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ON THE OPTIMAL CONTROL PROBLEM GOVERNED BY THE EQUATIONS OF VON KÁRMÁN
I. THE HOMOGENEOUS DIRICHLET BOUNDARY CONDITIONS

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A control of the system of nonlinear Kármán's equations for a thin elastic plate by means of the right-hand side of the equilibrium equation is considered. Thus the transversal loading plays the role of the control variable. In Part I we consider only Dirichlet boundary conditions, i.e. the edge of the plate is clamped.

Using some results of Ciarlet and Rabier [1], the set of admissible control functions is chosen so as to obtain the unique solvability of Kármán's system.

We prove the existence of the optimal control, the differentiability (Fréchet) of the state function with respect to the control variable, the uniqueness of the optimal control under certain conditions, and derive necessary conditions of the optimality.

1. FORMULATION OF THE PROBLEM

We shall be dealing with the system of equations

$$\Delta^2 y = [\Phi, y] - \lambda \Delta y + v, \tag{1.1}$$

$$\Delta^2 \Phi = -[y, y], \quad (x_1, x_2) \in \Omega, \tag{1.2}$$

where

$$[\varphi, \psi] = \varphi_{11}\psi_{22} + \varphi_{22}\psi_{11} - 2\varphi_{12}\psi_{12}, \tag{1.3}$$

$$\varphi_{ij} = \frac{\partial^2 \varphi}{\partial x_i \partial x_j}, \quad i, j = 1, 2.$$

Here $y = y(x_1, x_2)$ means the (reduced) deflection of the plate, $\Phi = \Phi(x_1, x_2)$ is the (reduced) Airy stress function, λ expresses the magnitude of the external forces acting on the boundary of the plate in the direction of the normal vector and v is the (reduced) load acting in the direction perpendicular to the middle surface of the plate.

We assume that the plate is clamped, i.e.

$$y = \frac{\partial y}{\partial n} = 0 \quad \text{on} \quad \partial\Omega\,. \tag{1.4}$$

The reduced Airy's function also satisfies the homogeneous boundary conditions

$$\Phi = \frac{\partial \Phi}{\partial n} = 0 \quad \text{on} \quad \partial\Omega\,. \tag{1.5}$$

We denote by $L^p(\Omega)$ $(1 \leqq p < \infty)$ the space of all real functions which are integrable in the Lebesgue sense with the power p on Ω. $L^2(\Omega)$ is a Hilbert space with the scalar product

$$(u, v)_0 = \int_\Omega uv \, dx_1 \, dx_2 \tag{1.6}$$

and the norm

$$\|u\|_0 = (u, u)_0^{1/2}\,. \tag{1.7}$$

Denoting

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2}}\,, \quad |\alpha| = \alpha_1 + \alpha_2\,,$$

we define for any integer $m \geqq 1$ the space

$$W^{m,p}(\Omega) = \{u \mid u \in L^p(\Omega),\ D^\alpha u \in L^p(\Omega) \text{ for } |\alpha| \leqq m\}$$

with the distributive derivatives. $W^{m,p}(\Omega)$ is a Banach space with the norm

$$\|u\|_{m,p} = \left(\int_\Omega |u|^p \, dx_1 \, dx_2 + \sum_{1 \leqq |\alpha| \leqq m} \int_\Omega |D^\alpha u|^p \, dx_1 \, dx_2\right)^{1/p}.$$

Further, we introduce the space

$$H_0^2(\Omega) = \left\{u \mid u \in W^{2,2}(\Omega)\,, \quad u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\right\},$$

where the boundary conditions are satisfied in the sense of traces on $\partial\Omega$ (see [4]). $H_0^2(\Omega)$ is a Hilbert space with the scalar product

$$((u, v)) = \sum_{|\alpha|=2} \int_\Omega D^\alpha u D^\alpha v \, dx_1 \, dx_2 = \int_\Omega \Delta u \, \Delta v \, dx_1 \, dx_2\,. \tag{1.8}$$

