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# ON IRROTATIONAL FLOWS THROUGH CASCADES OF PROFILES IN A LAYER OF VARIABLE THICKNESS

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## INTRODUCTION

In this paper we deal with the study of flows in blade rows, which is one of the most important subjects in the theory of blade machines (i.e. turbines, compressors, pumps etc.). Fig. 1 gives a simplified view of a part of a blade machine. It consists of a certain number of blades, periodically spaced round an axis of symmetry. These blades form the so-called blade row which is inserted into an axially symmetric channel.

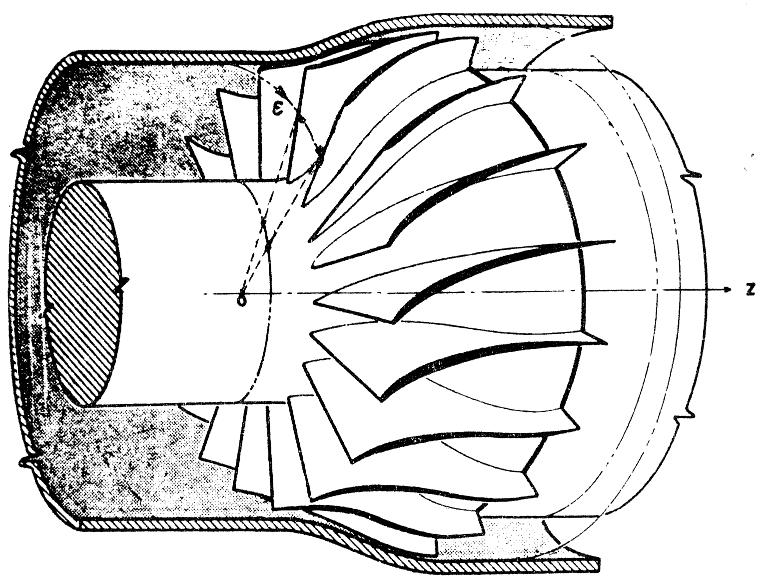


Fig. 1.

Very complicated (three-dimensional, non-stationary, rotational, turbulent) flows in blade rows are studied with the use of simplified boundary value problems. We can mention the widely used model of plane, irrotational, incompressible, non-viscous flows through cascades of profiles, represented e.g. by the well-known Martensen method ([22]). Significant results were also obtained by Polášek, Vlášek and other authors ([25, 33, 15, 16]). This model can be successfully applied if the walls of the channel, into which the blades are inserted, do not differ too much from concentric cylindrical surfaces.

Here we shall present new results concerning the more complex model of flows through cascades of profiles in a layer of variable thickness. This model takes account of the three-dimensional character of the stream field in a better way and can be used for the study of flows in blade rows inserted into channels with walls considerably differing from cylindrical surfaces.

A series of papers ([1, 4, 14, 17, 21, 26, 27, 30, 34]) is devoted to the study of irrotational, incompressible, non-viscous flows through cascades of profiles in a layer of variable thickness. The authors tried to apply the singularity method and the method of integral equations (used successfully by Martensen in [22] for the solution of plane flows) via a convenient iterative process.

In this paper we investigate general incompressible and also subsonic compressible, irrotational, non-viscous flows through cascades of profiles in a layer of variable thickness under complex boundary conditions. We introduce the stream function formulation of several boundary value problems that represent adequate two-dimensional models of stream fields in blade rows, and present a detailed analysis of their solvability.

## 1. FORMULATION OF FLOWS THROUGH CASCADES OF PROFILES

### 1.1. Geometry of the blade row and the cascade of profiles

Let us denote by $R_m$ an $m$-dimensional Euclidean space. If $A \subset R_m$, then $\bar{A}$ and $\partial A$ denote the closure and the boundary of the set $A$, respectively. In the space $R_3$ we shall use cylindrical coordinates $z$, $r$, $\varepsilon$ ($z$-axial, $r$-radial, $\varepsilon$-angular coordinates, $z \in R_1$, $r \in \langle 0, +\infty)$, $\varepsilon \in R_1$). If $A \subset R_m$ is an open set and $k \geqq 0$ is an integer, then $C^k(A)$ ($C^k(\bar{A})$) is the space of all functions that have continuous $k$-th order derivatives in $A$ (in $\bar{A}$).

Let $\Omega_M \subset R_2$ be a bounded domain lying in the upper half-plane $(z, r)$, $r > 0$. The boundary $\partial\Omega_M$ consists of arcs $L_1, L_2, \Gamma_I, \Gamma_O$, as is drawn in Fig. 2. By rotating the domain $\Omega_M$ round the axis $z$ we get a three-dimensional axially symmetric channel. We denote it by $\Omega_3$. The rotation of $L_i$ $(i = 1, 2)$, $\Gamma_I$ and $\Gamma_O$ round the axis $z$ gives the walls of the channel $\Omega_3$, the inlet (through which the fluid enters the channel) and the outlet (through which the fluid flows out from the channel), respectively.

Let us consider a blade row inserted into the channel $\Omega_3$, formed by $N$ blades periodically spaced in the direction $\varepsilon$ (see Fig. 1). Our aim is to approximate complicated three-dimensional stream fields in this blade row by a simplified model of flows past the blades in the space between two axially symmetric surfaces.

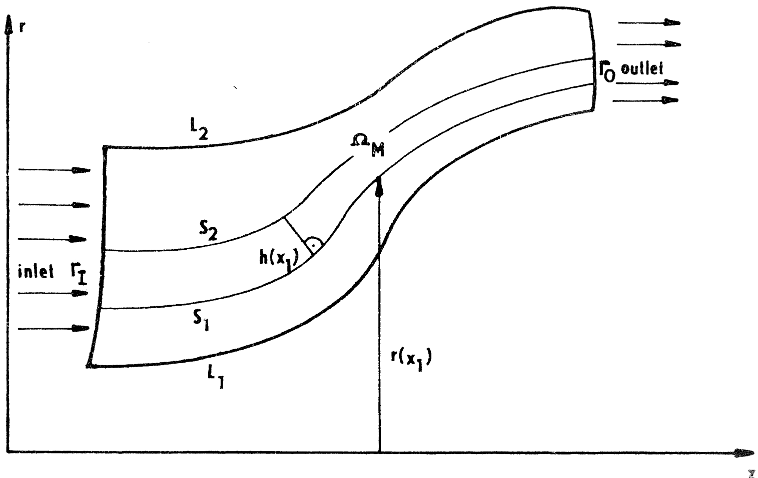


Fig. 2.

We start from the assumption that we have already calculated an axially symmetric flow in the channel $\Omega_3$ (without blades) and have obtained a family of axially symmetric stream surfaces. Let us consider two close surfaces $\mathscr{S}_1$, $\mathscr{S}_2$ from this family, represented in the meridional cross-section $\Omega_M$ by the curves $S_1$ and $S_2$ (see Fig. 2). The space between these surfaces is called a *layer of variable thickness.* Its geometry is determined by two quantities: $h$ – the distance of points lying on $\mathscr{S}_1$ from $\mathscr{S}_2$ measured in the direction normal to $\mathscr{S}_1$ and $r$ – the distance of these points from the axis of symmetry $z$. In a special case, when $r = \text{const.}$ on $\mathscr{S}_1$ and hence $\mathscr{S}_1$ is a cylindrical surface, $h$ is the so-called axial-velocity-ratio (abbr. AVR) factor (cf. [34]).

It is obvious that $h$ and $r$ can be considered as functions dependent on the length $s$ of the arc measured on the curve $S_1$ from its intersection with $\Gamma_I$ to the point in consideration lying on $S_1$. Under the assumption that $r = r(s)$ is continuous and $r > 0$ let us introduce a coordinate system $x_1, x_2$ on the surface $\mathscr{S}_1$, defined by the relations

$$(1.1) \qquad x_1 = r(0) \int_0^s \frac{d\xi}{r(\xi)}, \quad x_2 = r(0)\,\varepsilon$$

$$(x_1 \in \langle d_1, d_2 \rangle, \quad d_1 = 0, \quad d_2 = r(0) \int_0^{s_1} r^{-1}(\xi)\,d\xi,$$

$s_1$ = length of $S_1$, $x_2 \in R_1$) and express $h$ and $r$ as functions of $x_1$: $h = h(x_1)$, $r = r(x_1)$, $x_1 \in \langle d_1, d_2 \rangle$.

If we transform the surface $\mathscr{S}_1$ and its intersections with the blades into the $(x_1, x_2)$-plane, we get a two-dimensional domain $\Omega$ (shown in Fig. 3). The boundary $\partial\Omega$ of $\Omega$ is formed by two straight lines

$$K_i = \{(x_1, x_2); x_1 = d_i, x_2 \in R_1\}, \quad i = 1, 2 \tag{1.2}$$

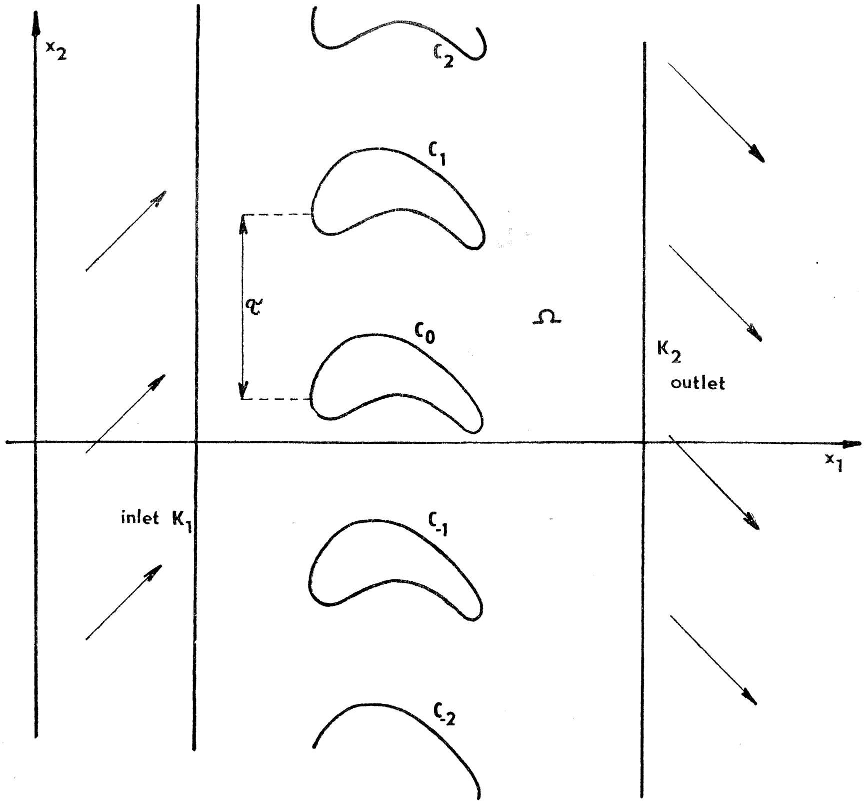


Fig. 3.

and by an infinite number of disjoint Jordan curves $C_k$, $k = 0, \pm 1, \pm 2, \ldots$, periodically spaced in the direction $x_2$ with the period $\tau = 2\pi\, r(0)/N$. The curves $C_k$ are given by the intersections of the blades with the surface $\mathscr{S}_1$ and form the so-called *cascade of profiles*. The lines $K_1$ and $K_2$ are called the *inlet* and the *outlet* of the cascade, since they represent the intersections of the surface $\mathscr{S}_1$ with the inlet and outlet of the channel $\Omega_3$, respectively.

The profile $C_k$ is obtained by moving $C_0$ in the direction $x_2$ by $k\tau$:

$$C_k = \{(x_1, x_2 + k\tau)\, ;\, (x_1, x_2) \in C_0\} \tag{1.3}$$

Hence, the domain $\Omega$ is periodic in the direction $x_2$ with the period $\tau$. It means that

$$(x_1, x_2) \in \bar{\Omega} \Leftrightarrow (x_1, x_2 + \tau) \in \bar{\Omega} . \tag{1.4}$$

We shall consider the following assumption concerning the profiles $C_k$:

Assumption (A1). The profile $C_0$ (and hence $C_k$, $k = \pm 1, \pm 2, \ldots$) is a piecewise smooth Jordan curve and the angles between neighbouring smooth parts of $C_0$ lie in the open interval $(0, 2\pi)$. ■

### 1.2. Equations describing the flows in a layer of variable thickness

In order to obtain a simplified two-dimensional model approximating the flows in the space between the surfaces $\mathscr{S}_1$, $\mathscr{S}_2$, we assume:

1) The surfaces $\mathscr{S}_1$, $\mathscr{S}_2$ are impermeable.
2) $\mathscr{S}_1$, $\mathscr{S}_2$ are "close enough" so that we can assume that the quantities describing the flow in the layer between $\mathscr{S}_1$ and $\mathscr{S}_2$ are constant in the direction normal to $\mathscr{S}_1$.
3) The blade row is not moving, blades are fixed and the flow is stationary.
4) The fluid is non-viscous.
5) The flow is irrotational.
6) Outer volume forces are neglected.
7) If the fluid is compressible, then the flow is subsonic and isentropic.

The system of equations describing the flow considered under the above assumptions consists of the *equation of continuity*, *condition of the irrotational flow* and *the equation for density*:

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i} (r(x_1)\, h(x_1)\, \varrho(x)\, v_i(x)) = 0 , \tag{1.5}$$

$$\frac{\partial(r(x_1)\, v_1(x))}{\partial x_2} - \frac{\partial(r(x_1)\, v_2(x))}{\partial x_1} = 0 , \tag{1.6}$$

(1.7) a) $\varrho(x) = \varrho_0$ ,

if the fluid is incompressible,

b) $$\varrho(x) = \varrho_0 \left(1 - \frac{\varkappa - 1}{2} \frac{v_1^2(x) + v_2^2(x)}{a_0^2}\right)^{1/(\varkappa - 1)} ,$$

if the fluid is compressible.

Here, we consider $x = (x_1, x_2) \in \Omega$ and use the following notation: $\varrho$ – density of the fluid, $v_i$ – velocity component in the direction $x_i$ $(i = 1, 2)$, $\mathbf{v} = (v_1, v_2)$ – velocity vector, $|\mathbf{v}|$ – absolute value of $\mathbf{v}$, $a = a_0(\varrho/\varrho_0)^{\varkappa - 1}$ – speed of sound, $M = |\mathbf{v}|/a$ – Mach number, $\varrho_0 > 0$, $a_0 > 0$, $\varkappa > 1$ – given constants. The equations (1.5)–(1.7) were derived e.g. in [21, 32] for incompressible flows and in [11] or [12] also in the case of compressible flows. They can be obtained from the general laws of fluid dynamics written in the integral form by neglecting the terms of higher orders in $h$.

