall also demonstrate some of its properties which prove the quality of this a posteriori bound for the error.

The contents of this paper is as follows: In Section II we explain the terms and symbols used. In order to be able to make use of the results of [8], we explain the meaning of some fundamental terms from the non-commutative integration theory for this particular case in Section III. The principal results of the paper are contained in Sections V and IV, in which the convergence and computation of the L_p-norms of a matrix will be investigated. Section IV contains a minor numerical illustration of some of the results which were derived in the previous sections.

II. SYMBOLS AND TERMINOLOGY

In this paper E_N will be used to denote the Hilbert space of all complex vectors of length N with the usual scalar product, and $B(E_N)$ will denote the space of all linear operators on E_N which, based on the well-known one-to-one correspondence, will be identified with the algebra of all $N \times N$-matrices.

Let $\mathbf{A} \in B(E_N)$, let $R(\mathbf{A})$ denote the range of the matrix $\mathbf{A}$, $r(\mathbf{A})$ its spectral radius and $\|\mathbf{A}\|_\infty$ its spectral norm (i.e. $\|\mathbf{A}\|_\infty = \sup_{\|\mathbf{x}\|=1} \|\mathbf{Ax}\|$). $\mathbf{A}^*$ will denote, as usual, the adjoint of the matrix $\mathbf{A}$. If $\mathbf{A} = \mathbf{A}^*$, we refer to the matrix $\mathbf{A}$ as an Hermitian matrix. The symbol $|\mathbf{A}|$ will be used to denote the absolute value of the matrix $\mathbf{A}$, i.e. the matrix $(\mathbf{A}^*\mathbf{A})^{1/2}$. The term *projection* will only be used for a matrix $\mathbf{P} \in B(E_N)$ which satisfies the relations $\mathbf{P}^2 = \mathbf{P}$ and $\mathbf{P}^* = \mathbf{P}$. The symbol $\mathbf{I}$ will be used for the unit matrix.

III. MATRICES AND NON-COMMUTATIVE INTEGRATION

Let us define a non-negative function m on the projections in $B(E_N)$ by $m(\mathbf{P}) = $ dimension $(R(\mathbf{P}))$. According to [4; p. 59], $\Gamma = (E_N, B(E_N), m)$ is then a regular gage space and, for $\mathbf{A} \in B(E_N)$, $m(\mathbf{A}) = \operatorname{tr}(\mathbf{A})$ (trace of matrix $\mathbf{A}$). (The definition of a regular gage space and other standard terms from the non-commutative integration theory can be found, e.g., in [5–8].) It is evident that in this gage space Γ the ring of elementary operators $E(\Gamma)$ is identical with the ring of measurable operators $\Lambda(\Gamma)$ and also with the L_p-spaces of integrable operators for $p \in \langle 1, \infty)$, i.e. $E(\Gamma) = B(E_N) = L_p(\Gamma) = \Lambda(\Gamma)$. Assume that $\mathbf{A} \in B(E_N)$ is a matrix with elements $\{a_{ij}\}_{i,j=1}^N$ and that $A_n \in B(E_N)$ $(n = 1, 2, \ldots)$ is a sequence of matrices with elements $\{a_{ij}^{(n)}\}_{i,j=1}^N$ $(n = 1, 2, \ldots)$. It is easy to prove that $\mathbf{A}_n \to \mathbf{A}$ almost everywhere if and only if $\lim_{n\to\infty} a_{ij}^{(n)} = a_{ij}$ for every $i, j = 1, 2, \ldots, N$. The L_p-norm of the matrix $\mathbf{A}$ is then given by the expression $\|\mathbf{A}\|_p = (\operatorname{tr}(|\mathbf{A}|^p))^{1/p} = (\sum_{i=1}^N \lambda_i^p)^{1/p}$, where λ_i $(i = 1, 2, \ldots, N)$ are the eigenvalues of the matrix $|\mathbf{A}|$.