The latter identity can be shown by integrating by parts and using the homogeneous boundary conditions. The norm in $H_0^2(\Omega)$,

$$\|u\| = ((u, u))^{1/2}\,, \tag{1.9}$$

is equivalent to the original norm $\|u\|_{2,2}$f In the sequel we denote by $H^{-2}(\Omega)$ the space of all linear continuous functionals over $H_0^2(\Omega)$ with the norm

$$(1.10) \qquad \|f\|_{-2} = \sup_{\substack{\|\varphi\| = 1 \\ \varphi \in H_0^2(\Omega)}} \langle f, \varphi \rangle .$$

Definition 1.1. *The couple* $(y, \Phi) \in (H_0^2(\Omega))^2$ *is called a weak solution of the problem* (1.1)–(1.5) *if*

$$(1.11) \qquad ((y, \varphi)) = ([\Phi, y], \varphi)_0 - \lambda(\Delta y, \varphi)_0 + (v, \varphi)_0 ,$$

$$(1.12) \qquad ((\Phi, \psi)) = -([y, y], \psi)_0$$

for all $(\varphi, \psi) \in (H_0^2(\Omega))^2$.

The expression $([\Phi, y], \varphi)_0$ is well defined, because $[\Phi, y] \in L^1(\Omega)$ and $\varphi \in C(\bar{\Omega})$ due to the continuous imbedding $H_0^2(\Omega) \hookrightarrow C(\bar{\Omega})$. The same holds for $([y, y], \psi)_0$.

It is more convenient for our further considerations to introduce a canonical form of the system (1.11), (1.12). Let us define operators $M : L^2(\Omega) \to H_0^2(\Omega)$, $B : (H_0^2(\Omega))^2 \to H_0^2(\Omega)$ by the relations

$$(1.13) \qquad ((Mv, \varphi)) = (v, \varphi)_0 ,$$

$$(1.14) \qquad ((B(y, \psi), \varphi)) = ([y, \psi], \varphi)_0$$

for all $v \in L^2(\Omega)$, $\varphi, y, \psi \in H_0^2(\Omega)$.

The operators M and B are well defined, because the right-hand sides of (1.13), (1.14) represent linear bounded functionals over $H_0^2(\Omega)$ for all $v \in L^2(\Omega)$ and $y, \psi \in H_0^2(\Omega)$.

Let $c > 0$ be such a number that

$$(1.15) \qquad \|\varphi\|_0 \leqq c\|\varphi\| \quad \text{for all} \quad \varphi \in H_0^2(\Omega) .$$

The existence of c follows from the equivalence of norms $\|\cdot\|$ and $\|\cdot\|_{2,2}$. We have, for $v \in L^2(\Omega)$,

$$\|Mv\|^2 = (v, Mv)_0 \leqq \|v\|_0 \, \|Mv\|_0 \leqq c\|v\|_0 \, \|Mv\| .$$

Hence the operator M is linear, bounded and

$$(1.16) \qquad \|Mv\| \leqq c\|v\|_0 , \quad v \in L^2(\Omega)$$

with c defined in (1.15).

Similarly, the operator $B : (H_0^2(\Omega))^2 \to H_0^2(\Omega)$ is bilinear over $(H_0^2(\Omega))^2$. In fact, if $(y, \eta) \in (H_0^2(\Omega))^2$, then we may write

$$\|B(y, \eta)\|^2 = \int_\Omega [y, \eta] \, B(y, \eta) \, d\Omega \leqq \int_\Omega |[y, \eta]| \, d\Omega \max_{x \in \bar{\Omega}} |B(y, \eta)| \leqq$$
$$\leqq C\|y\| \, . \, \|\eta\| \, . \, \|B(y, \eta)\|$$

using the form of $[y, \eta]$ and the continuous imbedding $H_0^2(\Omega) \hookrightarrow C(\bar{\Omega})$. Hence there exists a (finite) norm of B

(1.17)
$$\|B\| = \sup_{\substack{\|y\| = \|\eta\| = 1 \\ (y,\eta) \in (H_0^2(\Omega))^2}} \|B(y, \eta)\| .$$