In what follows we assume that

$$r, h \in C^1(\langle d_1, d_2 \rangle), \quad h, r > 0 \quad \text{in} \quad \langle d_1, d_2 \rangle. \tag{1.8}$$

With respect to the periodicity of the domain $\Omega$ we shall assume that the functions $v_1, v_2, \varrho$ are periodic in the direction $x_2$ with the period $\tau$:

$$\begin{aligned} &v_i(x_1, x_2 + \tau) = v_i(x_1, x_2), \quad i = 1, 2, \\ &\varrho(x_1, x_2 + \tau) = \varrho(x_1, x_2), \\ &(x_1, x_2) \in \bar{\Omega}. \end{aligned} \tag{1.9}$$

### 1.3. Stream function

It is convenient to introduce the so-called stream function $\psi : \bar{\Omega} \to R_1$ that satisfies the relations

$$\begin{aligned} &\frac{\partial \psi}{\partial x_1}(x) = -r(x_1)\, h(x_1)\, \varrho(x)\, v_2(x), \\ &\frac{\partial \psi}{\partial x_2}(x) = r(x_1)\, h(x_1)\, \varrho(x)\, v_1(x) \\ &\forall x \in (x_1, x_2) \in \Omega. \end{aligned} \tag{1.10}$$

The existence of the stream function can be proved on the basis of the equation (1.5) and the assumption that the blades are impermeable and fixed. From the periodicity conditions (1.9) it follows that

$$\psi(x_1, x_2 + \tau) = \psi(x_1, x_2) + Q \quad \forall (x_1, x_2) \in \bar{\Omega}. \tag{1.11}$$

The constant $Q$ is given by the total mass flux per second through the space bounded by the surfaces $\mathscr{S}_1$, $\mathscr{S}_2$ and two neighbouring blades.

If we substitute the relations (1.10) into (1.6), we get the equation

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\frac{1}{h\varrho} \frac{\partial \psi}{\partial x_i}\right) = 0 \quad \text{in} \quad \Omega. \tag{1.12}$$

For an incompressible fluid we have $\varrho = \varrho_0 = \text{const}$ and the equation (1.12) is linear and elliptic.

If the fluid is compressible, the situation is more complicated. From (1.10) and (1.7)b) we get

$$\varrho = \varrho_0 \left(1 - \frac{\varkappa - 1}{2} (a_0 r h \varrho)^{-2} (\nabla \psi)^2\right)^{1/(\varkappa - 1)}, \tag{1.13}$$

where $\nabla \psi = (\partial \psi / \partial x_1, \partial \psi / \partial x_2)$.

We see that the density is an implicit function dependent on $x$ and $\eta = (\nabla\psi)^2$. The equation (1.13) is solvable with respect to $\varrho$ for values of $\eta$ from a bounded interval only and for these $\eta$ there exist two solutions – one corresponding to subsonic and the other to supersonic flows.

These difficulties can be avoided, if we confine our considerations to subsonic flows with Mach number $M \leqq M^*$, where $M^* \in (0, 1)$ can be chosen arbitrarily close to one. Following the results from [5, 6, 10] we can construct the equation of the form

$$(1.14) \qquad \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(b(x, (\nabla\psi)^2)\frac{\partial\psi}{\partial x_i}\right) = 0 \quad \text{in } \Omega\,,$$

with "good" mathematical properties, which describes stream fields with $M \leqq M^*$. (The details are contained in [11, 12].)

1.3.1. **The function** $b$ is defined in the following way:

$$(1.15)\ \text{a)}\quad \lambda = \frac{2}{\varkappa - 1}\,(>0)\,,\quad \sigma_{kr} = \left(\frac{\lambda}{\lambda + 1}\right)^{\lambda},\quad \vartheta_{kr} = \frac{1}{\lambda + 1}\,\sigma_{kr}\,;$$

$$\text{b)}\quad \sigma^* = \left(\frac{M^{*2}}{\lambda} + 1\right)^{-\lambda} \in (\sigma_{kr}, 1)\,,$$

$$\vartheta^* = \sigma^* - \sigma^{*(1+1/\lambda)} \in (0, \vartheta_{kr})\,;$$

c) if $\vartheta \in \langle 0, \vartheta^*\rangle$, then $\sigma(\vartheta) \in \langle \sigma^*, 1\rangle$

is a (unique) solution of the equation

$$\sigma(\vartheta) = \left(1 - \frac{\vartheta}{\sigma(\vartheta)}\right)^{\lambda};$$

d) $\tilde{\sigma}: \langle 0, +\infty) \to \langle \sigma_0, 1\rangle\ (\sigma_0 \in (0, \sigma_{kr}\rangle)$

is a function with the following properties:

(i) $\tilde{\sigma}$ has a Lipschitz continuous $k$-th order derivative in $\langle 0, +\infty)$ $(k \geqq 1)$,

(ii) $\tilde{\sigma} \mid \langle 0, \vartheta^*\rangle = \sigma$,

(iii) $\tilde{\sigma}' \leqq 0$ in $\langle 0, +\infty)$,

(iv) there exists a constant $\hat{\vartheta} \geqq \vartheta_{kr}$ such that $\tilde{\sigma}(\vartheta) = \sigma_0\ \forall \vartheta \geqq \hat{\vartheta}$;

e) $b(x, \eta) = (\varrho_0\, h(x_1))^{-1}\,[\tilde{\sigma}(\lambda^{-1}(a_0\varrho_0\, r(x_1)\, h(x_1))^{-2}\,\eta)]^{-1/2}$
$\forall x = (x_1, x_2) \in \bar{\Omega}$, $\forall \eta \geqq 0$. ■

1.3.2. **Remark.** A simple example of the extension of the function $\sigma$ from the interval $\langle 0, \vartheta^*\rangle$ onto $\langle 0, +\infty)$, convenient for the numerical solution of the problem, can be found e.g. in [6]. ■

If the fluid is incompressible, we put

$$b = b(x) = \frac{1}{h(x_1)\,\varrho_0} \tag{1.16}$$

and the equation (1.12) can also be written in the form (1.14).

In the study of boundary value problems for the stream function we shall use the following properties of the function $b$:

1.3.3. **Lemma** 1) *The function $b$ is continuous in $\bar{\Omega} \times \langle 0, +\infty)$.*

2) *There exist constants $c_1, c_2 > 0$ such that*

$$c_1 \leqq b \leqq c_2 \quad in \quad \bar{\Omega} \times \langle 0, +\infty) . \tag{1.17}$$

3) *The function $b$ has continuous derivatives $\partial b/\partial \eta$ and $\partial b/\partial x_i$ $(i = 1, 2)$ in $\bar{\Omega} \times \langle 0, +\infty)$.*

4) *There exists $\hat{\eta} > 0$ such that*

$$\frac{\partial b}{\partial \eta}(x, \eta) = 0 \quad \forall x \in \bar{\Omega}, \quad \forall \eta \geqq \hat{\eta} . \tag{1.18}$$

(*If the fluid is incompressible, then of course $\partial b/\partial \eta = 0$ in $\bar{\Omega} \times \langle 0, +\infty)$.*)

5) *There exist constants $c_3, c_4 > 0$ such that*

$$0 \leqq \frac{\partial b}{\partial \eta} \leqq c_3 \quad in \quad \bar{\Omega} \times \langle 0, +\infty), \tag{1.19}$$

$$\left|\frac{\partial b}{\partial \eta}(x, \xi^2)\,\xi\right|, \quad \left|\frac{\partial b}{\partial \eta}(x, \xi^2)\,\xi^2\right| \leqq c_4 \quad \forall x \in \bar{\Omega}, \quad \forall \xi \in R_1 . \tag{1.20}$$

6) *If $\alpha_1 \in R_1$, $x \in \bar{\Omega}$, then the function $b(x, \alpha_1^2 + \xi^2)\,\xi$ of the variable $\xi$ is increasing in $R_1$.*

7) $b(x_1, x_2 + \tau, \eta) = b(x_1, x_2, \eta)$ $\forall (x_1, x_2) \in \bar{\Omega}$ *and* $\forall \eta \geqq 0$ .

Proof follows from the relations (1.15) e) or (1.16), the assumptions (1.8) and the properties (1.15) d) of the function $\tilde{\sigma}$. ■

1.3.4. **Remark.** On the basis of the detailed analysis ([11]) we can clarify the relation between the system (1.5)–(1.7) and the equation (1.14):

1) If $\psi: \bar{\Omega} \to R_1$ is a solution of the equation (1.14), where the function $b$ is defined by (1.16) or (1.15) for incompressible or subsonic compressible flows, respectively, then the functions $\varrho$, $v_1$, $v_2$ given by the relations

(1.21) a) $\varrho(x) = \varrho_0 = [h(x_1)\,b(x)]^{-1}$ or $\varrho(x) = [h(x_1)\,b(x, (\nabla\psi)^2(x))]^{-1}$

in the cases of incompressibility or compressibility, respectively,

$$\text{b)}\quad v_1(x) = \frac{1}{r(x_1)\,h(x_1)\,\varrho(x)}\,\frac{\partial\psi}{\partial x_2}(x)\,,$$

$$\text{c)}\quad v_2(x) = \frac{-1}{r(x_1)\,h(x_1)\,\varrho(x)}\,\frac{\partial\psi}{\partial x_1}(x)\,,$$

$$x \in \Omega\,,$$

form a solution of the equations (1.5), (1.6). If the condition (1.11) is fulfilled, then $\varrho$, $v_1$, $v_2$ satisfy the periodicity conditions (1.9). In the case of incompressible flows, (1.7) a) is equivalent to (1.21) a). If the fluid is compressible and

$$(1.22)\qquad (\nabla\psi)^2\,(x) \leqq \lambda[a_0\varrho_0\,r(x_1)\,h(x_1)]^2\,\vartheta^* \quad \forall x \in \bar{\Omega}\,,$$

then the equation (1.7) b) holds and $M \leqq M^*$ in $\bar{\Omega}$.

2) If $\varrho$, $v_1$, $v_2$ form a solution of (1.5)–(1.7) and if $M \leqq M^*$ in $\bar{\Omega}$ in the case of compressibility, then there exists a stream function $\psi$ satisfying the relations (1.10). This $\psi$ is a solution of the equation (1.14) in $\Omega$.

### 1.4. **Boundary conditions**

There exists a series of various boundary conditions that can be added to the equation (1.14) in order to characterize the behaviour of the stream fields on the boundary.

We shall denote by $\boldsymbol{n} = (n_1, n_2)$ the unit outer normal to $\partial\Omega$ and by $\partial/\partial n$ the derivative in the direction $\boldsymbol{n}$.

1.4.1. **Conditions on profiles.** Since the blades are fixed and impermeable, we have

$$(1.23)\qquad \psi \mid C_k = q_0 + kQ\,, \quad k = 0, \pm 1, \pm 2, \ldots,$$

where $Q$ is the given constant from (1.11). The constant $q_0$ may be unknown. ■

1.4.2. **Conditions on the inlet or outlet.** Here we have more possibilities.

a) If the quantity $rh\varrho v_1 \mid K_i$ is equal to a given function $\varphi_i$, then we consider the condition of the form

$$(1.24)\qquad \psi(d_i, x_2) = \Psi_i(x_2) + q_i\,, \quad x_2 \in R_1\,.$$

The functions $\varphi_i$ are $\tau$-periodic in $R_1$. $\Psi_i(x_2)$ are given by

$$(1.25)\qquad \Psi_i(x_2) = \int_0^{x_2} \varphi_i(\xi)\,\mathrm{d}\xi, \quad x_2 \in R_1\,, \quad i = 1, 2$$

and satisfy the conditions

$$\Psi_i(x_2 + \tau) = \Psi_i(x_2) + Q \quad \forall x_2 \in R_1\,, \quad i = 1, 2 \tag{1.26}$$

with

$$Q = \int_0^\tau \varphi_1(\xi)\,\mathrm{d}\xi = \int_0^\tau \varphi_2(\xi)\,\mathrm{d}\xi\,. \tag{1.27}$$

The constants $q_i$ may be unknown.

b) If the tangential component of the velocity $v_2 \mid K_i$ is given, then

$$\left[b(\cdot, (\nabla\psi)^2)\frac{\partial\psi}{\partial n}\right](d_i, x_2) = -\mu_i(x_2)\,, \quad x_2 \in R_1\,, \tag{1.28}$$

$$i = 1 \quad \text{or} \quad i = 2\,,$$

where $\mu_i : R_1 \to R_1$ is a given $\tau$-periodic function.

c) Sometimes we do not know the distribution of the tangential component of the velocity on $K_i$, but we can determine its average value. In this case we have the condition

$$\frac{1}{\tau}\int_{x_2}^{x_2+\tau}\left[b(\cdot, (\nabla\psi)^2)\frac{\partial\psi}{\partial n}\right](d_i, \xi)\,\mathrm{d}\xi = -\bar{\mu}_i\,, \quad x_2 \in R_1\,, \tag{1.29}$$

$$i = 1 \quad \text{or} \quad i = 2\,,$$

with a given constant $\bar{\mu}_i \in R_1$.

d) The constant $Q$ is determined either by (1.27) or from the given total mass flux per second through the space bounded by the surfaces $\mathscr{S}_1$, $\mathscr{S}_2$ and two neighbouring blades. ∎

1.4.3. **Complementary conditions.** a) If the circulation of the velocity round the blades is known, then we consider the following conditions with the line integrals along the curves $C_k$:

$$\int_{C_k} b(\cdot, (\nabla\psi)^2)\frac{\partial\psi}{\partial n}\,\mathrm{d}s = -\gamma\,, \quad k = 0, \pm 1, \pm 2, \ldots \tag{1.30}$$

$\gamma \in R_1$ is a given constant.

b) Usually, the circulation of the velocity is not known and then we consider the so-called trailing conditions which are more suitable from the physical point of view (cf. e.g. [7]):

$$\frac{\partial\psi}{\partial n}(z_k) = 0\,, \quad k = 0, \pm 1, \pm 2, \ldots \tag{1.31}$$

Here, $z_k = z_0 + (0, k\tau) \in C_k$ are given trailing points. ∎

## 1.5. **Classical formulation of the problem**

By a convenient choice of the above boundary conditions added to the equation (1.14) we get various boundary value problems describing the flows through cascades of profiles. We introduce here only several possibilities which seem to be the most convenient ones for technical practice.

I) *Let $\tau$-periodic functions $\varphi_1$, $\varphi_2 : R_1 \to R_1$ satisfying* (1.27) *be given. Let the constant $Q$ and functions $\Psi_1$, $\Psi_2$ be given by* (1.27) *and* (1.25), *respectively. Then $\Psi_1$, $\Psi_2$ and $Q$ satisfy* (1.26).

Problem (PSI. 1.1). *Given constants $\bar{\mu}_1$, $\bar{\mu}_2 \in R_1$, find $\psi \in C^2(\bar{\Omega})$ and constants $q_1$, $q_2 \in R_1$ satisfying the equation* (1.14) *in $\Omega$ and the conditions a)* (1.11), *b)* (1.23) *with $q_0 = 0$, c)* (1.24) *and* (1.29) *for $i = 1, 2$.*

Problem (PSI. 1.2). *Given constants $\bar{\mu}_1$, $\gamma \in R_1$, find $\psi \in C^2(\bar{\Omega})$ and constants $q_0$, $q_1$ satisfying* (1.14) *in $\Omega$ and the conditions a)* (1.11), *b)* (1.23) *(with $q_0$ unknown), c)* (1.24) *for $i = 1, 2$ with $q_1$ unknown and $q_2 = 0$, d)* (1.29) *for $i = 1$ and e)* (1.30).