Theorem 3.1 and its Corollary 3.2 from [8] then reduce to the following well-known propositions:

Proposition 3.1. *Let* $\mathbf{A} \in B(E_N)$. *Then* $\lim_{p\to\infty} (\operatorname{tr}(|\mathbf{A}|^p))^{1/p} = \|\mathbf{A}\|_\infty$.

Proposition 3.2. *Let* $\mathbf{A} \in B(E_N)$ *be a normal matrix. Then* $\lim_{k\to\infty} (\operatorname{tr}((\mathbf{A}^*\mathbf{A})^{2^{k-1}}))^{2^{-k}} = r(\mathbf{A})$.

Let $\mathbf{A} \in B(E_N)$ and $\mathbf{A} \neq \mathbf{0}$. Then according to [1; p. 18] the function $f(p) = \|\mathbf{A}\|_p$ is non-increasing in the interval $\langle 1, \infty)$. If we now put $R = \|\mathbf{A}\|_q^q / \|\mathbf{A}\|_\infty^q$ and apply [5; Corollary 1.1] we find

$$R = \|\mathbf{A}\|_q^q / \|\mathbf{A}\|_\infty^q \leqq (\|\mathbf{A}\|_\infty^q \, \|\mathbf{I}\|_q^q) / \|\mathbf{A}\|_\infty^q = \|\mathbf{I}\|_q^q = m(\mathbf{I}) = N \, .$$

According to the previous considerations, we have $\|A\|_q^q/\|A\|_\infty^q \geqq 1$. If we now put $p = q$, the following propositions can be inferred from [8; Theorem 3.3] and [8; Corollary 3.5].

Proposition 3.3. *Let* $A \in B(E_N)$. *Then, for* $p \geqq 2$,

$$|(\operatorname{tr}(|A|^p))^{1/p} - \|A\|_\infty| \leqq \frac{1}{p}\{(\operatorname{tr}(A^*A))^{1/2} . \ln N\} .$$

Proposition 3.4. *Let* $A \in B(E_N)$ *be a normal matrix. Then for* $k = 1, 2, \ldots$

$$|(\operatorname{tr}((A^*A)^{2^{k-1}}))^{2^{-k}} - r(A)| \leqq \frac{1}{2^k}\{(\operatorname{tr}(A^*A))^{1/2} . \ln N\} .$$

IV. COMPUTATION OF THE L_p-NORM OF A MATRIX

The possibility of an approximate computation of the spectral radius of an Hermitian matrix, as indicated by Proposition 3.2, is well-known, see e.g. [2]. Let A be a non-zero Hermitian matrix of the type $N \times N$. To compute $\operatorname{tr}(A^{2^k})$ $(k = 1, 2, \ldots)$ it is necessary to determine the sequence of matrices $\{A^{2^k}\}$ $(k = 1, 2, \ldots)$. Since during the calculation of the elements of matrices $\{A^{2^k}\}$ $(k = 1, 2, \ldots)$ on a computer we would frequently obtain numbers outside its range for sufficiently large values of k, we shall determine instead of the sequence $\{A^{2^k}\}$ $(k = 1, 2, \ldots)$ the sequence of matrices B_l $(l = 1, 2, \ldots)$ using the relations:

$$B_1 = A , \tag{1}$$

$$B_{2k} = B_{2k-1}^2 , \tag{2}$$

$$B_{2k+1} = B_{2k}/c_k , \tag{3}$$

where $k = 1, 2, \ldots$. Here $\{c_k\}$ $k = 1, 2, \ldots$ represents suitable sequence of non-zero numbers which we shall determine later. The equations (1), (2) and (3) imply

$$B_2 = A^2 , \tag{4}$$

$$B_{2k} = A^{2^k}/(c_1^{2^{k-1}} . c_2^{2^{k-2}} \ldots . c_{k-2}^{2^2} . c_{k-1}^2) , \tag{5}$$

where $k = 2, 3, \ldots$, and

$$B_{2k+1} = A^{2^k}/(c_1^{2^{k-1}} . c_2^{2^{k-2}} \ldots . c_{k-2}^{2^2} . c_{k-1}^2 . c_k) , \tag{6}$$