Further, we introduce operators $L: H_0^2(\Omega) \to H_0^2(\Omega)$ and $C : H_0^2(\Omega) \to H_0^2(\Omega)$ by the relations

(1.18)
$$Ly = M(-\Delta y),$$

(1.19)
$$C(y) = B(B(y, y), y), \quad y \in H_0^2(\Omega).$$

Operators C, L are compact, L is, moreover, selfadjoint and positive (see [1], L. 2.2.5, L. 2.3.1). Then (see [3], VII. Ch. 4, Th. 3.4) the number

(1.20)
$$\mu_1 = \|L\| = \sup_{\substack{\|y\| = 1 \\ y \in H_0^2(\Omega)}} ((Ly, y))$$

is the greatest eigenvalue of L. We denote by

(1.21)
$$\lambda_1 = \frac{1}{\mu_1} = \inf_{\substack{\|y\| = 1 \\ y \in H_0^2(\Omega)}} ((Ly, y))^{-1}$$

the smallest characteristic value of the operator L.

It can be easily seen that $(y, \Phi) \in (H_0(\Omega))^2$ is a solution of the system (1.11), (1.12) if and only if

(1.22)
$$y - \lambda Ly + C(y) = Mv,$$

(1.23)
$$\Phi = -B(y, y).$$

We proceed now to the formulation of the optimal control problem for the equation (1.22) which is (together with (1.23)) equivalent to the system (1.11), (1.12).

Let U_{ad} be the set of functions $v \in L^2(\Omega)$ such that the equation (1.22) has a unique solution $y = y(v)$. Let $J : U_{ad} \to R$ be the cost functional of the form

(1.24)
$$J(v) = \mathscr{J}(y(v)) + j(v),$$

where $\mathscr{J} : H_0(\Omega) \to R$, $j : L^2(\Omega) \to R$ are certain functionals.

Problem P. *To find such a control $u \in U_{ad}$ that*

(1.25)
$$J(u) = \min_{v \in U_{ad}} J(v),$$

(1.26)
$$y(u) - \lambda Ly(u) + C(y(u)) = Mu .$$

2. EXISTENCE AND UNIQUENESS OF A SOLUTION OF THE STATE PROBLEM

Theorem 2.1. *Let*

$$\|Mv\| < \|B\|^{-1}\,, \tag{2.1}$$

$$\lambda < \lambda_1(1 - \|B\|^{2/3}\,\|Mv\|^{2/3})\,. \tag{2.2}$$

where λ_1 *is defined in* (1.21). *Then there exists a unique solution* $y(v) \in H_0^2(\Omega)$ *of the equation* (1.26).

Moreover, $y(v)$ *satisfies the estimate*

$$\|y(v)\| \leqq K(\lambda)^{-1}\,\|Mv\|\,. \tag{2.3}$$

where

$$K(\lambda) = \min\left\{1, 1 - \frac{\lambda}{\lambda_1}\right\}. \tag{2.4}$$

Proof. The existence of a solution $y(v)$ is verified in [1], Th. 2.2.1. If $\lambda > 0$, then the uniqueness can be verified in the same manner as in [1], Th. 2.3.1. If $\lambda \leqq 0$, then we have from (1.26)*)

$$\|y(v)\| \leqq \|Mv\|\,, \tag{2.5}$$

by virtue of the positivity of the operator L and the relation ([1], Lemma 2.2.6)

$$0 \leqq ((Cy, y)) = \|B(y, y)\|^2 \quad \text{for all} \quad y \in H_0^2(\Omega)\,. \tag{2.6}$$