Problem (PSI. 1.3). *Given a constant $\bar{\mu}_1 \in R_1$ and trailing points $z_k = z_0 + (0, k\tau) \in C_k$. Find $\psi \in C^2(\bar{\Omega})$ and constants $q_0$, $q_1$ satisfying the equation* (1.14) *and the conditions a)* (1.11), *b)* (1.23) *(with $q_0$ unknown), c)* (1.24) *for $i = 1, 2$ with $q_1$ unknown and $q_2 = 0$, d)* (1.29) *for $i = 1$ and e)* (1.31). ■

II) *Given a constant $Q \in R_1$ and $\tau$-periodic functions $\mu_1, \mu_2 : R_1 \to R_1$.*

Problem (PSI. 2.1). *Find $\psi \in C^2(\bar{\Omega})$ satisfying the equation* (1.14) *in $\Omega$ and the conditions a)* (1.11), *b)* (1.23) *with $q_0 = 0$ and c)* (1.28) *for $i = 1, 2$.* ■

# 2. SOLVABILITY OF THE PROBLEM (PSI. 1.1)

Let $\Omega_\tau \subset \Omega$ be a curved strip of the width $\tau$ in the $x_2$-direction cut from the domain $\Omega$. Its boundary $\partial\Omega_\tau$ consists of two components – the inner component formed by the profile $C_0$ and the outer component, which is the union $\Gamma_1 \cup \Gamma_2 \cup \Gamma^- \cup \Gamma^+$. Here, $\Gamma_i = \{(d_i, x_2);\ x_2 \in \langle e_i, e_i + \tau \rangle\} \subset K_i$ is a segment of the length $\tau$, $\Gamma^-$ is a piecewise linear arc and $\Gamma^+ = \{(x_1, x_2 + \tau);\ (x_1, x_2) \in \Gamma^-\}$. See Fig. 4.

The initial points $(d_1, e_1)$, $(d_1, e_1 + \tau)$ of the arcs $\Gamma^-$, $\Gamma^+$ belong to $K_1$, their terminal points $(d_2, e_2)$, $(d_2, e_2 + \tau) \in K_2$ and all the other points of these arcs are elements of the domain $\Omega$. In view of the assumption (A 1) from Section 1.1, the boundary $\partial\Omega_\tau$ is Lipschitz-continuous and it is possible to define the one-dimensional Lebesgue measure on $\partial\Omega_\tau$ (see [18] or [24]). Let $\Omega_\tau^*$ de the bounded domain with $\delta\Omega_\tau^* = \Gamma_1 \cup \Gamma_2 \cup \Gamma^- \cup \Gamma^+$.

Let $\psi \in C^2(\bar{\Omega})$ be a solution of the equation (1.14). Let us multiply this equation by any function $v \in C^\infty(\bar{\Omega}_\tau)$ and integrate over the domain $\Omega_\tau$. By the application of Green's theorem we get

$$(2.1)\qquad 0 = \int_{\Omega_\tau} \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(b\,\frac{\partial\psi}{\partial x_i}\right) v\,\mathrm{d}x = \int_{\partial\Omega_\tau} b\,\frac{\partial\psi}{\partial n}\,v\,\mathrm{d}s - \int_{\Omega_\tau} b\,\nabla\psi\,.\,\nabla v\,\mathrm{d}x\,.$$

Hence,

$$\int_{\Omega_\tau} b\,\nabla\psi\,.\,\nabla v\,\mathrm{d}x = \int_{C_0\cup\Gamma_1\cup\Gamma_2\cup\Gamma^-\cup\Gamma^+} b\,\frac{\partial\psi}{\partial n}\,v\,\mathrm{d}s\,. \tag{2.2}$$

By a suitable choice of functions $v \in C^\infty(\bar{\Omega}_\tau)$ we get variational formulations of the particular problems for the stream function.

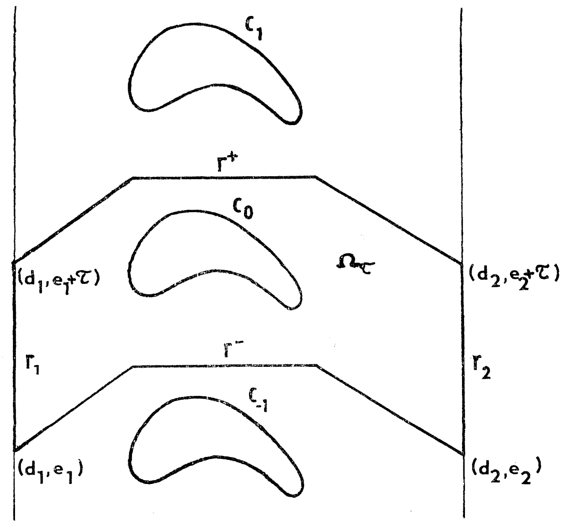


Fig. 4.

### 2.1. Variational formulation of the problem (PSI. 1.1)

If $x = (x_1, x_2) \in R_2$, then we shall use the simplified notation for the point $(x_1, x_2 + \tau)$:

$$x^\tau = (x_1, x_2 + \tau)\,. \tag{2.3}$$

Let us define

$$\mathscr{V}_\tau = \{v \in C^\infty(\bar{\Omega}_\tau);\ v \mid C_0 = 0,\ v \mid \Gamma_i = \text{const for } i = 1, 2,\ v(x^\tau) = \\ = v(x)\ \forall x \in \Gamma^-\}\,. \tag{2.4}$$

If $v \in \mathscr{V}_\tau$, then we denote by $v_{\Gamma_i}$ the constant value $v \mid \Gamma_i$.

Let the function $\psi$ and constants $q_1$, $q_2$ form a solution of the problem (PSI. 1.1). In (2.2) we shall consider the functions $v \in \mathscr{V}_\tau$. If we take into account (1.29), (2.4) and

$$\frac{\partial\psi}{\partial n}(x^\tau) = -\frac{\partial\psi}{\partial n}(x)\,, \tag{2.5}$$

$$b(x^\tau, (\nabla\psi)^2(x^\tau)) = b(x, (\nabla\psi)^2(x))\,,\quad x \in \Gamma^-$$

(obtained from (1.11) and the assertion 7) of Lemma 1.3.3), then for any $v \in \mathscr{V}_\tau$ we get

$$(2.6) \quad \text{a)} \int_{C_0} b \frac{\partial \psi}{\partial n} v \, ds = 0 ,$$

$$\text{b)} \int_{\Gamma_i} b \frac{\partial \psi}{\partial n} v \, ds = v \mid \Gamma_i \int_{\Gamma_i} b \frac{\partial \psi}{\partial n} \, ds = -\tau \bar{\mu}_i v_{\Gamma_i} ,$$

$$\text{c)} \int_{\Gamma^-} b \frac{\partial \psi}{\partial n} v \, ds = - \int_{\Gamma^+} b \frac{\partial \psi}{\partial n} v \, ds .$$

By (2.2) and (2.6) we find out that

$$(2.7) \qquad \int_{\Omega_\tau} b \, \nabla \psi \, . \, \nabla v \, dx = -\tau \sum_{i=1}^{2} \bar{\mu}_i v_{\Gamma_i} \quad \forall v \in \mathscr{V}_\tau .$$

Moreover, $\psi$ satisfies the conditions

$$(2.8) \quad \text{a)} \ \psi \in C^2(\bar{\Omega}_\tau) , \quad \text{b)} \ \psi \mid C_0 = 0 ,$$

$$\text{c)} \ \psi(x^\tau) = \psi(x) + Q \quad \forall x \in \Gamma^- ,$$

$$\text{d)} \ \psi \mid \Gamma_i = \Psi_i + q_i , \quad q_i = \text{const} , \quad i = 1, 2 .$$

((2.8) d) means that $\psi(d_i, x_2) = \Psi_i(x_2) + q_i$ for $(d_i, x_2) \in \Gamma_i$.)

The conditions (2.7) and (2.8) a)–d) (with the given constant $Q$ and functions $\Psi_1$, $\Psi_2$ satisfying (1.26)) form the so-called *variational formulation of the problem* (PSI. 1.1).

Now, let us assume that $\psi : \bar{\Omega}_\tau \to R_1$ and constants $q_1$, $q_2$ form a solution of (2.7), (2.8) a)–d). In the following we shall show that with the help of $\psi$ it is possible to construct a solution of the problem (PSI. 1.1):

a) By a suitable choice of $v \in \mathscr{V}_\tau$ such that $v \mid \partial\Omega_\tau = 0$ and by the application of Green's theorem we find out that $\psi$ is a solution of the equation (1.14) in $\bar{\Omega}_\tau$.

b) In virtue of (2.8) c),

$$(2.9) \qquad \frac{\partial \psi}{\partial t}(x^\tau) = \frac{\partial \psi}{\partial t}(x) \quad \forall x \in \Gamma^- - \{x^1, \ldots, x^s\} ,$$

where $\partial\psi/\partial t$ denotes the derivative of $\psi$ with respect to the arc $\Gamma^-$ and $x^1, \ldots, x^s$ are all points where $\Gamma^-$ is not smooth. Now let us consider any $v \in \mathscr{V}_\tau$ for which $v \mid \Gamma_1 \cup \Gamma_2 = 0$. Then on the basis of (2.1), (2.2) and (2.7) we find out that (2.6) c) holds for these $v$, which implies the relation

$$(2.10) \qquad b\left(x^\tau, \left(\frac{\partial \psi}{\partial t}(x^\tau)\right)^2 + \left(\frac{\partial \psi}{\partial n}(x^\tau)\right)^2\right) \frac{\partial \psi}{\partial n}(x^\tau) =$$

$$= -b\left(x, \left(\frac{\partial \psi}{\partial t}(x)\right)^2 + \left(\frac{\partial \psi}{\partial n}(x)\right)^2\right) \frac{\partial \psi}{\partial n}(x) \quad \forall x \in \Gamma^- - \{x^1, \ldots, x^s\} .$$

From (2.9), (2.10) and the assertions 6), 7) of Lemma 1.3.3 we get

$$\frac{\partial \psi}{\partial n}(x^\tau) = -\frac{\partial \psi}{\partial n}(x) \quad \forall x \in \Gamma^- - \{x^1, \ldots, x^s\}. \tag{2.11}$$

Hence, since $\psi \in C^2(\bar{\Omega}_\tau)$,

$$\nabla\psi(x^\tau) = \nabla\psi(x) \quad \forall x \in \Gamma^- . \tag{2.12}$$

The last two identities imply that (2.6) c) is valid for each $v \in \mathscr{V}_\tau$.

If we extend the function $\psi$ from $\bar{\Omega}_\tau$ onto the set $\bar{\Omega}$ so that the condition (1.11) is satisfied, we get a function which belongs to the space $C^1(\bar{\Omega})$. Let us denote it by $\psi_E$.

c) Let us consider any $v \in \mathscr{V}_\tau$ such that $v \mid \Gamma_1 = 1$ and $v \mid \Gamma_2 = 0$. If we substitute this function into (2.7), use (2.1), (2.2) and (2.6) c), then we have

$$\int_{\Gamma_1} b \frac{\partial \psi}{\partial n} \, ds = -\tau \bar{\mu}_1 ,$$

which implies that the extended function $\psi_E$ satisfies the condition (1.29), $i = 1$. Similarly, we prove also (1.29), $i = 2$.

d) In order to show that $\psi_E$ (together with $q_1, q_2$) is a solution of the problem (PSI. 1.1), we have to prove that $\psi \in C^2(\bar{\Omega})$.

In view of the assumption $\psi \in C^2(\bar{\Omega}^\tau)$, of (2.9) and (2.11),

$$\frac{\partial^2 \psi}{\partial t^2}(x^\tau) = \frac{\partial^2 \psi}{\partial t^2}(x), \tag{2.13}$$

$$\frac{\partial^2 \psi}{\partial t \, \partial n}(x^\tau) = \frac{\partial^2 \psi}{\partial n \, \partial t}(x^\tau) = -\frac{\partial^2 \psi}{\partial n \, \partial t}(x) = -\frac{\partial^2 \psi}{\partial t \, \partial n}(x)$$

$$\forall x \in \Gamma^- - \{x^1, \ldots, x^s\} .$$

If we express the equation (1.14) at any $x \in \Gamma^- \cup \Gamma^+ - \{x^1, (x^1)^\tau, \ldots, x^s, (x^s)^\tau\}$ with the use of the derivatives in the normal and tangential directions to $\Gamma^- \cup \Gamma^+$, we get

$$0 = a_{11}(x, \nabla\psi(x)) \frac{\partial^2 \psi}{\partial n^2}(x) + 2a_{12}(x, \nabla\psi(x)) \frac{\partial^2 \psi}{\partial t \, \partial n}(x) + \tag{2.14}$$

$$+ a_{22}(x, \nabla\psi(x)) \frac{\partial^2 \psi}{\partial t^2}(x) + a_{00}(x, \nabla\psi(x))$$

with $a_{11} > 0$. Moreover, $a_{ij}(x^\tau, \nabla\psi(x^\tau)) = (-1)^{i+j} a_{ij}(x, \nabla\psi(x))$ for $i, j = 0, 1, 2$ and $x \in \Gamma^-$. This relation, (2.13) and (2.14) immediately give the equality

$$\frac{\partial^2 \psi}{\partial n^2}(x^\tau) = \frac{\partial^2 \psi}{\partial n^2}(x) \quad \forall x \in \Gamma^- - \{x^1, \ldots, x^s\} , \tag{2.15}$$

which together with the above results already implies that $\psi_E \in C^2(\bar{\Omega})$.

Thus we have proved:

2.1.1 **Theorem.** *The problem* (PSI. 1.1) *is equivalent to* (2.7) *and* (2.8) *a)–d) in the following sense: If* $\psi, q_1, q_2$ *form a solution of the problem* (PSI. 1.1), *then* $\psi \mid \bar{\Omega}_\tau, q_1, q_2$ *satisfy* (2.7) *and* (2.8) *a)–d). On the other hand, on the basis of* $\psi : \bar{\Omega}_\tau \to R_1$ *and* $q_1, q_2$ *that solve the problem* (2.7), (2.8) *a)–d), we can construct a solution* $\psi_E : \bar{\Omega} \to R_1$ *of the problem* (PSI. 1.1) *such that* $\psi_E \mid \bar{\Omega}_\tau = \psi$. ■

2.1.2. **Remark.** If the solution of (2.7), (2.8) a)–d) is unique, then the solution $\psi_E$ does not depend on the choice of the strip $\Omega_\tau$. To prove this, let us consider two strips $\Omega_\tau, \Omega'_\tau$ and the corresponding variational problems with unique solutions $\psi : \bar{\Omega}_\tau \to R_1$ and $\psi' : \bar{\Omega}'_\tau \to R_1$, respectively. From $\psi$ and $\psi'$ we construct solutions $\psi_E$ and $\psi'_E$ of the problem (PSI. 1.1). In view of Theorem 2.1.1, $\psi'_E \mid \bar{\Omega}_\tau$ is also a solution of the problem (2.7), (2.8) a)–d), considered in the domain $\Omega_\tau$. As a consequence of the supposed unique solvability of this problem we have $\psi'_E \mid \bar{\Omega}_\tau = \psi$, which implies that $\psi_E = \psi'_E$. ■

### 2.2. Weak solutions

The preceding considerations lead us to the concept of generalized weak solutions to our stream function problem.