where $k = 1, 2, \ldots$. With a view to (5) we thus obtain the following algorithm for computing $\|A\|_{2^k}$:

$$d_k = (c_{k-1})^{1/2^{k-1}} . d_{k-1} , \quad d_1 = 1 , \tag{7}$$

$$(\operatorname{tr}(A^{2^k}))^{1/2^k} = (\operatorname{tr}(B_{\ k}))^{1/2^k} . d_k , \tag{8}$$

where $k = 2, 3, \ldots$. In virtue of (4), the relation (8) also holds for $k = 1$. It is therefore obvious that, in general, for every $k = 2, 3, \ldots$, to compute $(\operatorname{tr}(\mathbf{A}^{2^k}))^{1/2^k}$ we shall have to compute two roots: $(\operatorname{tr}(\mathbf{B}_{2k}))^{1/2^k}$ and $(c_{k-1})^{1/2^{k-1}}$. We shall prove later that, given a suitable choice of c_k, it is sufficient to calculate just one root for every $k = 1, 2, \ldots$, and that (7) and (8) can be replaced by a single recurrence relation.

Theorem 4.1. *Let $\mathbf{A} \in B(E_N)$, $\mathbf{A} = \mathbf{A}^*$ and $\mathbf{A} \neq \mathbf{0}$. Let matrices $\mathbf{B}_l$ $(l = 1, 2, \ldots)$ consist of the elements $\{b_{ij}^{(l)}\}_{i,j=1}^N$ $(l = 1, 2, \ldots)$ and be determined according to equations* (1), (2) *and* (3) *in which*

$$c_k = \operatorname{tr}(\mathbf{B}_{2k})\,. \tag{9}$$

Then

1) $c_k \neq 0$, $(k = 1, 2, \ldots)$.

2) *The algorithm for computing $\|\mathbf{A}\|_{2^k}$ using* (7) *and* (8) *then reduces to a single recurrence formula:*

$$d_{k+1} = (\operatorname{tr}(\mathbf{B}_{2k}))^{1/2^k} \,.\, d_k\,, \tag{10}$$

where

$$d_1 = 1\,, \quad d_{k+1} = (\operatorname{tr}(\mathbf{A}^{2^k}))^{1/2^k} \tag{11}$$

for $k = 1, 2, \ldots$.

3) *There exists a constant M (independent of l) such that*

$$\max_{\substack{i,j=1,2,\ldots,N\\ l=1,2,\ldots}} |b_{ij}^{(l)}| \leqq M\,.$$

Proof. 1) It is easy to prove that c_k $(k = 1, 2, \ldots)$ is non-zero by induction with respect to k, using equations (4) and (5).

2) We first rewrite equation (7) inserting $k + 1$ instead of k:

$$d_{k+1} = c^{1/2^k} \,.\, d_k\,, \quad d_1 = 1\,, \tag{7'}$$

where $k = 1, 2, \ldots$. With a view to (9) the right-hand sides of equations (7′) and (8) are equal and, therefore, also their left-hand sides are equal, i.e. $d_{k+1} = (\operatorname{tr}(\mathbf{A}^{2^k}))^{1/2^k}$ $(k = 1, 2, \ldots)$, which $(k = 1, 2, \ldots)$, which proves (11) and (10).

3) For $k = 2, 3, \ldots$ equations (10) and (11) yield $\operatorname{tr}(\mathbf{B}_{2k}) = (d_{k+1}/d_k)^2 = \operatorname{tr}(\mathbf{A}^{2^k}) / (\operatorname{tr}(\mathbf{A}^{2^{k-1}}))^2$. Also according to [5; Corollary 1.1], for $k = 2, 3, \ldots$ we obtain $\operatorname{tr}(\mathbf{A}^{2^k}) = \|\mathbf{A}^{2^k}\|_1 \leqq \|\mathbf{A}^{2^{k-1}}\|_\infty \|\mathbf{A}^{2^{k-1}}\|_1 \leqq \|\mathbf{A}^{2^{k-1}}\|_1^2 = (\operatorname{tr}(\mathbf{A}^{2^{k-1}}))^2$ and, therefore, $\operatorname{tr}(\mathbf{B}_{2k}) \leqq 1$ $(k = 2, 3, \ldots)$. With a view to (3) and (9), for $k = 1, 2, \ldots$, we obtain $\operatorname{tr}(\mathbf{B}_{2k+1}) = 1$ and, consequently, $\|\mathbf{B}_l\|_1 = \operatorname{tr}(\mathbf{B}_l) \leqq 1$ for $l = 3, 4, \ldots$. This implies that the sequence $\|\mathbf{B}_l\|_{\max} = \max_{i,j=1,2,\ldots,N} |b_{ij}^{(l)}|$ $(l = 1, 2, \ldots)$ must also be bounded.