Let y_1, y_2 be two solutions of (1.22). Then, for $\lambda < 0$,

$$\|y_1 - y_2\|^2 \leqq ((C(y_2) - C(y_1), y_1 - y_2))\,.$$

Following the method of the proof of [1], Th. 2.3.1 we arrive at

$$\|y_1 - y_2\|^2 \leqq \|B\|^2\,\|Mv\|^2\,\|y_1 - y_2\|^2\,.$$

This inequality can be true only if $y_1 = y_2 = y(v)$. It remains to verify the estimate (2.3). We obtained the estimate (2.5) for $\lambda \leqq 0$. If $\lambda > 0$, then from (1.26) and (1.21) we have the estimate

$$\left(1 - \frac{\lambda}{\lambda_1}\right)\|y(v)\|^2 \leqq \|Mv\|\,\|y(v)\|\,;$$

consequently,

$$\|y(v) \leqq \left(1 - \frac{\lambda}{\lambda_1}\right)^{-1}\|Mv\| \tag{2.7}$$

and (2.3) follows from the estimates (2.5), (2.7).

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*) Note that the case $\lambda \leqq 0$ is not discussed in [1].

Let us now define the admissible set of controls

$$U_{ad} = \left\{ v \in L^2(\Omega) : \|v\|_0 \leqq \frac{\alpha}{c} \|B\|^{-1} K(\lambda)^{3/2} \right\}, \tag{3.1}$$

where α is an arbitrary number satisfying $0 < \alpha < 1$, $c > 0$ is a constant defined in (1.15), and $K(\lambda)$ is defined in (2.4), $\lambda < \lambda_1$.

If $v \in U_{ad}$, then using (1.16) we obtain

$$\|Mv\| \leqq \alpha \|B\|^{-1} K(\lambda)^{3/2} \tag{3.2}$$

and the inequalities (2.1), (2.2) are fulfilled. Hence for every $v \in U_{ad}$ there exists a unique solution $y \in H_0^2(\Omega)$ of the state equation (1.25).

We can now formulate the existence theorem for the optimal control.

Theorem 3.1. *Let $\mathscr{J} : H_0^2(\Omega) \to R$, $j : L^2(\Omega) \to R$ be weakly lower semicontinuous. Then there exists a solution $u \in U_{ad}$ of Problem P.*

Proof. There exists a minimizing sequence $\{u_n\} \subset U_{ad}$,

$$\lim_{n \to \infty} J(u_n) = \inf_{v \in U_{ad}} J(v) . \tag{3.3}$$

The set U_{ad} is bounded in L^2 and hence there exists a subsequence (again denoted by) $\{u_n\}$ such that

$$u_n \rightharpoonup u \quad \text{(weakly) in} \quad L^2(\Omega), \quad u \in U_{ad}, \tag{3.4}$$

because U_{ad} (being convex and closed) is weakly sequentially closed in $L^2(\Omega)$.

Using the inequalities (2.3), (3.2) we obtain the estimate of all states

$$\|y(v)\| \leqq \alpha \|B\|^{-1} K(\lambda)^{1/2} \quad \text{for all} \quad v \in U_{ad} . \tag{3.5}$$

Hence the sequence $\{y(u_n)\}$ is bounded in $H_0^2(\Omega)$ and there exists a subsequence (again denoted by) $\{u_n\}$ such that

$$y_n \rightharpoonup y_0 \quad \text{in} \quad H_0^2(\Omega), \quad y_n = y(u_n) \tag{3.6}$$

and

$$y_n - \lambda L y_n + C(y_n) = M u_n . \tag{3.7}$$

The operators $L : H_0^2(\Omega) \to H_0^2(\Omega)$, $C : H_0^2(\Omega) \to H_0^2(\Omega)$ are compact and $Ly_n \to Ly_0$, $C(y_n) \to C(y_0)$ strongly in $H_0^2(\Omega)$. $M : L^2(\Omega) \to H_0^2(\Omega)$ is linear bounded and the relation (3.4) implies $Mu_n \rightharpoonup Mu$ (weakly) in $H_0^2(\Omega)$. Proceeding to a weak limit in (3.7), we arrive at

$$y_0 - \lambda L y_0 + C(y_0) = Mu . \tag{3.8}$$

From the uniqueness of the solution of (1.26) we have $y_0 = y(u)$ and $y_n \rightharpoonup y(u)$. The weak lower semicontinuity of $\mathcal{J}, j$ implies

$$J(u) = \mathcal{J}(y(u)) + j(u) \leqq \liminf_{n\to\infty} \mathcal{J}(y(u_n)) + \liminf_{n\to\infty} j(u_n) \leqq$$
$$\leqq \liminf_{n\to\infty} J(u_n) = \inf_{v\in U_{ad}} J(u)\,.$$

Consequently, u is a solution of Problem P.