First, we introduce some functional spaces. We denote by $L_2(\Omega_\tau)$ the space of all (equivalent classes of) measurable functions square integrable over $\Omega_\tau$. $H^1(\Omega_\tau)$ is the well-known Sobolev space formed by all $v \in L_2(\Omega_\tau)$ whose first order distribution derivatives belong also to $L_2(\Omega_\tau)$. $L_2(\Omega_\tau)$ and $H^1(\Omega_\tau)$ are Hilbert spaces with scalar products

(2.16) $$(u, v)_{L_2(\Omega_\tau)} = \int_{\Omega_\tau} uv \, \mathrm{d}x \,, \quad u, v \in L_2(\Omega_\tau)$$

and

(2.17) $$(u, v)_H = \int_{\Omega_\tau} (uv + \nabla u \,.\, \nabla v) \, \mathrm{d}x, \; u, v \in H^1(\Omega_\tau) \,,$$

which induce the normes

(2.18) $$\|v\|_{L_2(\Omega_\tau)} = (v, v)^{1/2}_{L_2(\Omega_\tau)} \quad \text{in} \quad L_2(\Omega_\tau)$$

and

(2.19) $$\|v\|_H = (v, v)^{1/2}_H \quad \text{in} \quad H^1(\Omega_\tau) \,,$$

respectively.

Since $\partial\Omega_\tau$ is Lipschitz-continuous, it is possible to define the space $L_2(\partial\Omega_\tau)$ of all (equivalent classes of) measurable functions on $\partial\Omega_\tau$ square integrable over $\partial\Omega_\tau$:

$$\int_{\partial\Omega_\tau} v^2 \, \mathrm{d}s < +\infty \quad \text{for} \quad v \in L_2(\partial\Omega_\tau) \,.$$

$L_2(\partial\Omega_\tau)$ is equipped with the scalar product

$$(2.20) \qquad (u, v)_{L_2(\partial\Omega_\tau)} = \int_{\partial\Omega_\tau} uv \, ds \,, \quad u, v \in L_2(\partial\Omega_\tau)$$

and the norm

$$(2.21) \qquad \|v\|_{L_2(\partial\Omega_\tau)} = (v, v)^{1/2}_{L_2(\partial\Omega_\tau)} \,, \quad v \in L_2(\partial\Omega_\tau) \,.$$

If $\alpha, \beta \in R_1$, $\alpha < \beta$, then of course $L_2((\alpha, \beta))$ is the space of all (equivalent classes of) measurable functions square integrable over the interval $(\alpha, \beta)$.

$\mathscr{D}(\Omega_\tau)$ denotes the set formed by all functions $v \in C^\infty(\bar{\Omega}_\tau)$ with compact supports $\operatorname{supp} v \subset \Omega_\tau$. By $H^1_0(\Omega_\tau)$ we denote the closure of $\mathscr{D}(\Omega_\tau)$ in the topology of the space $H^1(\Omega_\tau)$.

We shall often use the important theorem on traces:

2.2.1. **Theorem.** *There exists a continuous linear mapping $\theta : H^1(\Omega_\tau) \to L_2(\partial\Omega_\tau)$ such that $\theta u = u \mid \partial\Omega_\tau$ for every $u \in C^\infty(\bar{\Omega}_\tau)$. Hence,*

$$(2.21) \qquad \|\theta u\|_{L_2(\partial\Omega_\tau)} \leqq k_\theta \|u\|_H \quad \forall u \in H^1(\Omega_\tau)$$

*with a constant $k_\theta$ independent of $u$.* ■

In the following we shall write $\theta u = u \mid \partial\Omega_\tau$ for $u \in H^1(\Omega_\tau)$. The space $H^1_0(\Omega_\tau)$ can be characterized as

$$(2.22) \qquad H^1_0(\Omega_\tau) = \{u \in H^1(\Omega_\tau);\ u \mid \partial\Omega_\tau = 0\} \,.$$

(Cf. [13, 18, 24].)

2.2.2. **Lemma.** *Let the functions $\varphi_i$, $i = 1, 2$, from 1.4.2. a) be $\tau$-periodic in $R_1$, satisfy the condition (1.27) and let $\varphi \mid (0, \tau) \in L_2((0, \tau))$. Then there exists $\psi^* \in H^1(\Omega_\tau)$ with the following properties:*

$$(2.23) \qquad \begin{aligned} &\text{a)}\ \psi^*(x^\tau) = \psi^*(x) + Q \,, && x \in \Gamma^- \,, \\ &\text{b)}\ \psi^* \mid \Gamma_i = \Psi_i \,, && i = 1, 2 \,, \\ &\text{c)}\ \psi^* \mid C_0 = 0 \,, && \end{aligned}$$

where $\Psi_i$ and $Q$ are defined by (1.25) and (1.27), respectively.

Proof. The function $\Psi_i$ $(i = 1, 2)$ can be written in the form $\Psi_i(x_2) = (Q/\tau)\, x_2 + g_i(x_2)$ $(x_2 \in R_1)$, where $g_i$ is $\tau$-periodic in $R_1$. Moreover, $g_i$ is an indefinite integral of the function $\beta_i = \varphi_i - Q/\tau : R_1 \to R_1$, $\beta_i \mid (0, \tau) \in L_2((0, \tau))$. Let us consider the infinitely differentiable transformation

$$(2.24) \qquad \begin{aligned} (x_1, x_2) \to (\tilde{x}_1, \tilde{x}_2) &= F(x_1, x_2) = (F_1(x_1, x_2), F_2(x_1, x_2)) = \\ &= (\exp(2\pi x_1/\tau) \cos(2\pi x_2/\tau), \exp(2\pi x_1/\tau) \sin(2\pi x_2/\tau)) \,, \end{aligned}$$

$\tau$-periodic in the direction $x_2$, which maps the strip $\mathscr{P} = \{(x_1, x_2);\ d_1 < x_1 < d_2,\ x_2 \in R_1\}$ onto the domain $F(\mathscr{P}) = \{(\tilde{x}_1, \tilde{x}_2); \exp(2\pi d_1/\tau) < \sqrt{(\tilde{x}_1^2 + \tilde{x}_2^2)} < \exp(2\pi d_2/\tau)\}$. The boundary $\partial F(\mathscr{P})$ is infinitely differentiable, since it is formed by the disjoint circle lines $\tilde{K}_i = F(K_i)$, $i = 1, 2$ ($K_i$ are straight lines defined by (1.2)).

The inverse $F_{-1} = ((F_{-1})_1, (F_{-1})_2)$ to $F$ can be considered as an infinitely valued analytic function. Let us put $\tilde{g}_i = g_i \circ ((F_{-1})_2 \mid \tilde{K}_i) : \tilde{K}_i \to R_1$ $(i = 1, 2)$. From the properties of $g_i$ and $F$ it follows that $\tilde{g}_i$ is a single-valued function which can be written as an indefinite integral along $\tilde{K}_i$ of a function $\tilde{\beta}_i \in L_2(\tilde{K}_i)$. If we define $\tilde{g} : \partial F(\mathscr{P}) \to R_1$ by the relations $\tilde{g} \mid \tilde{K}_i = \tilde{g}_i$, $i = 1, 2$, then according to [23] there exists $\tilde{\mathscr{E}} \in H^1(F(\mathscr{P}))$ such that $\tilde{\mathscr{E}} \mid \partial F(\mathscr{P}) = \tilde{g}$. Hence, $\tilde{\mathscr{E}} \mid \tilde{K}_i = \tilde{g}_i$, $i = 1, 2$.

Now let us put $\mathscr{E} = \tilde{\mathscr{E}} \circ F$. We see that $\mathscr{E}$ is $\tau$-periodic in the direction $x_2$ and $\mathscr{E}(d_i, x_2) = g_i(x_2)$, $x_2 \in R_1$, $i = 1, 2$. Since $F \mid \Omega_\tau^*$ is a one-to-one mapping of the domain $\Omega_\tau^*$ onto $\tilde{\Omega} = F(\mathscr{P}) - F(\Gamma^-)$, $F \in C^\infty(\bar{\Omega}_\tau^*)$ and the Jacobian determinant $|(DF/Dx)(x)| \neq 0$ for all $x \in \bar{\Omega}_\tau^*$, the restriction $F \mid \Omega_\tau^*$ and its inverse are Lipschitz-continuous. It is evident that $\tilde{\mathscr{E}} \in H^1(\tilde{\Omega})$. By the direct application of results from [24] (Ch. 2, § 3, page 66) we find out that $\mathscr{E} \mid \Omega_\tau \in H^1(\Omega_\tau)$.

Further, it is easy to see that there exists $\vartheta \in C^\infty(\bar{\Omega}_\tau)$ such that $\vartheta = 1$ in a neighbourhood of the outer component of $\partial\Omega_\tau$, formed by the union $\Gamma_1 \cup \Gamma_2 \cup \Gamma^- \cup \Gamma^+$, and $\vartheta = 0$ in a neighbourhood of $C_0$. If we put

$$(2.25) \qquad \psi^*(x_1, x_2) = \left(\frac{Q}{\tau} x_2 + \mathscr{E}(x_1, x_2)\right) \vartheta(x_1, x_2), \quad (x_1, x_2) \in \bar{\Omega}_\tau,$$

then $\psi^*$ is the sought function with the properties (2.23) a)–c). ∎

2.2.3. **Remark.** We can prove even a stronger result. In virtue of the properties of $\tilde{g}$, we have $\tilde{g} \in W_2^1(\partial F(\mathscr{P}))$ (cf. e.g. [18] or [24]). This and the results from [18], Ch. 8 imply the existence of

$$\tilde{\mathscr{E}} \in W_2^{1+1/2}(F(\mathscr{P}))\ (\subset H^1(F(\mathscr{P})) \cap C(\overline{F(\mathscr{P})}))$$

such that $\tilde{\mathscr{E}} \mid \partial F(\mathscr{P}) = \tilde{g}$. Now it is possible to show that $\mathscr{E} = \tilde{\mathscr{E}} \circ F \in W_2^{1+1/2}(\Omega_\tau^*)$ and thus, the function $\psi^*$ defined in $\bar{\Omega}_\tau$ by (2.25) satisfies the conditions (2.23) a)–c) and $\psi^* \in W_2^{1+1/2}(\Omega_\tau)$. (For the complete proof, see Appendix.) ∎

Next, in $H^1(\Omega_\tau)$ we define a subspace $V(\Omega_\tau)$ by

$$(2.26) \quad V(\Omega_\tau) = \{v \in H^1(\Omega_\tau);\ v \mid \Gamma_i = \text{const},\ i = 1, 2,\ v \mid C_0 = 0,\ v(x^\tau) = v(x) \text{ for almost every } x \in \Gamma^-\}.$$

(The concept "almost every $x \in \Gamma^-$" is considered here in the sense of the one-dimensional Lebesgue measure on $\partial\Omega_\tau$.)

2.2.4. **Lemma.** 1) $\mathscr{V}_\tau \subset V(\Omega_\tau)$. 2) *$V(\Omega_\tau)$ is a closed subspace of $H^1(\Omega_\tau)$.*

Proof. 1) Assertion 1) is evident. 2) We want to prove that the closure $\bar{V}(\Omega_\tau)$ of the space $V(\Omega_\tau)$ in the topology of $H^1(\Omega_\tau)$ is $V(\Omega_\tau)$. If $v \in \bar{V}(\Omega_\tau)$, then there exist $v_n \in V(\Omega_\tau)$, $n = 1, 2, \ldots$, such that $v_n \to v$ in $H^1(\Omega_\tau)$, if $n \to +\infty$. Let us prove that $v \in V(\Omega_\tau)$.

From Theorem 2.2.1 on traces we get

$$\lim_{n \to +\infty} \int_{\partial\Omega_\tau} |v_n - v|^2 \, ds = 0 \tag{2.27}$$

and thus,

$$\int_{C_0} |v_n - v|^2 \, ds \to 0 \,, \quad \int_{\Gamma_i} |v_n - v|^2 \, ds \to 0 \quad \text{for} \quad n \to +\infty \,.$$

This and the definition of the space $V(\Omega_\tau)$ imply that $v \mid C_0 = 0$ and $v \mid \Gamma_i = \text{const} = \lim_{n \to +\infty} v_{n\Gamma_i}$ (here $v_{n\Gamma_i} = v_n \mid \Gamma_i = \text{const}$), $i = 1, 2$. Next, by (2.27) we have

$$0 \leqq \left(\int_{\Gamma^-} |v(x^\tau) - v(x)|^2 \, ds\right)^{1/2} = \left(\int_{\Gamma^-} |v(x^\tau) - v_n(x^\tau) + v_n(x) - v(x)|^2 \, ds\right)^{1/2} \leqq$$

$$\leqq \left(\int_{\Gamma^-} |v(x^\tau) - v_n(x^\tau)|^2 \, ds\right)^{1/2} + \left(\int_{\Gamma^-} |v(x) - v_n(x)|^2 \, ds\right)^{1/2} =$$

$$= \left(\int_{\Gamma^+} |v - v_n|^2 \, ds\right)^{1/2} + \left(\int_{\Gamma^-} |v - v_n|^2 \, ds\right)^{1/2} \leqq$$

$$\leqq 2\left(\int_{\partial\Omega_\tau} |v_n - v|^2 \, ds\right)^{1/2} \to 0 \quad \text{for} \quad n \to +\infty \,.$$

This yields $v(x^\tau) = v(x)$ for almost every $x \in \Gamma^-$. If we summarize our results, we see that $v \in V(\Omega_\tau)$. ∎

For $u \in H^1(\Omega_\tau)$ we put

$$\|u\|_V = \left(\int_{\Omega_\tau} (\nabla u)^2 \, dx\right)^{1/2} . \tag{2.28}$$

2.2.5. **Lemma.** *The function $\|\cdot\|_V$ is a norm in the space $V(\Omega_\tau)$, equivalent to the norm $\|\cdot\|_H$. It means that there exist constant $c_5, c_6 > 0$ such that*

$$c_5\|u\|_V \leqq \|u\|_H \leqq c_6\|u\|_V \quad \forall u \in V(\Omega_\tau) \,.$$

Proof. Since the one-dimensional measure of $C_0$ (defined on $\partial\Omega_\tau$) is positive and $v \mid C_0 = 0$ for each $v \in V(\Omega_\tau)$, this lemma is a consequence of the well-known Fridrichs inequality. (Cf. [18, 24, 28].) ∎

By $V^*(\Omega_\tau)$ we denote the dual to the space $V(\Omega_\tau)$ (i.e. the space of all continuous linear functionals defined on $V(\Omega_\tau)$). If $f \in V^*(\Omega_\tau)$, $v \in V(\Omega_\tau)$, then $\langle f, v\rangle$ denotes the value of the functional $f$ at the point $v$. The norm of $f$ in $V^*(\Omega_\tau)$ is defined by the relation

$$\|f\|_{V^*} = \sup_{\substack{v \in V(\Omega_\tau) \\ \|v\|_V = 1}} |\langle f, v\rangle| . \tag{2.29}$$

2.2.6. **Lemma.** *$V(\Omega_\tau)$ is a Hilbert space, whose norm is induced by the scalar product*

$$(u, v)_V = \int_{\Omega_\tau} \nabla u \,.\, \nabla v \,\mathrm{d}x\,, \quad u, v \in V(\Omega_\tau)\,. \tag{2.30}$$

*If $f \in V^*(\Omega_\tau)$, then there exists exactly one $\varphi \in V(\Omega_\tau)$ such that*

$$\langle f, v\rangle = (\varphi, v)_V \quad \forall v \in V(\Omega_\tau)\,. \tag{2.31}$$

(Cf. e.g. [13, 20].) ■

Let us remark that the functions "$u, v \to (u, v)$"$_v$ and "$u \to \|u\|_V$" are also defined for $u, v \in H^1(\Omega_\tau)$. However, $\|\cdot\|_V$ is only a seminorm on $H^1(\Omega_\tau)$. In the following considerations it will be more convenient to work with the norm $\|\cdot\|_V$ in the space $V(\Omega_\tau)$ instead of the norm $\|\cdot\|_H$.