Theorem 4.2. *Let $\mathbf{A} \in B(E_N)$, $\mathbf{A} = \mathbf{A}^*$ and $\mathbf{A} \neq \mathbf{0}$. Assume that the number of eigenvalues of the matrix $\mathbf{A}$, whose absolute values are equal to $r(\mathbf{A})$, is t. Then*

1) $t \leqq 1/\operatorname{tr}(\mathbf{B}_{2k}) \leqq N$ *for* $k = 2, 3, \ldots$;

2) $\lim_{k\to\infty} 1/\mathrm{tr}(\mathbf{B}_{2k}) = t$;

3) *The sequence* $1/\mathrm{tr}\,(\mathbf{B}_{2k})$, $k = 2, 3, \dots$ *is non-increasing.*

Proof. According to [5; Theorem 1] we obtain $(\mathrm{tr}\,(\mathbf{A}^{2^{k-1}}))^2 = \|\mathbf{A}^{2^{k-1}} \cdot \mathbf{I}\|_1^2 \leqq \leqq \|\mathbf{A}^{2^{k-1}}\|_2^2 \, \|\mathbf{I}\|_2^2 = \|\mathbf{A}^{2^k}\|_1 \cdot N$ and using (10) and (11) we find

$$1/\mathrm{tr}\,(\mathbf{B}_{2k}) = (\mathrm{tr}\,(\mathbf{A}^{2^{k-1}}))^2/\mathrm{tr}\,(\mathbf{A}^{2^k}) \leqq N\,.$$

Let $\{\lambda_i\}$ $(i = 1, 2, \dots, N)$ represent the eigenvalues of the matrix $\mathbf{A}$ and let us assume that $|\lambda_1| = |\lambda_2| = \dots = |\lambda_t| > |\lambda_{t+1}| \geqq \dots \geqq |\lambda_N| \geqq 0$. Denote $\alpha_i = |\lambda_i|/|\lambda_1|$, $i = 1, 2, \dots, N$. For $k = 2, 3, \dots$ we then obtain

$$1/\mathrm{tr}\,(\mathbf{B}_{2k}) = (\sum_{i=1}^{N} \lambda_i^{2^{k-1}})^2/\sum_{i=1}^{N} \lambda_i^{2^k} = (\lambda_1^{2^{k-1}}(t + \sum_{i=t+1}^{N} \alpha_i^{2^{k-1}}))^2/(\lambda_1^{2^k}(t + \sum_{i=t+1}^{N} \alpha_i^{2^k})) =$$

$$= t\left\{\left(t + 2\sum_{i=t+1}^{N} \alpha_i^{2^{k-1}} + \frac{1}{t}(\sum_{i=t+1}^{N} \alpha_i^{2^{k-1}})^2\right)\Big/\left(t + \sum_{i=t+1}^{N} \alpha_i^{2^k}\right)\right\} = t\,.\,G(k, t)\,.$$

Since evidently $G(k, t) \geqq 1$ $(k = 2, 3, \dots)$ and $\lim_{k\to\infty} G(k, t) = 1$, relations 1) and 2) are proved. The third part of the assertion of the theorem follows from the next lemma and from the relations

$$1/\mathrm{tr}\,(\mathbf{B}_{2k}) = (\sum_{i=1}^{N} \lambda_i^{2^{k-1}})^2/\sum_{i=1}^{N} \lambda_i^{2^k} = (\sum_{\substack{i=1\\ \alpha_i \neq 0}}^{N} \alpha_i^{2^{k-1}})^2/\sum_{\substack{i=1\\ \alpha_i \neq 0}}^{N} \alpha_i^{2^k}\,.$$