4. DIFFERENTIABILITY OF THE STATE FUNCTION

Let us show that the mapping $y(\cdot) : U_{ad} \to H_0^2(\Omega)$, $y(v) - \lambda Ly(v) + C(y(v)) = Mv$, $v \in U_{ad}$, is Fréchet differentiable and the derivative $y'(v) : L^2(\Omega) \to H_0^2(\Omega)$ is a solution of the problem

$$\text{(4.1)} \qquad [I - \lambda L + C'(y(v))]\, y'(v)\, h = Mh\,, \quad h \in L^2(\Omega)\,,$$

where

$$\text{(4.2)} \qquad C'(y)\,\eta = 2B(B(y, \eta, y) + B(B(y, y), \eta)\,, \quad y, \eta \in H_0^2(\Omega)\,,$$

$C'(y) \in \mathscr{L}(H_0^2(\Omega); H_0(\Omega))$ is the Gâteaux derivative of the operator C at the point $y \in H_0^2(\Omega)$.

Lemma 4.1. *The operator* $A(y(v)) : H_0^2(\Omega) \to H_0^2(\Omega)$, $A(y(v)) = I - \lambda L + C'(y(v))$, *is linear, symmetric and positive definite.*

Proof. The linearity follows from (4.2). The symmetry of $A(y(v))$ results from the symmetry of the operator L and the properties

$$\text{(4.3)} \qquad B(y, z) = B(z, y)\,,$$

$$\text{(4.4)} \qquad ((B(y, z), w)) = ((B(y, w), z))$$

for all $y, w, z \in H_0^2(\Omega)$ (see [1], Lemma 2.2.1, 2.2.3).

Let $w \in H_0^2(\Omega)$. Then by virtue of (4.3), (4.4) and (3.5) we may write

$$((A(y(v))\, w, w)) = \|w\|^2 - \lambda((Lw, w)) + 2\|B(y(v), w)\|^2 +$$
$$+ ((B(y(v), y(v)), B(w, w))) \geqq K(\lambda)\, \|w\|^2 - \|B\|^2\, \|y(v)\|^2\, \|w\|^2 \geqq K(\lambda)\,(1 - \alpha^2)\, \|w\|^2\,.$$

Hence

$$\text{(4.5)} \qquad ((A(y(v))\, w, w)) \geqq C(\lambda)\, \|w\|^2 \quad \text{for all} \quad w \in H_0^2(\Omega)\,,$$

where

$$\text{(4.6)} \qquad C(\lambda) = (1 - \alpha^2)\, K(\lambda) > 0\,,$$

which concludes the proof.

By virtue of Lemma 4.1 there exists a unique solution $z(h) \in H_0^2(\Omega)$ of the equation

$$A(y(v))\, z(h) = [I - \lambda L + C'(y(v))]\, z(h) = Mh \tag{4.7}$$

for every $h \in L^2(\Omega)$.

Let

$$w = w(h) = y(v + h) - y(v) - z(h)\,; \quad v, v + h \in U_{ad}\,. \tag{4.8}$$

If we verify that $\|w\| = o(h)$, we shall obtain $z(h) = y'(v)\, h$.