Now, let us define the form $\boldsymbol{a} : H^1(\Omega_\tau) \times H^1(\Omega_\tau) \to R_1$:

$$\boldsymbol{a}(\psi, v) = \int_{\Omega_\tau} b(\cdot, (\nabla\psi)^2)\, \nabla\psi \,.\, \nabla v \,\mathrm{d}x\,, \quad \psi, v \in H^1(\Omega_\tau) \tag{2.32}$$

and the function $\mu : V(\Omega_\tau) \to R_1$:

$$\mu(v) = -\tau \sum_{i=1}^{2} \bar{\mu}_i v_{\Gamma_i}\,, \quad v \in V(\Omega_\tau)\,. \tag{2.33}$$

From the continuity and boundedness of the function $b$ (see Lemma 1.3.3) it follows that for any $\psi, v \in H^1(\Omega_\tau)$ the finite integral in (2.32) exists.

Main properties of $\boldsymbol{a}$ and $\mu$:

2.2.7. **Theorem.** 1) *If $\psi \in H^1(\Omega_\tau)$, then the mapping "$v \in V(\Omega_\tau) \to \boldsymbol{a}(\psi, v) \in R_1$" represents a continuous linear functional defined on $V(\Omega_\tau)$. It means that we can write*

$$\boldsymbol{a}(\psi, v) = \langle \mathscr{A}(\psi), v\rangle\,, \quad \psi \in H^1(\Omega_\tau)\,, \quad v \in V(\Omega_\tau)\,, \tag{2.34}$$

*where $\mathscr{A}(\psi) \in V^*(\Omega_\tau)$. Hence, $\mathscr{A} : H^1(\Omega_\tau) \to V^*(\Omega_\tau)$.*

2) *The mapping $\mu : V(\Omega_\tau) \to R_1$ is a continuous linear functional on $V(\Omega_\tau)$:*

$$\langle \mu, v\rangle = \mu(v) = -\tau \sum_{i=1}^{2} \bar{\mu}_i v_{\Gamma_i}\,, \quad v \in V(\Omega_\tau)\,. \tag{2.35}$$

3) *There exists a constant $\alpha > 0$ such that*

$$\langle \mathscr{A}(\psi_1) - \mathscr{A}(\psi_2), \psi_1 - \psi_2\rangle \geqq \alpha\|\psi_1 - \psi_2\|_V^2 \quad \forall \psi_1, \psi_2 \in H^1(\Omega_\tau)\,, \\ \psi_1 - \psi_2 \in V(\Omega_\tau)\,. \tag{2.36}$$

4) *There exists a constant* $K > 0$ *such that*

$$(2.37) \qquad |\langle \mathscr{A}(\psi_1) - \mathscr{A}(\psi_2), v\rangle| \leqq K\|\psi_1 - \psi_2\|_V \|v\|_V \quad \forall \psi_1, \psi_2 \in H^1(\Omega_\tau), \quad \forall v \in V(\Omega_\tau).$$

5) *Let us put* $T(u) = \mathscr{A}(\psi^* + u)$ *for* $u \in V(\Omega_\tau)$, *where* $\psi^* \in H^1(\Omega_\tau)$ *is a function with the properties* (2.23) *a*)–*c*). *Then* $T: V(\Omega_\tau) \to V^*(\Omega_\tau)$ *and*

$$(2.38) \qquad \langle T(u_1) - T(u_2), u_1 - u_2\rangle \geqq \alpha\|u_1 - u_2\|_V^2 \quad \forall u_1, u_2 \in V(\Omega_\tau)$$

(*T is strongly monotone in* $V(\Omega_\tau)$),

$$(2.39) \qquad |\langle T(u_1) - T(u_2), v\rangle| \leqq K\|u_1 - u_2\|_V \|v\|_V \quad \forall u_1, u_2, v \in V(\Omega_\tau).$$

*Thus, in view of* (2.29),

$$(2.40) \qquad \|T(u_1) - T(u_2)\|_{V^*} \leqq K\|u_1 - u_2\|_V \quad \forall u_1, u_2 \in V(\Omega_\tau)$$

(*T is Lipschitz-continuous in* $V(\Omega_\tau)$).

6) *There exists exactly one mapping* $\mathscr{H} : V(\Omega_\tau) \to V(\Omega_\tau)$ *defined by the relation*

$$(2.41) \qquad \langle T(u), v\rangle = (\mathscr{H}(u), v)_V \quad \forall u, v \in V(\Omega_\tau).$$

*Next, there exists a uniquely determined* $\tilde{\mu} \in V(\Omega_\tau)$ *such that*

$$(2.42) \qquad \langle \mu, v\rangle = (\tilde{\mu}, v)_V \quad \forall v \in V(\Omega_\tau).$$

*The following inequalities hold*:

$$(2.43) \qquad (\mathscr{H}(u_1) - \mathscr{H}(u_2), u_1 - u_2)_V \geqq \alpha\|u_1 - u_2\|_V^2 \quad \forall u_1, u_2 \in V(\Omega_\tau)$$

($\mathscr{H}$ *is strongly monotone*),

$$(2.44) \qquad \|\mathscr{H}(u_1) - \mathscr{H}(u_2)\|_V \leqq K\|u_1 - u_2\|_V \quad \forall u_1, u_2 \in V(\Omega_\tau)$$

($\mathscr{H}$ *is Lipschitz-continuous*).

Proof. 1) If $\psi \in H^1(\Omega_\tau)$ and $v \in V(\Omega_\tau)$, then (1.17) and the Cauchy inequality yield

$$|\boldsymbol{a}(\psi, v)| = \left|\int_{\Omega_\tau} b(\cdot, (\nabla\psi)^2\, \nabla\psi \,.\, \nabla \mathrm{d}x\right| \leqq \int_{\Omega_\tau} |b(\cdot, (\nabla\psi)^2)\, \nabla\psi \,.\, \nabla v|\, \mathrm{d}x \leqq$$

$$\leqq c_2 \left(\int_{\Omega_\tau} (\nabla\psi)^2\, \mathrm{d}x\right)^{1/2} \left(\int_{\Omega_\tau} (\nabla v)^2\, \mathrm{d}x\right)^{1/2} = c_2\|\psi\|_V \|v\|_V = k_\psi\|v\|_V.$$

Moreover, it is evident that $\boldsymbol{a}(\psi, v)$ is linear with respect to $v$, which proves the assertion 1).

2) It is easy to see that the function $\mu$ is linear. Let us prove its boundedness. If $v \in V(\Omega_\tau)$, then

$$|\mu(v)| = \left|\tau \sum_{i=1}^{2} \bar{\mu}_i v_{\Gamma_i}\right| \leqq \tau \sum_{i=1}^{2} |\bar{\mu}_i| \, |v_{\Gamma_i}| \, .$$

Due to the relations

$$|v_{\Gamma_i}| = \frac{1}{\operatorname{meas}(\Gamma_i)} \int_{\Gamma_i} |v| \, \mathrm{d}s \, , \quad \operatorname{meas}(\Gamma_i) = \tau$$

($\operatorname{meas}(\Gamma_i)$ denotes the one-dimensional Lebesgue measure (defined on $\partial\Omega_\tau$) of the set $\Gamma_i$), to Theorem 2.2.1, Lemma 2.2.5 and the Cauchy inequality, we get the estimates

$$|\mu(v)| \leqq \sum_{i=1}^{2} |\bar{\mu}_i| \int_{\Gamma_i} |v| \, \mathrm{d}s \leqq (|\bar{\mu}_1| + |\bar{\mu}_2|) \int_{\partial\Omega_\tau} |v| \, \mathrm{d}s \leqq$$

$$\leqq (|\bar{\mu}_1| + |\bar{\mu}_2|) \left( \int_{\partial\Omega_\tau} 1 \, \mathrm{d}s \right)^{1/2} \left( \int_{\partial\Omega_\tau} v^2 \, \mathrm{d}s \right)^{1/2} =$$

$$= (|\bar{\mu}_1| + |\bar{\mu}_2|) (\operatorname{meas}(\partial\Omega_\tau))^{1/2} \, \|v\|_{L_2(\partial\Omega_\tau)} \leqq$$

$$\leqq (|\bar{\mu}_1| + |\bar{\mu}_2|) (\operatorname{meas}(\partial\Omega_\tau))^{1/2} \, k_\theta c_6 \|v\|_V = k_\mu \|v\|_V \quad \forall v \in V(\Omega_\tau) \, .$$

Hence, $\mu \in V^*(\Omega_\tau)$.

3) Let $\xi, \tilde{\xi}, \vartheta \in R_2$, $h = \tilde{\xi} - \xi$, $x \in \Omega_\tau$, $t \in R_1$ .

If we denote

$$g(t) = b(x, (\xi + th)^2) (\xi + th) \, . \, \vartheta \, , \tag{2.45}$$

then

$$g(1) - g(0) = [b(x, \tilde{\xi}^2) \, \tilde{\xi} - b(x, \xi^2) \, \xi] \, . \, \vartheta \, . \tag{2.46}$$

From the properties of the function $b$ (see Lemma 1.3.3) it follows that in the interval $\langle 0, 1 \rangle$ the derivative

$$\begin{aligned} g'(t) = {} & b(x, (\xi + th)^2) \, h \, . \, \vartheta \, + \\ & + 2 \frac{\partial b}{\partial \eta} (x, (\xi + th)^2) \, [(\xi + th) \, . \, h] \, [(\xi + th) \, . \, \vartheta] \end{aligned} \tag{2.47}$$

exists and is finite.

a) Let us put $\vartheta = h$. Then in view of (1.17) and (1.19), we have

$$g'(t) \geqq c_1 h^2 \quad \forall t \in \langle 0, 1 \rangle \, .$$

By the Mean Value Theorem, $g(1) - g(0) = \int_0^1 g'(t) \, \mathrm{d}t \geqq c_1 h^2$ and thus,

$$[b(x, \tilde{\xi}^2) \, \tilde{\xi} - b(x, \xi^2) \, \xi] \, . \, (\tilde{\xi} - \xi) \geqq c_1 (\tilde{\xi} - \xi)^2 \, . \tag{2.48}$$

Now, if $\psi_1, \psi_2 \in H^1(\Omega_\tau)$ ($\psi_1 - \psi_2 \in V(\Omega_\tau)$), then from (2.48) we get the inequality

$$\begin{aligned} & [b(x, (\nabla\psi_1)^2 (x)) \, \nabla\psi_1(x) - b(x, (\nabla\psi_2)^2 (x)) \, \nabla\psi_2(x)] \, . \\ & . \, (\nabla\psi_1(x) - \nabla\psi_2(x)) \geqq c_1 (\nabla\psi_1(x) - \nabla\psi_2(x))^2 \, , \end{aligned} \tag{2.49}$$

which holds for almost every $x \in \Omega_\tau$. (2.49) immediately yields

$$\langle \mathscr{A}(\psi_1) - \mathscr{A}(\psi_2), \psi_1 - \psi_2 \rangle =$$

$$= \int_{\Omega_\tau} [b(\cdot, (\nabla\psi_1)^2) \nabla\psi_1 - b(\cdot, (\nabla\psi_2)^2) \nabla\psi_2] \cdot (\nabla\psi_1 - \nabla\psi_2) \, dx \geqq$$

$$\geqq c_1 \int_{\Omega_\tau} [\nabla(\psi_1 - \psi_2)]^2 \, dx = c_1 \|\psi_1 - \psi_2\|_V^2,$$

which is (2.36) with $\alpha = c_1$.

b) Let us go back to (2.45)–(2.47), which give

$$|[b(x, \tilde{\xi}^2) \tilde{\xi} - b(x, \xi^2) \xi] \cdot \vartheta| = |g(1) - g(0)| \leqq$$

$$\leqq \int_0^1 |g'(t)| \, dt \leqq \int_0^1 b(x, (\xi + th)^2) |h| |\vartheta| \, dt +$$

$$+ 2 \int_0^1 \frac{\partial b}{\partial \eta}(x, (\xi + th)^2) (\xi + th)^2 |h| |\vartheta| \, dt \leqq$$

$$\leqq (c_2 + 2c_4) |h| |\vartheta| = K|h| |\vartheta| = K|\tilde{\xi} - \xi| |\vartheta| .$$

Hence we already derive the estimate (2.37): If $\psi_1, \psi_2 \in H^1(\Omega_\tau)$, $v \in V(\Omega_\tau)$, then for almost every $x \in \Omega_\tau$

$$|[b(x, (\nabla\psi_1)^2 (x)) \nabla\psi_1(x) - b(x, (\nabla\psi_2)^2 (x)) \nabla\psi_2(x)] \cdot \nabla v(x)| \leqq$$

$$\leqq K|\nabla\psi_1(x) - \nabla\psi_2(x)| |\nabla v(x)| ,$$

so that

$$|\langle \mathscr{A}(\psi_1) - \mathscr{A}(\psi_2), v \rangle| =$$

$$= \left| \int_{\Omega_\tau} [b(\cdot, (\nabla\psi_1)^2) \nabla\psi_1 - b(\cdot, (\nabla\psi_2)^2) \nabla\psi_2] \cdot \nabla v \, dx \right| \leqq$$

$$\leqq K \int_{\Omega_\tau} |\nabla\psi_1 - \nabla\psi_2| |\nabla v| \, dx \leqq$$

$$\leqq K \left( \int_{\Omega_\tau} (\nabla\psi_1 - \nabla\psi_2)^2 \, dx \right)^{1/2} \left( \int_{\Omega_\tau} (\nabla v)^2 \, dx \right)^{1/2} = K\|\psi_1 - \psi_2\|_V \|v\|_V .$$

This completes the proof of (2.37).