Lemma 4.3. *Assume that real numbers* α_i $(i = 1, 2, \dots, N)$ *satisfy the relations* $1 = \alpha_1 = \alpha_2 = \dots = \alpha_t > \alpha_{t+1} \geqq \alpha_{t+2} \geqq \dots \geqq \alpha_M > 0$, *where* $1 \leqq t \leqq M \leqq N$, Then the sequence of numbers

$$(\sum_{i=1}^{M} \alpha_i^k)^2/\sum_{i=1}^{M} \alpha_i^{2k}\,, \quad k = 2, 3, \dots$$

is non-increasing.

Proof. Let us define the function

$$H(k) = (\sum_{i=1}^{M} \alpha_i^k)^2/\sum_{i=1}^{M} \alpha_i^{2k}$$

for $k \in \langle 2, \infty)$ and put $N(k) = (\sum_{i=1}^{M} \alpha_i^k)^2$ and $D(k) = \sum_{i=1}^{M} \alpha_i^{2k}$. We then obtain

$$D^2(k) \cdot H'(k) = 2\sum_{i=1}^{M} \alpha_i^k \sum_{i=1}^{M} \alpha_i^k \ln \alpha_i \sum_{i=1}^{M} \alpha_i^{2k} - \sum_{i=1}^{M} 2\alpha_i^{2k} \ln \alpha_i (\sum_{i=1}^{M} \alpha_i^k)^2 =$$

$$= 2\sum_{i=1}^{M} \alpha_i^k (\sum_{i=1}^{M} \alpha_i^k \ln \alpha_i \sum_{j=1}^{M} \alpha_j^{2k} - \sum_{i=1}^{M} \alpha_i^{2k} \ln \alpha_i \sum_{j=1}^{M} \alpha_j^k) =$$

$$= 2\sum_{i=1}^{M} \alpha_i^k \{ \sum_{i=t+1}^{M} \sum_{j=1}^{t} \alpha_i^k \ln \alpha_i (1 - \alpha_i^k) +$$

$$+ \sum_{j=t+1}^{M} \sum_{i=t+1}^{M} \alpha_i^k \alpha_j^k (\ln \alpha_i - \ln \alpha_j) (\alpha_j^k - \alpha_i^k)\} \leqq 0$$

and, therefore, $H(k)$ is a non-increasing function in the interval $\langle 2, \infty)$.

V. ESTIMATE OF THE RATE OF CONVERGENCE

Theorem 5.1. *Let* $\mathbf{A} \in B(E_N)$, $\mathbf{A} = \mathbf{A}^*$ *and* $\mathbf{A} \neq \mathbf{0}$. *Then for* $k = 2, 3, \ldots,$

$$| \, \|\mathbf{A}\|_{2^k} - r(\mathbf{A})| \leqq \frac{1}{2^k} \{\|\mathbf{A}\|_{2^k} \ln (1/\mathrm{tr}\,(\mathbf{B}_{2k}))\} \,.$$

Proof. Assume that k is an arbitrary but fixed integer, $k \geqq 2$. According to [5; Corollary 1.1] we have

$$\|\mathbf{A}\|_{2^k}^{2^k} = \|\mathbf{A}^{2^k}\|_1 \leqq \|\mathbf{A}^{2^{k-1}}\|_\infty \|\mathbf{A}^{2^{k-1}}\|_1 \leqq \|\mathbf{A}\|_\infty^{2^{k-1}} \|\mathbf{A}\|_{2^{k-1}}^{2^{k-1}} \,.$$

