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ON THE DISTANCE SPECTRUM OF A CYCLE

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The distance polynomial $\Delta(G)$ of a graph G has been recently considered in this journal [1]. The distance spectrum of a complete graph, a complete bipartite graph $K_{m,n}$ and a star have been determined in [1]. Regarding a cycle the following proposition has been proved [1]: if G is an even cycle, then at least one root of $\Delta(G)$ equals zero.

The full treatment of the distance spectrum of a cycle is given in the present paper.

For a graph G with **n** vertices the distance matrix $\mathbf{D} = \mathbf{D}(G)$ is a square matrix of order **n** whose elements are defined by: $d_{rr} = 0$ and $d_{rs} =$ the length of the shortest path between the vertices **r** and **s**.

The eigenvalue problem for **D** reads as follows:

$$\mathbf{D}\mathbf{Y}_j = x_j \mathbf{Y}_j\,, \quad j = 1, 2, \ldots, n\,, \tag{1}$$

where $x_j = x_j(\mathbf{D})$ are the eigenvalues and $\mathbf{Y}_j$ are the eigenvectors of **D**. The collection of x_j's is called the distance spectrum of G and denoted by $Sp_D(G)$. The eigenvalues of **D** are at the same time the roots of the distance polynomial $\Delta(G) = \Delta(G, x)$ which is defined by $\det(X\mathbf{I} - \mathbf{D})$ where **I** is the unit matrix of order **n**.

A cycle C_n with n vertices is treated in what follows. By using the cyclic properties of $\mathbf{D}(C_n)$, the coordinates of the jth eigenvector $\mathbf{Y}_j$ are [2, 3]

$$Y_{rj} = \frac{1}{\sqrt{(2\pi)}}\,\omega_j^r\,, \quad r = 1, 2, \ldots, n\,, \tag{2}$$

where

$$\omega_j = \exp(i\theta_j) \tag{3}$$

and

$$\theta_j = j\,\frac{2\pi}{n}\,, \quad j = 1, 2, \ldots, n\,. \tag{4}$$

By means of Eqs. (1)–(4) the following expressions for $Sp_D(C_n)$ are obtained:

$$x_j = \begin{cases} 2\sum_{r=1}^{k} r\cos r\theta_j - k(-1)^j & \text{for } n = 2k\,, \quad \text{(5a)} \\ 2\sum_{r=1}^{k} r\cos r\theta_j & \text{for } n = 2k+1\,. \quad \text{(5b)} \end{cases}$$

Following the procedure of Polansky [4] we obtain the sums of sines and cosines in the form

(6a) $$I_0(\theta) = \sum_{r=m}^{n} \cos r\theta = \frac{1}{2\sin\frac{1}{2}\theta}\left[\sin\left(n+\tfrac{1}{2}\right)\theta - \sin\left(m-\tfrac{1}{2}\right)\theta\right],$$

(6b) $$J_0(\theta) = \sum_{r=m}^{n} \sin r\theta = \frac{1}{2\sin\frac{1}{2}\theta}\left[-\cos\left(n+\tfrac{1}{2}\right)\theta + \cos\left(m-\tfrac{1}{2}\right)\theta\right].$$

Accordingly, one derives

(7) $$I_1(\theta) = \frac{\mathrm{d}J_0(\theta)}{\mathrm{d}\theta} = \sum_{r=m}^{n} r\cos r\theta =$$

$$= \frac{1}{2\sin\frac{1}{2}\theta}\left[n\sin\left(n+\tfrac{1}{2}\right)\theta - m\sin\left(m-\tfrac{1}{2}\right)\theta\right] + \frac{1}{4\sin^2\frac{1}{2}\theta}\left(\cos n\theta - \cos m\theta\right).$$

Case 1. Let us consider C_n with an even n, $n = 2k$. Because of Eqs. (5a) and (7), the eigenvalues of C_{2k} are given by

(8) $$x_j = \frac{1}{\sin\frac{1}{2}\theta_j}\left[(k-1)\sin\left(k-\tfrac{1}{2}\right)\theta_j - \sin\tfrac{1}{2}\theta_j\right] +$$

$$+ \frac{1}{2\sin^2\frac{1}{2}\theta_j}\left[\cos(k-1)\theta_j - \cos\theta_j\right] + k(-1)^j\,,$$

where $\theta_j = j\pi/k$, $j = 1, 2, \ldots, 2k$.

In particular, for $j = n$ one has

(9) $$x_{2k} = k^2\,.$$

Further, for $j = k$ we have $\theta_k = \pi$, and one easily derives

(10) $$x_k = \begin{cases} 0 & \text{for } k = \text{even}\,, \\ -1 & \text{for } k = \text{odd}\,. \end{cases}$$

Note that $\theta_{2k-j} = 2\pi - \theta_j$ and consequently,

(11) $$x_{2k-j} = x_j$$

holds.

Let us first consider even j's, $j = 2l \neq 2k$. In this case one easily obtains that

(12) $$x_{2l} = x_{2(k-l)} = 0\,, \quad l = 1, 2, \ldots, \left[\tfrac{1}{2}(k-1)\right]$$

where $[a]$ denotes the integer part of a.