4) The rest of Theorem 2.2.7 is an easy consequence of the assertions 1)–4) we have already proved and of Riesz's theorem on the representation of a linear functional defined on a Hilbert space (see Lemma 2.2.6). ■

Now, we can introduce the following

2.2.8. **Definition of the weak solution.** *A function* $\psi : \bar{\Omega}_\tau \to R_1$ *is called a weak solution of the problem* (PSI. 1.1), *if*

$$(2.50)\quad \text{a)}\ \psi \in H^1(\Omega_\tau)\,, \qquad \text{b)}\ \psi - \psi^* \in V(\Omega_\tau)\,,$$
$$\text{c)}\ \langle \mathscr{A}(\psi), v \rangle = \langle \mu, v \rangle \quad \forall v \in V(\Omega_\tau)\,,$$

*where $\psi^* \in H^1(\Omega_\tau)$ is a function satisfying the conditions* (2.23) *a)–c).* ■

(2.50) a)–c) is a generalized analogue of the variational formulation (2.7), (2.8) a) – d). The condition (2.50) a) represents the generalization of the assumption (2.8) a), (2.50) b) expresses the conditions (2.8) b)–d) and (2.50) c) is a weak form of (2.7).

2.2.9. **Remark.** The problem (2.50) a)–c) is formally equivalent to the problem (PSI. 1.1) in the following sense: If $\psi \in C^2(\bar{\Omega}_\tau)$ is a solution of the problem (2.50) a)–c), then (since $\mathscr{V}_\tau \subset V(\Omega_\tau)$), $\psi$ obviously satisfies (2.7), (2.8) a)–d) with some $q_1, q_2$ and in view of 2.1.1, it induces the solution $\psi_E, q_1, q_2$ of the problem (PSI. 1.1). On the other hand, if $\psi \in C^2(\bar{\Omega})$, $q_1, q_2 \in R_1$ form a solution of the problem (PSI. 1.1), then, as we have already proved, $\psi \mid \bar{\Omega}_\tau$ satisfies (2.7), (2.8) a)–d). In order to show that this function is also a solution of (2.50) a)–c), it is necessary to prove that the set $\mathscr{V}_\tau$ is dense in $V(\Omega_\tau)$. This property is also important for the numerical solution of our problem by the finite element method ([8, 9]). Therefore, we shall prove the following:

2.2.10. **Theorem.** *The set $\mathscr{V}_\tau$ is dense in the space $V(\Omega_\tau)$.*

Proof of this theorem is based on the partition of unity and on regularization. Let us denote by $\Omega_E$ the domain whose closure $\bar{\Omega}_E$ is the extension of $\bar{\Omega}_\tau$ obtained by adding the sets $E_0, E_1, E_2, E^-, E^+$ to $\bar{\Omega}_\tau$, as is shown in Fig. 5; $E_0 = \overline{\text{Int } C_0}$*),

$$E^- = \{(x_1, x_2 - \delta); (x_1, x_2) \in \Gamma^-, \delta \in \langle 0, \tilde{\delta} \rangle\}\,,$$
$$E^+ = \{(x_1, x_2 + \delta); (x_1, x_2) \in \Gamma^+, \delta \in \langle 0, \tilde{\delta} \rangle\}\,,$$
$$E_i = \{(d_i, x_2) - \delta \mathbf{k}_i; x_2 \in \langle e_i - \tilde{\delta}, e_i + \tau + \tilde{\delta} \rangle, \delta \in \langle 0, \tilde{\delta} \rangle\}\,,$$
$$\tilde{\Gamma}_i = \{(d_i, x_2) - \tilde{\delta} \mathbf{k}_i, x_2 \in \langle e_i - \tilde{\delta}, e_i + \tau + \tilde{\delta} \rangle\}\,, \quad i = 1, 2\,.$$

Here, $\boldsymbol{k}_i$ is a unit vector parallel to $\Gamma^-$ and $\Gamma^+$ near $\Gamma_i$ and pointing into $\Omega_\tau$ (i.e., $\boldsymbol{k}_i \,.\, \boldsymbol{n} < 0$ on $\Gamma_i$, $\boldsymbol{n}$ being the unit outer normal to $\partial\Omega_\tau$). $\tilde{\delta} > 0$ is sufficiently small so that $\bar{\Omega}_E \cap C_k = \emptyset$ for $k = \pm 1, \pm 2, \ldots$. We can write $\partial\Omega_E = \tilde{\Gamma}_1 \cup \tilde{\Gamma}_2 \cup \tilde{\Gamma}^- \cup \cup \tilde{\Gamma}^+$, as is shown in Fig. 5.

Further, let $B_i$, $i = 0, 1, 2, 3$, be open sets such that $E_i \subset B_i$ for $i = 0, 1, 2$, $\bar{B}_i \cap \bar{B}_0 = \emptyset$ for $i = 1, 2$, $B_0 \subset \Omega_\tau \cup E_0$, $\bar{\Omega}_E \subset \bigcup_{i=0}^{3} B_i$. The sets $\partial B_i \cap \bar{\Omega}_E$ $(i = 1, 2)$ nad $\partial B_3 \cap \bar{\Omega}_E - B_0$ are straight lines parallel to the axis $x_2$ and intersecting the arcs $\Gamma^-$, $\Gamma^+$ near $\Gamma_i$, where $\Gamma^-$, $\Gamma^+$ have the direction $\boldsymbol{k}_i$. $\tilde{\delta} > 0$ and $B_0$ are chosen in such a way that dist $(\partial B_0, \Gamma^-)$ and dist $(\partial B_0, \Gamma^+) > \tilde{\delta}$. We shall consider a parti-

*) Int $C_0$ is the bounded component of $R_2 - C_0$.

tion $\{\varphi_i\}_{i=0}^{3}$ of unity corresponding to the covering $\{B_i\}_{i=0}^{3}$ of the set $\bar{\Omega}_E$ (see [24], Ch. 1, § 2.4). It means that $\varphi_i \in C^\infty(R_2)$, $0 \leqq \varphi_i \leqq 1$, the support supp $\varphi_i \subset B_i$ and $\sum_{i=0}^{3} \varphi_i(x) = 1$ for every $x \in \bar{\Omega}_E$. Choosing the sets $B_i$ in the above way and following the proof concerning the existence of a partition of unity contained in [24], we can assume that $\varphi_i(x^\tau) = \varphi_i(x)$ for all $x = (x_1, x_2)$, $x^\tau = (x_1, x_2 + \tau)$ such that $x, x^\tau \in \bar{\Omega}_E$ and for $i = 0, \ldots, 3$.

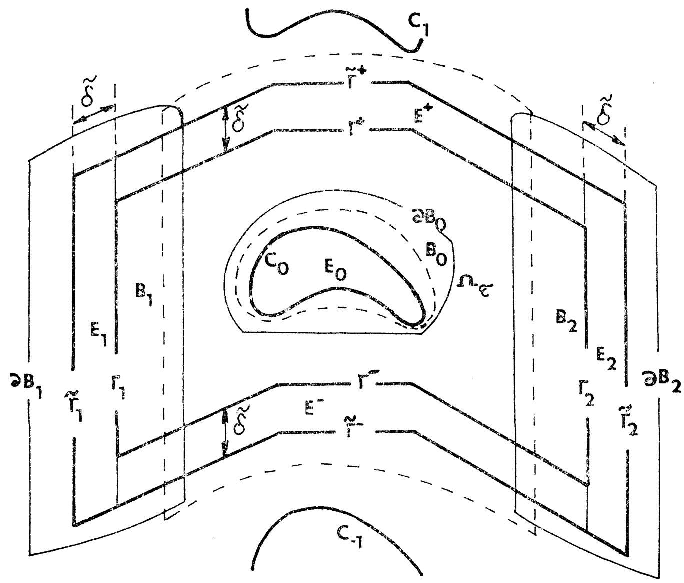


Fig. 5.

Now, let us consider an arbitrary $v \in V(\Omega_\tau)$. In order to prove the assertion of the theorem we need to show that

$$\forall \varepsilon > 0 \quad \exists v^\varepsilon \in \mathscr{V}_\tau \quad \|v - v^\varepsilon\|_V < \varepsilon \,. \tag{2.51}$$

We shall proceed in the following way: We denote by $v_E$ the extension of $v$ to the set $\Omega_E$, defined by

$$\begin{gathered} v_E \mid \Omega_\tau = v \,, \quad v_E \mid E_0 = 0 \,, \quad v_E \mid E_i = v_{\Gamma_i} \,, \quad i = 1, 2 \,; \\ v_E(x_1, x_2) = v_E(x_1, x_2 + \tau) \,, \quad (x_1, x_2) \in E^- \\ v_E(x_1, x_2) = v_E(x_1, x_2 - \tau) \,, \quad (x_1, x_2) \in E^+ \,. \end{gathered} \tag{2.52}$$

($v$ is extended onto $E_0, E_1, E_2$ by constants, and "periodically" onto $E^-, E^+$.) According to (2.52), $v_E \in H^1(\Omega_E)$.

Now, we put $v_i = v_E \varphi_i$ in $\bar{\Omega}_E$. It is easy to see that

(2.53) a) $v_i(x^\tau) = v_i(x) \quad \text{for} \quad x, x^\tau \in \bar{\Omega}_E$,

b) $v_E = \sum_{i=0}^{3} v_i$,

c) $v = \sum_{i=0}^{3} v_i \mid \Omega_\tau$,

d) $v_i \in H^1(\Omega_E)$,

e) $v_i \mid \Omega_\tau \in V(\Omega_\tau)$.

From (2.53) c) it follows that the verification of the condition (2.51) can be converted to the proof of the assertion

$$(2.54) \qquad \forall \varepsilon > 0 \quad \exists v_i^\varepsilon \in \mathscr{V}_\tau \quad \|v_i \mid \Omega_\tau - v_i^\varepsilon\|_V < \frac{\varepsilon}{4} \quad (i = 0, \ldots, 3).$$

a) Let us begin with $i = 0$. Since $v_0 \mid \Omega_\tau \in H_0^1(\Omega_\tau) \subset V(\Omega_\tau)$, there exists a sequence $\{v_0^n\}_{n=1}^{+\infty}$, $v_0^n \in \mathscr{D}(\Omega_\tau)\ (\subset \mathscr{V}_\tau)$ such that $v_0^n \to v_0$ in $H_0^1(\Omega_\tau)$ (and thus, in $V(\Omega_\tau)$), which proves (2.54), $i = 0$.

b) Now, we consider $i = 1$ or $i = 2$. Let us define the function $v_{i,\delta}(x) = v_i(x + \delta \mathbf{k}_i)$ for all $x + \delta \mathbf{k}_i \in \Omega_E$. It is evident that if $\delta \in (0, \delta_1)$, where $\delta_1 > 0$ is sufficiently small, $v_{i,\delta}$ is defined for (almost) every $x$ from a certain neighbourhood $\mathfrak{U}$ of $\bar{\Omega}_\tau$, $v_{i,\delta} = v_{\Gamma_i}$ in a neighbourhood of the arc $\Gamma_i$, $v_{i,\delta}(x^\tau) = v_{i,\delta}(x)$ if $x, x^\tau \in \mathfrak{U}$, $v_{i,\delta} = 0$ in neighbourhoods of $C_0$ and $\Gamma_j$ if $j \neq i$. Moreover, $v_{i,\delta} \mid \mathfrak{U} \in H^1(\mathfrak{U})$, $v_{i,\delta} \mid \mathfrak{U} \xrightarrow[\delta \to 0+]{} v_i \mid \mathfrak{U}$ in $H^1(\mathfrak{U})$, which implies that $v_{i,\delta} \mid \Omega_\tau \in V(\Omega_\tau)$ and $v_{i,\delta} \mid \Omega_\tau \xrightarrow[\delta \to 0+]{} v_i \mid \Omega_\tau$ in $V(\Omega_\tau)$. Let us choose and fix $\delta > 0$ so small that

$$(2.55) \qquad \|(v_{i,\delta} - v_i) \mid \Omega_\tau\|_V < \frac{\varepsilon}{8}.$$

Next, we apply the regularization process. Let $\omega_h$ be a mollifier, i.e., $h > 0$,

$$\omega_h(x) = \exp \frac{|x|^2}{|x|^2 - h^2} \quad \text{for} \quad |x| < h,$$

$$\omega_h(x) = 0 \quad \text{for} \quad |x| \geqq h \quad (x \in R_2),$$

$$\varkappa = \int_{|x| < 1} \omega_1(x) \, \mathrm{d}x.$$

For $h > 0$ sufficiently small we define

$$(2.56) \qquad v_{i,\delta,h}(x) = \varkappa^{-1} h^{-2} \int_{|x-y| \leqq h} \omega_h(x - y)\, v_{i,\delta}(y) \, \mathrm{d}y, \quad x \in \bar{\Omega}_\tau.$$

From the properties of $v_{i,\delta}$ mentioned above it follows that $v_{i,\delta,h} \in C^\infty(\bar{\Omega}_\tau)$, $v_{i,\delta,h} \mid C_0 = 0$, $v_{i,\delta,h} \mid \Gamma_i = v_{\Gamma_i}$, $v_{i,\delta,h} \mid \Gamma_j = 0$ if $j \neq i$, $v_{i,\delta,h}(x^\tau) = v_{i,\delta,h}(x)$ for all $x \in \Gamma^-$. Thus, $v_{i,\delta,h} \in \mathscr{V}_\tau$. According to the results in [24, 29], $v_{i,\delta,h} \to v_{i,\delta} \mid \Omega_\tau$ in $V(\Omega_\tau)$ if $h \to 0+$. If we choose $h > 0$ so small that

$$\|v_{i,\delta,h} - v_{i,\delta} \mid \Omega_\tau\|_V < \frac{\varepsilon}{8}$$

and put $v_i^\varepsilon = v_{i,\delta,h}$, then (2.55) implies $\|v_i^\varepsilon - v_i \mid \Omega_\tau\| < \varepsilon/4$, which we wanted to prove.

c) Finally, let $i = 3$. In this case we can define directly

$$v_{3,h} = \varkappa^{-1} h^{-2} \int_{|x-y| \leqq h} \omega_h(x - y)\, v_3(y)\, \mathrm{d}y\,, \quad x \in \bar{\Omega}_\tau\,. \tag{2.57}$$

It is easy to verify that for sufficiently small $h > 0$ we have $v_{3,h} \in C^\infty(\bar{\Omega}_\tau)$, $v_{3,h} \mid \Gamma_1 \cup \Gamma_2 = 0$, $v_{3,h} \mid C_0 = 0$ and $v_{3,h}(x^\tau) = v_{3,h}(x)$ for all $x \in \Gamma^-$, so that $v_{3,h} \in \mathscr{V}_\tau$. Moreover, $v_{3,h} \to v_3$ in $V(\Omega_\tau)$ if $h \to 0+$. It means that we have verified (2.54) for $i = 3$, and this completes the proof of Theorem 2.2.10. ∎

2.2.11. **Remark.** Let us notice that from the proof of Theorem 2.2.10 it follows that an arbitrary $v \in V(\Omega_\tau)$ can be approximated with a given accuracy $\varepsilon > 0$ by an element $v_\varepsilon \in \mathscr{V}_\tau$ that is equal to zero in a neighbourhood of $C_0$ and equal to $v_{\Gamma_i}$ in a neighbourhood of $\Gamma_i$. We can also remark that the assertion 2) of Lemma 2.2.4 is a consequence of Theorem 2.2.10. ∎

Now, we come to the study of the solvability of the problem (2.50) a)–c). Let us notice that the solution of this problem can be sought in the form $\psi = \psi^* + u$, where $u \in V(\Omega_\tau)$. With respect to Theorem 2.2.7 we get the following equivalent formulations of this problem:

$$\langle \mathscr{A}(\psi^* + u), v \rangle = \langle \mu, v \rangle \quad \forall v \in V(\Omega_\tau)\,, \tag{2.50*}$$

$$\langle T(u), v \rangle = \langle \mu, v \rangle \quad \forall v \in V(\Omega_\tau)\,, \tag{2.50**}$$

$$(\mathscr{H}(u), v)_V = (\tilde{\mu}, v)_V \quad \forall v \in V(\Omega_\tau) \tag{2.50***}$$

for an unknown function $u \in V(\Omega_\tau)$. The last equation can be written as the operator equation

$$\mathscr{H}(u) = \tilde{\mu} \tag{2.50****}$$

in the space $V(\Omega_\tau)$ for an unknown $u \in V(\Omega_\tau)$.