Let us put $q = 2^k$ and $R = \|\mathbf{A}\|_q^q / \|\mathbf{A}\|_\infty^q$. By the preceding relation we find

$$R \leqq (\|\mathbf{A}\|_{2^k}^{2^k} \,.\, \|\mathbf{A}\|_{2^{k-1}}^{2^k}) / \|\mathbf{A}\|_{2^k}^{2^{k+1}} = (\mathrm{tr}\,(\mathbf{A}^{2^{k-1}}))^2 / \mathrm{tr}(\mathbf{A}^{2^k}) = 1/\mathrm{tr}\,(\mathbf{B}_{2k}) \,.$$

Let $\{\lambda_i\}$ $(i = 1, 2, \ldots, N)$ represent the eigenvalues of the matrix $\mathbf{A}$ and assume that $|\lambda_1| = |\lambda_2| = \ldots = |\lambda_t| > |\lambda_{t+1}| \geqq \ldots \geqq |\lambda_N|$. Then $|\lambda_i|$ $(i = 1, 2, \ldots, N)$ are the eigenvalues of the matrix $|\mathbf{A}|$ and if we denote by $\mathbf{S}$ the projection onto the eigenspace of $|\mathbf{A}|$ corresponding to the eigenvalue $r(\mathbf{A})$, then $\mathrm{tr}\,(\mathbf{S}) = t$. The inequality to be proved is now obtained from [8; Corrolary 3.5] by using Theorem 4.2.

The following Theorem 5.2 expresses the asymptotic behaviour of the actual error and of its a posteriori bound from Theorem 5.1.

Theorem 5.2. *Let* $\mathbf{A} \in B(E_N)$, $\mathbf{A} = \mathbf{A}^*$ *and* $\mathbf{A} \neq \mathbf{0}$. *Let* $\{\lambda_i\}$ $(i = 1.2 \ldots, N)$ *represent the eigenvalues of the matrix* $\mathbf{A}$ *and let* $|\lambda_1| \geqq |\lambda_2| \geqq \ldots \geqq |\lambda_N| \geqq 0$. *Then*

1) $| \, \|\mathbf{A}\|_{2^k} - r(\mathbf{A})| = O\left(\frac{1}{2^k} \left|\frac{\lambda_2}{\lambda_1}\right|^{2^k}\right),$

2) $\frac{1}{2^k} \{\|\mathbf{A}\|_{2^k} \ln (1/\mathrm{tr}\,(\mathbf{B}_{2k}))\} = O\left(\frac{1}{2^{k-1}} \left|\frac{\lambda_2}{\lambda_1}\right|^{2^{k-1}}\right).$

Proof. Again set $\alpha_i = |\lambda_i| / |\lambda_1|$, $i = 1, 2, \ldots, N$; we then find

$$\|\mathbf{A}\|_{2^k} - r(\mathbf{A}) = |\lambda_1| \{(\sum_{i=1}^{N} \alpha_i^{2^k})^{1/2^k} - 1\} =$$

$$= |\lambda_1| \frac{1}{2^k} \ln \sum_{i=1}^{N} \alpha_i^{2^k} + O\left(\left\{\frac{1}{2^k} \ln \sum_{i=1}^{N} \alpha_i^{2^k}\right\}^2\right).$$

If $|\lambda_1| = |\lambda_2|$, relation 1) is clearly satisfied. Therefore, let $|\lambda_1| > |\lambda_2|$; for sufficiently large values of k we get

$$\|\mathbf{A}\|_{2^k} - r(\mathbf{A}) = |\lambda_1| \frac{1}{2^k} \sum_{i=2}^{N} \alpha_i^{2^k} + O\left(\frac{1}{2^k}\left\{\sum_{i=2}^{N} \alpha_i^{2^k}\right\}^2\right)$$

and relation 1) is again satisfied. If $|\lambda_1| = |\lambda_2|$, relation 2) follows immediately. Let $|\lambda_1| > |\lambda_2|$. For sufficiently large values of k we then obtain

$$\ln(1/\operatorname{tr}(\mathbf{B}_{2k})) = \ln\left(\left(\sum_{i=1}^{N} \alpha_i^{2^{k-1}}\right)^2 / \sum_{i=1}^{N} \alpha_i^{2^k}\right) =$$

$$= 2\left\{\sum_{i=2}^{N} \alpha_i^{2^{k-1}} + O\left(\left(\sum_{i=2}^{N} \alpha_i^{2^{k-1}}\right)^2\right)\right\} - \left\{\sum_{i=2}^{N} \alpha_i^{2^k} + O\left(\left(\sum_{i=2}^{N} \alpha_i^{2^k}\right)^2\right)\right\}$$

proving relation 2).