Using (1.26) and (4.7) we have

$$A(y(v))\, w = [I - \lambda L + C'(y(v))]\, w = C'(y(v))\,(y(v + h) - y(v)) -$$

$$- [C(y(v + h) - C(y(v))] = \int_0^1 [C'(y(v)) - C'(y(v) + s\eta)]\, \eta \, ds\,,$$

where

$$\eta = y(v + h) - y(v)\,. \tag{4.9}$$

As $\|y(v)\|$, $\|y(v + h)\|$ are bounded for all $v, v + h \in U_{ad}$ (see (3.5)), by rewriting the expression

$$[C'(y(v)) - C'(y(v) + s\eta)]\, \eta$$

and using the boundedness of B and the positive definiteness of $A(y(v))$ we obtain the estimate

$$\|w\| \leqq C_2(\lambda)\, \|\eta\|^2\,. \tag{4.10}$$

Let us estimate $\|\eta\| = \|y(v + h) - y(v)\|$. The function $\eta \in H_0^2(\Omega)$ fulfils the equation

$$\eta - \lambda L\eta + C(y(v + h)) - C(y(v)) = Mh\,. \tag{4.11}$$

Using Lemma 2.25 of [1] and estimates similar to that established above, we obtain

$$K(\lambda)\, \|\eta\|^2 - \|B\|^2 \max \{\|yy(v)\|^2, \|y(v + h)\|^2\}\, \|\eta\|^2 \leqq c\|h\|_0\, \|\eta\|$$

with the constant c defined in (1.15). Estimating the values $\|y(v)\|^2$, $\|y(v + h)\|^2$ as in (3.5), we arrive at

$$\|\eta\| \leqq c[K(\lambda)\,(1 - \alpha^2)]^{-1}\, \|h\|_0 = C_3(\lambda)\, \|h\|_0 \tag{4.12}$$

and from (4.8), (4.10),

$$\|w\| = \|y(v + h) - y(v) - z(h)\| = 0(h)$$

follows. Hence we have verified the following theorem.

Theorem 4.1. *The mapping* $y(\cdot) : U_{ad} \to H_0^2(\Omega)$ *determined by the equation* $y(v) - \lambda Ly(v) + C(y(v)) = Mv\,, \quad v \in U_{ad}$, *is Fréchet differentiable for all functions*

$v \in \mathring{U}_{ad}$ *and*

$$(4.13) \qquad [I - \lambda L + C'(y(v))] \, y'(v) \, h = Mh$$

holds for all $h \in L^2(\Omega)$. ($\mathring{U}_{ad}$ *denotes the interior of* U_{ad}.)

5. UNIQUENESS OF THE OPTIMAL CONTROL

Let us assume that the functionals $\mathscr{J}$, j are, moreover, Fréchet differentiable and satisfy the conditions

$$(5.1) \qquad \langle \mathscr{J}'(y_1) - \mathscr{J}'(y_2), y_1 - y_2 \rangle \geqq m\|y_1 - y_2\|^2 , \quad m > 0 ,$$

for all $y_1, y_2 \in H_0^2(\Omega)$,

$$(5.2) \qquad \langle j'(v_1) - j'(v_2), v_1 - v_2 \rangle_0 \geqq N\|v_1 - v_2\|_0^2 ; \quad N > 0 ,$$

for all $v_1, v_2 \in L^2(\Omega)$, while $\mathscr{J}'$ satisfies the condition

$$(5.3) \qquad \|\mathscr{J}'(y)\|_* \leqq c_0\|y\| + c_1 \quad \text{for all} \quad y \in H_0^2(\Omega) ; \quad c_0 > 0 .$$

If $u \in U_{ad}$ is the optimal control, i.e. a solution of Problem P, then $\langle J'(u), v - u \rangle_0 \geqq \geqq 0$ for all $v \in U_{ad}$. Let u_1, u_2 be two optimal controls. Then

$$(5.4) \qquad \langle J'(u_1), v - u_1 \rangle_0 = \langle \mathscr{J}'(y(u_1)), y'(u_1)(v - u_1) \rangle + \\ + \langle j'(u_1), v - u_1 \rangle_0 \geqq 0 ,$$

$$\langle J'(u_2), v - u_2 \rangle_0 = \langle \mathscr{J}'(y(u_2)), y'(u_2)(v - u_2) \rangle + \\ + \langle j'(u_2), v - u_2 \rangle_0 \geqq 0$$

for all $v \in U_{ad}$.