In the case of odd j's, $j = 2l + 1$, Eq. (8) reduces to

$$(13) \qquad x_{2l+1} = x_{2k-(2l+1)} = -\frac{1}{\sin^2 \frac{(2l+1)\pi}{2k}}, \quad l = 0, 1, 2, \ldots, [\tfrac{1}{2}k] - 1 .$$

We summarize Eqs. (9)–(13) as follows: The distance spectrum of an *even* cycle $C_n = C_{2k}$ is given by

$$(14) \qquad x_1 = x_{2k-1} = -1/\sin^2 \frac{\pi}{2k} < x_3 = x_{2k-3} = -1/\sin^2 \frac{3\pi}{2k} <$$

$$< \ldots < x_{2l+1} = x_{2k-(2l+1)} = -1/\sin^2 \frac{(2l+1)\pi}{2k} <$$

$$< \ldots < x_2 = x_{2k-2} = x_4 = x_{2k-4} = \ldots = 0 < x_{2k} = k^2 .$$

In other words, among $2k$ eigenvalues of $\mathbf{D}(C_{2k})$ there are k negative eigenvalues, the zero eigenvalue whose degeneracy equals $(k - 1)$, and only one positive eigenvalue which is equal to k^2. Among k negative eigenvalues there are $[k/2]$ mutually distinct, doubly degenerate eigenvalues, and in addition, for k being and odd number, there is also a single negative eigenvalue which is equal to -1.

Case 2. Let us consider C_n with an *odd* n, $n = 2k + 1$. By applying Eqs. (5b) and (7) the following expression for the eigenvalues of C_{2k+1} is obtained:

$$(15) \qquad x_j = \frac{1}{\sin \frac{1}{2}\theta_j} [k \sin (k + \tfrac{1}{2})\theta_j - \sin \tfrac{1}{2}\theta_j] + \frac{1}{2 \sin^2 \frac{1}{2}\theta_j} (\cos k\theta_j - \cos \theta_j) ,$$

where: $\theta_j = j\, 2\pi/(2k + 1)$, $j = 1, 2, \ldots, 2k + 1$.

In particular, for $j = 2k + 1$ one has

$$(16) \qquad x_{2k+1} = k(k + 1) .$$

Because of $\theta_{2k+1-j} = \theta_j$ one obtains

$$(17) \qquad x_j = x_{2k+1-j} ,$$

i.e., now the eigenvalues with even and odd indices go together in pairs. Simple algebra immediately yields

$$(18) \qquad x_{2l} = x_{2k+1-2l} = -\frac{1}{4 \cos^2 \frac{l\pi}{2k+1}}, \quad l = 1, 2, \ldots, k .$$

We summarize Eqs. (16)–(18) as follows: The distance spectrum of an *odd* cycle

$C_n = C_{2k+1}$ is given by

$$(19) \quad x_1 = x_{2k} = -\frac{1}{4\cos^2 \frac{k\pi}{2k+1}} < x_3 = x_{2k-2} = -\frac{1}{4\cos^2 \frac{(k-1)\pi}{2k+1}} <$$

$$< \ldots < x_{2l+1} = x_{2(k-l)} = -\frac{1}{4\cos^2 \frac{(k-l)}{2k+1}} < \ldots < x_{2k-1} =$$

$$= x_2 = -\frac{1}{4\cos^2 \frac{\pi}{2k+1}} < x_{2k+1} = k(k+1)\,.$$

In other words, among $2k + 1$ eigenvalues of $\mathbf{D}(C_{2k+1})$ there are k mutually distinct, doubly degenerate negative eigenvalues and only one positive eigenvalue which is equal to $k(k + 1)$.

Numerical data $Sp_D(C_n)$, $n = 3, 4, \ldots, 10$, are presented below:

$Sp_D(C_3) = \{-1., -1., +2.\}$,

$Sp_D(C_4) = \{-2., -2., 0., +4.\}$,

$Sp_D(C_5) = \{-2.618\,0340, -2.618\,0340, -0.381\,9660, -0.381\,966, +6.\}$,

$Sp_D(C_6) = \{-4., -4., -1., 0., 0., +9.\}$

$Sp_D(C_7) = \{-5.048\,917\,3, -5.048\,917\,3, -0.643\,104\,1, -0.643\,104\,1,$
$-0.307\,978\,5, -0.307\,978\,5, +12.\}$,

$Sp_D(C_8) = \{-6.828\,427\,1, -6.828\,427\,1, -1.171\,572\,9, -1.171\,572\,9, 0., 0., 0.,$
$+16.\}$,

$Sp_D(C_9) = \{-8.290\,859\,3, -8.290\,859\,3, -1., -1., -0.426\,022\,0, -0.426\,022\,0,$
$-0.283\,118\,6, -0.283\,118\,6, +20.\}$,

$Sp_D(C_{10}) = \{-10.472\,136\,0, -10.472\,136\,0, -1.527\,864\,0, -1.527\,864\,0, -1., 0.,$
$0., 0., 0., +25.\}$.

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Souhrn

O DISTANČNÍM SPEKTRU CYKLU

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V práci jsou odvozeny analytické výrazy pro kořeny distančního polynomu cyklů.

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