### 2.3. Existence and uniqueness of the weak solution

2.3.1. **Theorem.** *Let the following assumptions be satisfied*: 1) $\varphi_i$, $i = 1, 2$, *are $\tau$-periodic functions in $R_1$*, $\varphi_i \mid (0, \tau) \in L_2((0, \tau))$.

2) *The functions $\Psi_i$, $i = 1, 2$, and the constant $Q$ are defined by* (1.25) *and* (1.27), *respectively* (*and satisfy* (1.26)).

3) $\bar{\mu}_1, \bar{\mu}_2 \in R_1$ *are given constants. Then there exists exactly one weak solution $\psi$ of the problem* (PSI. 1.1). *This solution does not depend on the choice of the function $\psi^* \in H^1(\Omega_\tau)$ satisfying the conditions* (2.23) *a)–c)*.

Proof 1) The solvability can be proved on the basis of the monotone operator theory ([2], [19], [31]). However, it is not necessary to apply this powerful method. If we take into consideration the Hilbert structure of the space $V(\Omega_\tau)$, then the proof of the solvability of the equation (2.50****) becomes quite elementary. For the sake of completeness of our theory we reproduce here this well-known approach (cf. e.g. [3, 13]).

For $\nu > 0$ let us put

$$F_\nu(u) = u - \nu(\mathscr{H}(u) - \tilde{\mu}) , \quad u \in V(\Omega_\tau) . \tag{2.58}$$

It is evident that the equation (2.50****) has a solution $u \in V(\Omega_\tau)$ if and only if

$$u = F_\nu(u) , \tag{2.59}$$

i.e. if and only if $u$ is a fixed point of the mapping $F_\nu : V(\Omega_\tau) \to V(\Omega_\tau)$. To prove the existence and uniqueness of the solution of the equation (2.50****), it is sufficient to verify that $F_\nu$ is contractive for some $\nu$.

Let $u, v \in V(\Omega_\tau)$. Then, using (2.43) and (2.44), we get

$$\|F_\nu(u) - F_\nu(v)\|_V^2 = (F_\nu(u) - F_\nu(v), F_\nu(u) - F_\nu(v))_V =$$
$$= \|u - v\|_V^2 - 2\nu(\mathscr{H}(u) - \mathscr{H}(v), u - v)_V + \nu^2 \|\mathscr{H}(u) - \mathscr{H}(v)\|_V^2 \leqq$$
$$\leqq (1 - 2\nu\alpha + \nu^2 K^2) \|u - v\|_V^2$$

and thus,

$$\|F_\nu(u) - F_\nu(v)\|_V \leqq q\|u - v\|_V \quad \forall u, v \in V(\Omega_\tau) \tag{2.60}$$

with $q = (1 - 2\nu\alpha + \nu^2 K^2)^{1/2}$. It is easy to find out that $0 < q < 1$ if $0 < \nu < 2\alpha/K^2$ and $F_\nu$ is contractive.

2) We have just proved that the equation (2.50****) has a unique solution $u \in V(\Omega_\tau)$, from which we get a solution $\psi = \psi^* + u$ of the problem (2.50) a)–c). However, since the operator $\mathscr{H}$ depends on the function $\psi^*$, we have to prove the uniqueness of this solution $\psi$.

Let $\psi_1^*, \psi_2^* \in H^1(\Omega_\tau)$ be two functions satisfying the conditions (2.23) a)–c). Then $\psi_1^* - \psi_2^* \in V(\Omega_\tau)$. Let $u_i \in V(\Omega_\tau)$ be the (unique) solution of the problem

$$\langle \mathscr{A}(\psi_i^* + u_i), v \rangle = \langle \mu, v \rangle \quad \forall v \in V(\Omega_\tau) . \tag{2.61$_i$}$$

Then $\psi_i = \psi_i^* + u_i$, $i = 1, 2$, are weak solutions of the problem (PSI. 1.1). Let us substract the equation $(2.61)_2$ from $(2.62)_1$, substitute $v = \psi_1 - \psi_2 = \psi_1^* - \psi_2^* + u_1 - u_2 \in V(\Omega_\tau)$ and apply (2.36). Then

$$0 = \langle \mathscr{A}(\psi_1) - \mathscr{A}(\psi_2), \psi_1 - \psi_2 \rangle \geqq \alpha \|\psi_1 - \psi_2\|_V^2 ,$$

so that $\|\psi_1 - \psi_2\|_V = 0$. Since $\psi_1 - \psi_2 \in V(\Omega_\tau)$, we get $\psi_1 = \psi_2$. This completes the proof of Theorem 2.3.1. ∎

## 3. VARIATIONAL FORMULATION AND SOLUTION OF THE PROBLEMS (PSI. 1.2) AND (PSI. 2.1)

Now, we shall proceed more briefly, since the situation is quite analogous as in the preceding sections.

### 3.1. **Problem (PSI. 1.2)**

Let us put

$$\mathscr{V}_\tau = \{v \in C^\infty(\bar{\Omega}_\tau);\ v \mid \Gamma_2 = 0,\ v \mid \Gamma_1 = \text{const}\,, \quad v \mid C_0 = \text{const},\ v(x^\tau) = v(x) \quad \forall x \in \Gamma^-\}\,. \tag{2.4*}$$

Then the problem (PSI. 1.2) is equivalent to the problem of determining a function $\psi$ and constants $q_0$, $q_1$ satisfying the following conditions:

$$\int_{\Omega_\tau} b(\cdot, (\nabla\psi)^2)\, \nabla\psi \,.\, \nabla v \,\mathrm{d}x = -\gamma v \mid C_0 - \tau\bar{\mu}_1 v \mid \Gamma_1 \quad \forall v \in \mathscr{V}_\tau\,, \tag{2.7*}$$

(2.8*) a) $\psi \in C^2(\bar{\Omega}_\tau)$,

b) $\psi \mid C_0 = q_0$,

c) $\psi(x^\tau) = \psi(x) + Q\,, \quad x \in \Gamma^-$,

d) $\psi \mid \Gamma_1 = \Psi_1 \mid \Gamma_1 + q_1$,

e) $\psi \mid \Gamma_2 = \Psi_2 \mid \Gamma_2$.

Let $\psi^* \in H^1(\Omega_\tau)$ be a function with the properties (2.23) a)–c), whose existence is ensured by Lemma 2.2.2. We define

$$V(\Omega_\tau) = \{v \in H^1(\Omega_\tau);\ v \mid \Gamma_2 = 0,\ v \mid \Gamma_1 = \text{const}, \quad v \mid C_0 = \text{const},\ v(x_\tau) = v(x) \text{ for almost every } x \in \Gamma^-\}\,. \tag{2.26*}$$

3.1.1. **Lemma.** 1) $\mathscr{V}_\tau \subset V(\Omega_\tau)$. 2) $V(\Omega_\tau)$ *is a closed subspace of* $H^1(\Omega_\tau)$. 3) $\mathscr{V}_\tau$ *is dense in* $V(\Omega_\tau)$. 4) $V(\Omega_\tau)$ *is a Hilbert space with the norm defined by* (2.28). 5) *The mapping "*$v \in V(\Omega_\tau) \to -\gamma v \mid C_0 - \tau\bar{\mu}_1 v \mid \Gamma_1$*" is a linear continuous functional* *defined on the space $V(\Omega_\tau)$. Let us denote it by $\mu$, so that*

$$\langle \mu, v\rangle = -\gamma\, v \mid C_0 - \tau \bar{\mu}_1\, v \mid \Gamma_1\,, \quad v \in V(\Omega_\tau)\,. \tag{2.35*}$$

Proof of the assertions 1), 2), 4), 5) can be carried out by modifying the proofs of the corresponding analogous assertions from Section 2.2.

Let us prove the assertion 3). If $v \in V(\Omega_\tau)$, then $v \mid \Gamma_1 = v_{\Gamma 1} = \text{const}$, $v \mid C_0 = v_{C_0} = \text{const}$ and $v \mid \Gamma_2 = 0$. Let us put $w = v - v_{C_0}$. Then $w \mid C_0 = 0$, $w \mid \Gamma_1 = v_{\Gamma_i} - v_{C_0} = \text{const}$, $w \mid \Gamma_2 = -v_{C_0} = \text{const}$. Of course, $w(x^\tau) = w(x)$ for almost every $x \in \Gamma^-$. Hence, we see that $w$ is an element of the space $V(\Omega_\tau)$ defined by (2.26) in the preceding section. It means that there exists a sequence $\{w_n\}_{n=1}^{+\infty}$ with elements that belong to the space $\mathscr{V}_\tau$ defined by (2.4) such that $\|w_n - w\|_V \underset{n\to+\infty}{\to} 0$. Moreover, in view of Remark 2.2.11, we can assume that $w_n \mid \Gamma_2 = w \mid \Gamma_2 = -v_{C_0}$. Therefore, the functions $v_n = w_n + v_{C_0}$, $n = 1, 2, \ldots$, satisfy the conditions $v_n \in C^\infty(\bar{\Omega}_\tau)$, $v_n \mid C_0 = \text{const}$, $v_n \mid \Gamma_1 = \text{const}$, $v_n \mid \Gamma_2 = 0$, $v_n(x^\tau) = v_n(x)$ for every $x \in \Gamma^-$. This means that $v_n \in \mathscr{V}_\tau$ ($\mathscr{V}_\tau$ defined by (2.4*)) and $\|v_n - v\|_V \underset{n\to+\infty}{\to} 0$. Hence, the set $\mathscr{V}_\tau$ is dense in $V(\Omega_\tau)$. Moreover, from the above considerations and Remark 2.2.11 it follows that $v \in V(\Omega_\tau)$ can be approximated with a given accuracy $\varepsilon > 0$ by $v_\varepsilon \in \mathscr{V}_\tau$ that is equal to $v_{C_0}$ in a neighbourhood of $C_0$. ■

If we define the form $\boldsymbol{a}$ again by (2.32), then the assertions 1), 3)–6) of Theorem 2.2.7 remain valid. Under the above notation, the problem (PSI. 1.2) is formally equivalent to the problem written in the form (2.50) a)–c).

The solvability results proved in the same way as in 2.3 can be formulated as follows:

3.1.2. **Theorem.** *Let us assume that the assumptions* 1)–2) *of Theorem* 2.3.1 *are satisfied. Moreover, let $\bar{\mu}_1, \gamma \in R_1$ be given constants. Then there exists exactly one weak solution $\psi$ of the problem* (PSI. 1.2). *This solution is independent of the choice of the function $\psi^* \in H^1(\Omega_\tau)$ with the properties* (2.23) *a)–c).*

3.1.3. **Remark.** Using Green's theorem we can easily prove that the classical problem (PSI. 1.2) can be transformed to the problem (PSI. 1.1), if we put

$$\bar{\mu}_2 = -\bar{\mu}_1 - \frac{\gamma}{\tau}\,. \tag{3.1}$$

However, if we know nothing about the regularity of the weak solutions of these problems, we cannot assert the equivalence of their weak formulations. Therefore, if we define the numerical solution of our problems by approximating the spaces $H^1(\Omega_\tau)$ and $V(\Omega_\tau)$ in the weak formulation (2.50) a)–c), we do not recommend to solve numerically the problem (PSI. 1.1) (with $\bar{\mu}_2$ given by (3.1)) instead of the problem (PSI. 1.2). This is the reason why we study the solvability of each problem separately. ■

3.1.4. **Remark.** If we consider trailing conditions instead of a given velocity circulation round the profiles $C_k$, we get the problem (PSI. 1.3), which from the physical point of view describes the flows round the profiles $C_k$ probably better than the problem (PSI. 1.2). On the other hand, the mathematical study of the problem (PSI. 1.3) is more difficult. Because of the discrete trailing conditions, the problem (PSI. 1.3) has not a variational formulation in a usual sense and it is necessary to consider directly the classical solutions. Some results concerning plane incompressible (generally rotational) flows were obtained in [7] on the basis of appropriate à priori estimates and the strong maximum principle. ■

### 3.2. Problem (PSI. 2.1.)

We assume that $\tau$-periodic functions $\mu_1, \mu_2 : R_1 \to R_1$ and a constant $Q \in R_1$ are given. Let us denote

$$(2.4^{**}) \qquad \mathscr{V}_\tau = \{v \in C^\infty(\Omega_\tau);\, v \mid C_0 = 0,\, v(x_\tau) = v(x)\ \forall x \in \Gamma^-\}\,.$$

The (classical) problem (PSI. 2.1) is equivalent to the following variational formulation: To find $\psi : \bar{\Omega}_\tau \to R_1$ such that

$$(2.7^{**}) \qquad \int_{\Omega_\tau} b(\cdot, (\nabla\psi)^2)\, \nabla\psi \,.\, \nabla v \,\mathrm{d}x = -\sum_{i=1}^{2} \int_{\Gamma_i} \mu_i v \,\mathrm{d}s \quad \forall v \in \mathscr{V}_\tau\,,$$

$$(2.8^{**}) \quad \text{a)}\ \psi \in C^2(\bar{\Omega}_\tau)\,,$$

$$\text{b)}\ \psi \mid C_0 = 0\,,$$

$$\text{c)}\ \psi(x^\tau) = \psi(x) + Q\,, \quad x \in \Gamma^-\,.$$

Let $\psi^* \in H^1(\Omega_\tau)$ be a function satisfying the conditions

$$(2.23^{**}) \quad \text{a)}\ \psi^* \mid C_0 = 0\,,$$

$$\text{b)}\ \psi^*(x^\tau) = \psi^*(x) + Q \quad \text{for almost every} \quad x \in \Gamma^-.$$

The existence of this $\psi^*$ is obvious.