Inserting u_2, u_1 in (5.4) and adding we obtain

$$(5.5) \qquad 0 \leqq \langle \mathscr{J}'(y(u_1)) - \mathscr{J}'(y(u_2)), y(u_1) - y(u_2) \rangle + \\ + \langle j'(u_1) - j'(u_2), u_2 - u_1 \rangle_0 - \\ - \langle \mathscr{J}'(y(u_1)), y(u_2) - y(u_1) - y'(u_1)(u_2 - u_1) \rangle - \\ - \langle \mathscr{J}'(y(u_2)), y(u_1) - y(u_2) - y'(u_2)(u_1 - u_2) \rangle .$$

Let us denote

$$(5.6) \qquad w_1 = y(u_2) - y(u_1) - y'(u_1)(u_2 - u_1) , \\ w_2 = y(u_1) - y(u_2) - y'(u_2)(u_1 - u_2) , \quad \eta = y(u_2) - y(u_1) .$$

We derive an estimate for w_1, using (1.26) and (4.7):

$$(5.7) \qquad [I - \lambda L + C'(y(u_1))] \, w_1 =$$

$$= C(y(u_1)) - C(y(u_2)) + C'(y(u_1))(y(u_2) - y(u_1)) =$$

$$= \int_0^1 [C'(y(u_1)) - C'(y(u_1) + s\eta)]\, \eta \,\mathrm{d}s = \psi\,.$$

Using the mean value theorem we arrive at the relations

$$\|\psi\| = \sup_{\|h\|=1} \left|\left(\left(\int_0^1 [C'(y(u_2)) - C'(y(u_2) + s\eta)]\, \eta \,\mathrm{d}s,\, h\right)\right)_2\right| = \tag{5.8}$$

$$= \sup_{\|h\|=1} \left|\left(\left(\int_0^1 C''(y(u_2 + \tau(s)\,\eta)(\eta, \eta)\, s \,\mathrm{d}s,\, h\right)\right)_2\right|,$$

where $\tau(s) \in (0, s)$. The second derivative C'' has the form

$$C''(y)(\eta, \eta) = 2B(B(\eta, \eta), y) + 4B(B(y, \eta), \eta) \tag{5.9}$$

for all $y, \eta \in H_0^2(\Omega)$.

Using the estimate (3.5) we have

$$\|\psi\| \leqq 3\alpha \|B\|\, K(\lambda)^{1/2}\, \|\eta\|^2\,. \tag{5.10}$$

By virtue of positive definiteness in (4.5), (4.6) we obtain, from (5.7) and (5.10),

$$\|w_i\| \leqq 3\alpha\, C(\lambda)^{-1}\, K(\lambda)^{1/2}\, \|B\|\,.\, \|\eta\|^2\,. \tag{5.11}$$

From (5.5), (5.6), (3.5), (5.1), (5.2), (5.3), (5.11) we have the inequality

$$0 \leqq [-m + (c_0\, \alpha \|B\|^{-1}\, K(\lambda)^{1/2} + c_1)\,.\, 6\alpha\, C(\lambda)^{-1}\, K(\lambda)^{1/2}\, \|B\|]\,. \tag{5.12}$$
$$.\, \|\eta\|^2 - N\|u_1 - u_2\|_0^2\,.$$

Setting $h = u_1 - u_2$ in (4.12) we obtain

$$\|\eta\| \leqq c[K(\lambda)(1 - \alpha^2)]^{-1}\, \|u_1 - u_2\|_0 \tag{5.13}$$

and combining it with (5.12) we are led to the inequality

$$0 \leqq \{[-m + (c_0\, \alpha \|B\|^{-1}\, K(\lambda)^{1/2} + c_1)\,.\, 6\alpha\, C(\lambda)^{-1}\, K(\lambda)^{1/2}\, \|B\|]\,. \tag{5.14}$$
$$.\, c^2[K(\lambda)(1 - \alpha^2)]^{-2} - N\}\, \|u_1 - u_2\|_0^2\,.$$

Now it is easily to derive sufficient conditions for the uniqueness.