Several other properties of the a posteriori bound for the error, which prove its quality, are contained in the following theorem:

Theorem 5.3. *Let* $\mathbf{A} \in B(E_N)$, $\mathbf{A} = \mathbf{A}^*$ *and* $\mathbf{A} \neq \mathbf{0}$. *Let* $\{\lambda_i\}$ $(i = 1, 2, \ldots, N)$ *be the eigenvalues of the matrix* $\mathbf{A}$ *and let* $|\lambda_1| \geqq |\lambda_2| \geqq \ldots \geqq |\lambda_N| \geqq 0$. *Denote*

$$E(\mathbf{A}, k) = \frac{1}{2^k}\{\|\mathbf{A}\|_{2^k} \,.\, \ln(1/\operatorname{tr}(\mathbf{B}_{2k}))\}\,.$$

Then

1) $\|\mathbf{A}\|_{2^k} - r(\mathbf{A}) \leqq E(\mathbf{A}, k) \leqq \|\mathbf{A}\|_{2^{k-1}} - r(\mathbf{A}) \quad (k = 2, 3, \ldots);$

2) *if* $|\lambda_1| > |\lambda_2|$, *then* $\displaystyle\lim_{k\to\infty} \frac{E(\mathbf{A}, k)}{\|\mathbf{A}\|_{2^{k-1}} - r(\mathbf{A})} = 1\,;$

3) *if* $|\lambda_1| = |\lambda_2|$, *then* $\displaystyle\lim_{k\to\infty} \frac{E(\mathbf{A}, k)}{\|\mathbf{A}\|_{2^k} - r(\mathbf{A})} = 1\,.$

Proof. For $k = 2, 3, \ldots$ we have

$$E(\mathbf{A}, k) = \|\mathbf{A}\|_{2^k} \,.\, \ln(1/(\operatorname{tr}(\mathbf{B}_{2k}))^{1/2^k}) =$$

$$= \|\mathbf{A}\|_{2^k} \ln\left(\frac{\|\mathbf{A}\|_{2^{k-1}}}{\|\mathbf{A}\|_{2^k}}\right) \leqq \|\mathbf{A}\|_{2^k}\left(\frac{\|\mathbf{A}\|_{2^{k-1}}}{\|\mathbf{A}\|_{2^k}} - 1\right) =$$

$$= \|\mathbf{A}\|_{2^{k-1}} - \|\mathbf{A}\|_{2^k} \leqq \|\mathbf{A}\|_{2^{k-1}} - r(\mathbf{A})$$

and statement 1) is proved.

Let $|\lambda_1| > |\lambda_2|$ and once again put $\alpha_i = |\lambda_i|/|\lambda_1|$ $(i = 1, 2, \ldots, N)$. Denote $S(k) = (1/2^k) \ln \sum_{i=1}^{N} \alpha_i^{2^k}$. We then obtain

$$\lim_{k\to\infty} \frac{E(\mathbf{A}, k)}{\|\mathbf{A}\|_{2^{k-1}} - r(\mathbf{A})} = \lim_{k\to\infty} \frac{\|\mathbf{A}\|_{2^k}}{|\lambda_1|} \frac{S(k-1) - S(k)}{e^{S(k-1)} - 1} = 1\,,$$

because

$$0 \leqq \lim_{k\to\infty} \frac{S(k)}{e^{S(k-1)} - 1} \leqq \lim_{k\to\infty} \frac{S(k)}{S(k-1)} \leqq$$

$$\leqq \lim_{k\to\infty} \frac{1}{2} \frac{\sum_{i=2}^{N} \alpha_i^{2^k}}{\ln \sum_{i=1}^{N} \alpha_i^{2^{k-1}}} \leqq \lim_{k\to\infty} \frac{\sum_{i=2}^{N} \alpha_i^{2^k}}{\sum_{i=2}^{N} \alpha_i^{2^{k-1}}} = 0\,,$$

which proves that the second part of the assertion of the theorem is valid.