We define

$$(2.26^{**}) \qquad V(\Omega_\tau) = \{v \in H^1(\Omega_\tau);\, v \mid C_0 = 0\,,$$

$$v(x^\tau) = v(x) \text{ for almost every } x \in \Gamma^-\}\,.$$

3.2.1. **Lemma.** 1) $\mathscr{V}_\tau \subset V(\Omega_\tau)$. 2) *$V(\Omega_\tau)$ is a closed subspace of $H^1(\Omega_\tau)$.* 3) *$\mathscr{V}_\tau$ is dense in $V(\Omega_\tau)$.* 4) *$V(\Omega_\tau)$ is a Hilbert space with the norm defined by* (2.28). 5) *Let $\mu_i \mid (0, \tau) \in L_2((0, \tau))$, $i = 1, 2$. Then the mapping*

$$\text{“}v \in V(\Omega_\tau) \to -\sum_{i=1}^{2} \int_{\Gamma_i} \mu_i v \,\mathrm{d}s = -\sum_{i=1}^{2} \int_{e_i}^{e_i+\tau} \mu_i(x_2)\, v(d_i, x_2)\,\mathrm{d}x_2\text{”}$$

*is a linear continuous functional defined on the space $V(\Omega_\tau)$. If we denote it by $\mu$, then*

$$(2.35^{**}) \qquad \langle \mu, v \rangle = - \sum_{i=1}^{2} \int_{\Gamma_i} \mu_i v \, ds \,, \quad v \in V(\Omega_\tau) \,.$$

Proof. All assertions of this lemma can be verified similarly as in Section 2.2 ∎

Under the above notation we can define the weak solution of the problem (PSI. 2.1) by (2.50) a)–c). On the basis of Theorem 2.2.7, Lemma 3.2.1 and by the same argument as in the proof of Theorem 2.3.1, we can prove the solvability also in this case:

3.2.2. **Theorem.** *Let $\mu_i : R_1 \to R_1$ be given $\tau$-periodic functions, $\mu_i \mid (0, \tau) \in L_2((0, \tau))$ $(i = 1, 2)$ and let $Q \in R_1$ be a given constant. Then there exists exactly one weak solution $\psi$ of the problem* (PSI. 2.1). *This solution does not depend on the choice of the function $\psi^* \in H^1(\Omega_\tau)$ with the properties* (2.23**) *a), b).* ∎

## 4. CONCLUDING REMARKS

The paper partially solves one of the problems formulated by E. Meister and J. Polášek at the conference "Mathematical Methods in Fluid Mechanics" held in 1981 at Oberwolfach: the study of flows through cascades of blades with variable inlet and oulet velocity distributions.

The theory presented here can be generalized to the problem of flows through a group of cascades (e.g. a cascade of profiles with a tandem cascade) and also through moving cascades.

In another paper to appear, special attention will be devoted to flows through cascades of profiles with given trailing conditions (i.e. to the problem (PSI. 1.3)). Survey of the results concerning the numerical solution of the problem by the finite element method can be found in [8, 9].

In order to complete the solution of the problem formulated by E. Meister and J. Polášek, the results of this paper can be generalized to rotational flows. For brief information see [8].

## 5. APPENDIX

Here we show that the function $\psi^*$ constructed in the proof of Lemma 2.2.2 is an element of the space $W_2^{1+1/2}(\Omega_\tau)$.

Let us use the symbol $F_{-1}$ to denote the inverse to $F \mid \Omega_\tau^*$. Both $F \mid \Omega_\tau^* : \Omega_\tau^* \xrightarrow{\text{onto}} \tilde{\Omega}$ and $F_{-1} : \tilde{\Omega} \xrightarrow{\text{onto}} \Omega_\tau^*$ are one-to-one mappings, infinitely differentiable, with bounded partial derivatives of all orders. Hence, $F$, $F_{-1}$ and their derivatives are Lipschitz continuous.

In view of the results from 2.2.2 and 2.2.3, it will do to prove that $\mathscr{E} \mid \Omega_\tau^* \in W_2^{1+1/2}(\Omega_\tau^*)$ ($\mathscr{E} = \tilde{\mathscr{E}} \circ F$, $\tilde{\mathscr{E}} \in W_2^{1+1/2}(\tilde{\Omega})$), which consists in verifying the condition

$$(5.1) \qquad I = \int_{\Omega_\tau^*} \left( \int_{\Omega_\tau^*} \frac{\left| \frac{\partial \mathscr{E}}{\partial x_i}(x) - \frac{\partial \mathscr{E}}{\partial x_i}(y) \right|^2}{|x - y|^3} \, dx \right) dy \quad < +\infty, i = 1, 2 .$$

If we use the substitution $\tilde{x} = F(x)$, $\tilde{y} = F(y)$, so that $x = F_{-1}(\tilde{x})$, $y = F_{-1}(\tilde{y})$ and

$$(5.2) \qquad \frac{\partial \mathscr{E}}{\partial x_i}(x) = \sum_{j=1}^{2} \frac{\partial \tilde{\mathscr{E}}}{\partial \tilde{x}_j}(\tilde{x}) \frac{\partial F_j}{\partial x_i}(F_{-1}(\tilde{x})) , \quad i = 1, 2 ,$$

we get

$$(5.3) \qquad I = \int_{\tilde{\Omega}} \int_{\tilde{\Omega}} \frac{\left| \sum_{j=1}^{2} \left[ \frac{\partial \tilde{\mathscr{E}}}{\partial \tilde{x}_j}(\tilde{x}) \frac{\partial F_j}{\partial x_i}(F_{-1}(\tilde{x})) - \frac{\partial \tilde{\mathscr{E}}}{\partial \tilde{x}_j}(\tilde{y}) \frac{\partial F_j}{\partial x_i}(F_{-1}(\tilde{y})) \right] \right|^2}{|F_{-1}(\tilde{x}) - F_{-1}(\tilde{y})|^3} \times$$
$$\times \left| \frac{DF_{-1}(\tilde{x})}{D\tilde{x}} \right| \left| \frac{DF_{-1}(\tilde{y})}{D\tilde{x}} \right| d\tilde{x} \, d\tilde{y} .$$

In view of the relations $(a_1 + a_2)^2 \leqq 2(a_1^2 + a_2^2)$ and

$$(5.4) \qquad \left| \frac{DF_{-1}(\tilde{x})}{D\tilde{x}} \right| \leqq \text{const} \quad \text{for all} \quad \tilde{x} \in \tilde{\Omega} ,$$

it holds

$$(5.5) \qquad I \leqq \text{const} \sum_{j=1}^{2} I_j ,$$

where

$$(5.6) \qquad I_j = \int_{\tilde{\Omega}} \int_{\tilde{\Omega}} \frac{\left| \frac{\partial \tilde{\mathscr{E}}}{\partial \tilde{x}_j}(\tilde{x}) \frac{\partial F_j}{\partial x_i}(F_{-1}(\tilde{x})) - \frac{\partial \tilde{\mathscr{E}}}{\partial \tilde{x}_j}(\tilde{y}) \frac{\partial F_j}{\partial x_i}(F_{-1}(\tilde{y})) \right|^2}{|F_{-1}(\tilde{x}) - F_{-1}(\tilde{y})|^3} \, d\tilde{x} \, d\tilde{y} .$$

Further, since the mapping $F$ is Lipschitz-continuos, which means that

$$(5.7) \qquad \frac{|\tilde{x} - \tilde{y}|}{|F_{-1}(\tilde{x}) - F_{-1}(\tilde{y})|} = \frac{|F(x) - F(y)|}{|x - y|} \leqq \text{const} \quad \forall \tilde{x}, \tilde{y} \in \tilde{\Omega} ,$$

we have

$$(5.8) \qquad I_j \leqq \text{const} \int_{\tilde{\Omega}} \int_{\tilde{\Omega}} \frac{\left| \frac{\partial \tilde{\mathscr{E}}}{\partial \tilde{x}_j}(\tilde{x}) \frac{\partial F_j}{\partial x_i}(F_{-1}(\tilde{x})) - \frac{\partial \tilde{\mathscr{E}}}{\partial \tilde{x}_j}(\tilde{y}) \frac{\partial F_j}{\partial x_i}(F_{-1}(\tilde{y})) \right|^2}{|\tilde{x} - \tilde{y}|^3} \, d\tilde{x} \, d\tilde{y} .$$

If we use the inequality

$$(5.9)\qquad |ab - cd|^2 \leqq (|a - c|\,|b| + |c|\,|b - d|)^2 \leqq$$
$$\leqq 2(|a - c|^2\,|b|^2 + |c|^2\,|b - d|^2)$$

and take into consideration that the mapping $F_{-1}$ and the derivatives $\partial F_i/\partial x_i$ are Lipschitz-continuous and bounded, we can derive the following estimates:

$$(5.10)\qquad I_j \leqq \text{const}\,(I_j^* + I_j^{**}),$$

$$(5.11)\qquad I_j^* = \int_{\tilde{\Omega}}\int_{\tilde{\Omega}} \frac{\left|\frac{\partial\tilde{\mathscr{E}}}{\partial \tilde{x}_j}(\tilde{x}) - \frac{\partial\tilde{\mathscr{E}}}{\partial \tilde{x}_j}(\tilde{y})\right|^2 \left|\frac{\partial F_j}{\partial x_i}(F_{-1}(\tilde{x}))\right|^2}{|\tilde{x} - \tilde{y}|^3}\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y} \leqq$$

$$\leqq \text{const}\int_{\tilde{\Omega}}\int_{\tilde{\Omega}} \frac{\left|\frac{\partial\tilde{\mathscr{E}}}{\partial \tilde{x}_j}(\tilde{x}) - \frac{\partial\tilde{\mathscr{E}}}{\partial \tilde{x}_j}(\tilde{y})\right|^2}{|\tilde{x} - \tilde{y}|^3}\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y} \leqq \text{const}\,\|\tilde{\mathscr{E}}\|^2_{W_2^{1+1/2}(\tilde{\Omega})} < +\infty\,;$$

$$(5.12)\qquad I_j^{**} = \int_{\tilde{\Omega}}\int_{\tilde{\Omega}} \frac{\left|\frac{\partial\tilde{\mathscr{E}}}{\partial \tilde{x}_j}(\tilde{y})\right|^2 \left|\frac{\partial F_j}{\partial x_i}(F_{-1}(\tilde{x})) - \frac{\partial F_j}{\partial x_i}(F_{-1}(\tilde{y}))\right|^2}{|\tilde{x} - \tilde{y}|^3}\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y} \leqq$$

$$\leqq \text{const}\int_{\tilde{\Omega}}\int_{\tilde{\Omega}} \frac{\left|\frac{\partial\tilde{\mathscr{E}}}{\partial \tilde{x}_j}(\tilde{y})\right|^2}{|\tilde{x} - \tilde{y}|}\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}\,.$$

Finally, we shall estimate the integral

$$(5.13)\qquad I_j^{***} = \int_{\tilde{\Omega}}\int_{\tilde{\Omega}} \frac{\left|\frac{\partial\tilde{\mathscr{E}}}{\partial \tilde{x}_j}(\tilde{y})\right|^2}{|\tilde{x} - \tilde{y}|}\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y} =$$
$$= \int_{\tilde{\Omega}}\left\{\left|\frac{\partial\tilde{\mathscr{E}}}{\partial \tilde{x}_j}(\tilde{y})\right|^2 \int_{\tilde{\Omega}} \frac{\mathrm{d}\tilde{x}}{|\tilde{x} - \tilde{y}|}\right\}\mathrm{d}\tilde{y}\,.$$

Let $\delta > 0$ be arbitrary and fixed. If $\tilde{y} \in \bar{\tilde{\Omega}}$, then

$$(5.14)\qquad \int_{\tilde{\Omega}} \frac{\mathrm{d}\tilde{x}}{|\tilde{x} - \tilde{y}|} \leqq \int_{\tilde{\Omega} - B_\delta(\tilde{y})} \frac{\mathrm{d}\tilde{x}}{|\tilde{x} - \tilde{y}|} + \int_{B_\delta(\tilde{y})} \frac{\mathrm{d}\tilde{x}}{|\tilde{x} - \tilde{y}|}\,,$$

where $B_\delta(\tilde{y}) = \{\tilde{x}; |\tilde{x} - \tilde{y}| \leqq \delta\}$. For $x \in \tilde{\Omega} - B_\delta(\tilde{y})$ we have $|\tilde{x} - \tilde{y}| \geqq \delta$ so that

$$\int_{\tilde{\Omega} - B_\delta(\tilde{y})} \frac{\mathrm{d}\tilde{x}}{|\tilde{x} - \tilde{y}|} \leqq \frac{1}{\delta} \operatorname{meas}(\tilde{\Omega}) . \tag{5.15}$$

The integral

$$I_\delta = \int_{B_\delta(\tilde{y})} \frac{\mathrm{d}\tilde{x}}{|\tilde{x} - \tilde{y}|} \tag{5.16}$$

can be calculated by introducing the polar coordinates $R$, $\varphi$ with the origine at $\tilde{y}$. We get

$$I_\delta = \int_0^{2\pi} \left( \int_0^\delta \mathrm{d}R \right) \mathrm{d}\varphi = 2\pi\delta \tag{5.17}$$

and then, in virtue of (5.14)–(5.17),

$$\int_{\tilde{\Omega}} \frac{\mathrm{d}\tilde{x}}{|\tilde{x} - \tilde{y}|} \leqq k_\delta = \frac{1}{\delta} \operatorname{meas}(\tilde{\Omega}) + 2\pi\delta . \tag{5.18}$$

Now let us go back to the integral $I_j^{***}$ from (5.13). With respect to (5.18),

$$I_j^{***} \leqq k_\delta \int_{\tilde{\Omega}} \left| \frac{\partial \tilde{\mathscr{E}}}{\partial \tilde{x}_j} (\tilde{y}) \right|^2 \mathrm{d}\tilde{y} \leqq k_\delta \|\tilde{\mathscr{E}}\|^2_{H^1(\tilde{\Omega})} \leqq k_\delta \|\tilde{\mathscr{E}}\|^2_{W_2^{1+1/2}(\tilde{\Omega})} < +\infty . \tag{5.19}$$

From (5.5), (5.10)–(5.13) and (5.19) it finally follows that $I < +\infty$, which we wanted to prove.

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## Souhrn

## NEVÍŘIVÉ PROUDĚNÍ PROFILOVYMI MŘÍŽEMI VE VRSTVĚ PROMĚNNÉ TLOUŠŤKY

MILOSLAV FEISTAUER

Článek se zabývá studiem nevazkého, nevířivého, podzvukového proudění v lopatkových mřížích na osově symetrické proudoploše ve vrstvě proměnné tloušťky. Na rozdíl od řady jiných prací věnovaných této problematice a používajících metodu singularit a integrálních rovnic zde zavádíme proudovou funkci a formulujeme několik okrajových úloh, které představují adekvátní dvourozměrné modely proudových polí v lopatkových kolech. V článku je zaveden pojem slabého řešení a je provedeno podrobné vyšetření řešitelnosti uvažovaných problémů. Na výsledky obsažené v této práci navážou články věnované numerickému řešení proudění lopatkovými mřížemi metodou konečných prvků.

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