Theorem 5.1. *Let the functionals $\mathscr{J}$, j be lower bounded, weakly lower semi-continuous with Fréchet derivatives satisfying the conditions* (5.1), (5.2), (5.3). *If*

$$N > \left[-m + (c_0\, \alpha \|B\|^{-1}\, K(\lambda)^{1/2} + c_1)\, \frac{6\alpha}{1 - \alpha^2}\, K(\lambda)^{-1/2} \|B\|\right].$$
$$.\, c^2[(1 - \alpha^2)\, K(\lambda)]^{-2}\,,$$

where $K(\lambda) = \min\{1, 1 - \lambda/\lambda_1\}$, c is defined in (1.15), then there exists a unique solution u of Problem P.

If $J(u) = \min_{v \in U_{ad}} J(v)$, then

$$\langle \mathscr{J}'(y(u)), y'(u)(v - u) \rangle + \langle j'(u), v - u \rangle_0 \geqq 0 \quad \forall v \in U_{ad} . \tag{6.1}$$

where

$$[I - \lambda L + C'(y(u))] \, y'(u) \, h = Mh \quad \forall h \in L^2(\Omega) . \tag{6.2}$$

As we have seen above, the operator $A(y(u)) = I - \lambda L + C'(y(u))$ is linear symmetric. The system (6.1), (6.2) can have the form

$$(p + R_0 \, j'(u), v - u)_0 \geqq 0 \quad \text{for all} \quad v \in U_{ad} , \tag{6.3}$$

$$[I - \lambda L + C'(y(u))] \, p = R \, \mathscr{J}'(y(u)) , \tag{6.4}$$

where $R_0 : (L^2(\Omega))^* \to L^2(\Omega)$, $R : H^{-2}(\Omega) \to H_0^2(\Omega)$ are the Riesz representative operators and we have used the relations

$$\begin{aligned} \langle \mathscr{J}'(y(u)), y'(u)(v - u) \rangle &= ((R \, \mathscr{J}'(y(u)), y'(u)(v - u))) = \\ = ((A(y(u)) \, p, y'(u)(v - u)) &= ((p, A(y(u)) \, y'(u)(v - u))) = \\ &= ((p, M(v - u))) = (p, v - u)_0 . \end{aligned}$$

Further,

$$y(u) - \lambda L(u) + C(y(u)) = Mu . \tag{6.5}$$

The system (6.3), (6.4), (6.5) is the optimality system for Problem P. The problem (6.4) is the adjoint problem to the problem (6.5).

Remark. It is possible to obtain similar results for the optimal control problem

$$J(\lambda) = \min_{\mu \in U_{ad}} J(\mu) = \min_{\mu \in U_{ad}} [\mathscr{J}(y(\mu)) + j(\mu)] , \tag{6.6}$$

where

$$y(\mu) - \mu L y(\mu) + C(y(\mu)) = F ; \tag{6.7}$$

$$F \in H_0^2(\Omega), \; y(\mu) \in H_0^2(\Omega), \; U_{ad} \subset (-\infty, \lambda_1(1 - \|B\|^{2/3} \, \|F\|^{2/3})) ,$$

$$\|F\| < \|B\|^{-1} .$$

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Souhrn

O PROBLÉMU OPTIMÁLNÍHO ŘÍZENÍ PRO KÁRMÁNOVY ROVNICE I. HOMOGENNÍ DIRICHLETOVY OKRAJOVÉ PODMÍNKY

IGOR BOCK, IVAN HLAVÁČEK, JÁN LOVÍŠEK

Je studována úloha řízení systému nelineárních Kármánových rovnic pro tenkou desku prostřednictvím pravé strany rovnice rovnováhy. Okraj desky se uvažuje dokonale vetknutý. Pomocí některých výsledků práce [1] je vybrána množina přípustných funkcí tak, aby stavová úloha byla jednoznačně řešitelná.

Dokazuje se existence řešení optimalizační úlohy, diferencovatelnost řešení stavové úlohy vzhledem k řídící proměnné, jednoznačnost za určitých podmínek a odvozují se nutné podmínky optimality.

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