If $|\lambda_1| = |\lambda_2|$, we obtain

$$\lim_{k\to\infty} \frac{E(\boldsymbol{A}, k)}{\|\boldsymbol{A}\|_{2^k} - r(\boldsymbol{A})} = \lim_{k\to\infty} \frac{\|\boldsymbol{A}\|_{2^k}}{|\lambda_1|} \left\{\frac{S(k-1) - S(k)}{e^{S(k)} - 1}\right\} =$$

$$= \lim_{k\to\infty} \left\{\frac{S(k-1)}{e^{S(k)} - 1} - \frac{S(k)}{e^{S(k)} - 1}\right\} = 1\,.$$

because

$$\lim_{k\to\infty} \frac{S(k-1)}{e^{S(k)} - 1} = 2\,,$$

and the proof is complete.

VI. NUMERICAL EXAMPLE

Let $\boldsymbol{A}$ be the following 5×5 – matrix

$$\boldsymbol{A} = \begin{bmatrix} 10 & 1 & 2 & 3 & 4 \\ 1 & 9 & -1 & 2 & -3 \\ 2 & -1 & 7 & 3 & -5 \\ 3 & 2 & 3 & 12 & -1 \\ 4 & -3 & -5 & -1 & 15 \end{bmatrix}$$

and let λ_i $(i = 1, \ldots, 5)$ be its eigenvalues. According to [3], we have

$$\lambda_1 \doteq 19.175420\,2773\,,$$

$$\lambda_2 \doteq 15.808920\,7645\,.$$

In the tables which follow we give the values of $\|\boldsymbol{A}\|_{2^k}$, the values of the expression $1/\text{tr}\,(\boldsymbol{B}_{2k})$, the actual error of the approximation $r(\boldsymbol{A})$ found with the aid of $\|\boldsymbol{A}\|_{2^k}$ and the a posteriori bound for the error according to Theorem 5.1. All the values given in the tables have been rounded off to a suitable number of digits.

Table I

k	$\|\mathbf{A}\|_{2^k}$	$1/\mathrm{tr}(\mathbf{B}_{2k})$
1	27.513632 9844	
2	21.349559 3822	2.7582 657
3	19.651941 8274	1.9402 941
4	19.228893 5539	1.4165 072
5	19.176662 4826	1.0909 395
6	19.175421 5674	1.0041 501
7	19.175420 2773	1.0000 086

Table II

k	Error of approximation $r(\mathbf{A})$ with the aid of $\|\mathbf{A}\|_{2^k}$	A posteriori bound for the error
1	8.34	
2	2.17	5.42
3	0.48	1.63
4	0.53×10^{-1}	0.42
5	0.12×10^{-2}	0.52×10^{-1}
6	0.13×10^{-5}	0.12×10^{-2}
7	less than 10^{-10}	0.13×10^{-5}

Clearly, with regard to the number of arithmetical operations, the procedure of approximate computation of $r(\mathbf{A})$ given here is unsuitable particularly for large and sparse matrices. However, the advantage of this procedure is the simple algorithm and the possibility of easy and effective use of a vector computer. Moreover, convergence is guaranteed without any assumptions about the distribution of the matrix spectrum.

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Souhrn

KONVERGENCE L_p-NOREM MATICE

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V článku je odvozen rekurentní vztah pro výpočet L_p-normy hermitovské matice a stanoven výraz, který aproximuje počet vlastních čísel matice rovných v absolutní hodnotě spektrálnímu poloměru. Dále je dokázán apriorní a aposteriorní odhad aproximace spektrálního poloměru hermitovské matice pomocí jejich L_p-norem a ukázáno několik vlastností tohoto aposteriorního odhadu.

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