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# NOTE ON TYPE II COUNTER PROBLEM

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## 1. INTRODUCTION

The mathematical theory of particle counters is concerned with the formulation and study of stochastic processes associated with the registration of particles due to radioactive substances by a counting device designed to detect and record them and placed within the range of a radioactive material. The general problem can be described as follows.

First we consider a sequence of random events consisting of the arrivals of the emitted particles. This sequence is called the primary sequence of events or the primary process. We suppose that any arriving particle generates an impulse of a random length (may be constant one, too). Due to the inertia of the counting device, it is possible that all particles will not be counted. The time during which the device is unable to record is called the dead time. The sequence of registered particles forms a secondary process which is selected from the primary sequence according to the type of the counter employed.

The basic problem in the counter theory is to determine the distribution function of the distance between two successive registered particles if the distribution function of the primary process, distribution of impulses and the counter type are known.

Our main aim in this note is to determine the joint Laplace transform of the above mentioned distribution, and the generating function of the number of particles arriving in the counting device during the dead time for the so called Type II counter, and to make some remarks on the registrations of $m$ types of particles ($m \geqq 1$).

## 2. NOTATION AND KNOWN RESULTS

The mathematical and physical literature on the counter theory deals mainly with two types of models. A Type I counter (counter with nonprolonging dead time) is one in which dead time occurs only after impulses of particles have been registered.

A Type II counter (counter with prolonging dead time) is one in which dead time occurs after registration of all impulses of emitted particles. Examples of Type I and Type II counters are the Geiger-Müller counters and electron multipliers, respectively. An extensive bibliography of the counter theory is found in Takács [24] and Smith [21]. From the physical literature dealing on this object see, for example, the monograph [11].

Let us suppose that particles arrive at a counter at the instances $0 = \tau_0 < \tau_1 < < \tau_2 < \ldots < \infty$, where the inter-arrival times $\tau_n - \tau_{n-1}$ $(n = 1, 2, \ldots)$ are identically distributed, independent, positive random variables with the distribution function

$$F(x) = P(\tau_n - \tau_{n-1} < x) .$$

Denote by $\chi_n$ the duration of the impulse starting at $\tau_n$, $(n = 0, 1, 2, \ldots)$. It is supposed that $\{\chi_n\}$ is a sequence of identically distributed, independent, positive random variables with the distribution function

$$H(x) = P(\chi_n) < x) ,$$

and independent of $\{\tau_n\}$.

At any instant $t$ there are two mutually exclusive states in which the counter may be, state $E_0$ when no impulse covers the instant, and state $E_1$ otherwise. The interval when the counter is in the state $E_1$ corresponds to the dead time, and the interval when it is in $E_0$ corresponds to an idle time. The particles are registered only if the counter is idle, and let us suppose that the registration process starts from $\tau_0 = 0$.

In the Type II counter the $n$th particle $(n = 1, 2, \ldots)$ is registered iff $\tau_m + \chi_m < \tau_n$ for any $m = 0, 1, \ldots, n - 1$. If we define, following Pyke [15], $n_0 = 0$,

$$n_j = \min \{k : k > n_{j-1}, \tau_k > \tau_r + \chi_r, \ r = n_{j-1}, \ldots, k - 1\}$$

for $j = 1, 2, \ldots$, then $\{n_j\}_{j=0}^{\infty}$ is a sequence of indices of registered particles. Since the primary process is a recurrent one, the secondary process

$$Z_j = \tau_{n_j} - \tau_{n_{j-1}}, \quad j = 1, 2, \ldots,$$

is recurrent as well.

The main problem is to determine the distribution function

$$G(x) = P(Z_j < x) ,$$

or, equivalently, its Laplace transform

$$\gamma(s) = M(\mathrm{e}^{-sZ_j}), \quad s \geqq 0 .$$

This very interesting problem has been studied by several authors. The particular case of the Poisson primary process is discussed by Takács [22–24, 26, 28], Pollaczek [14], Smith [21], Sankaranarayanan [17–19], Albert and Nelson [4], Afanaseva and Michajlova [1, 2].

Although in the physical practice we deal mainly with the Poisson primary process, due to the repeated handling of particles by several counters, the initial process input at the last counter will not be Poisson, but only a recurrent one.

The recurrent primary process with a constant length of impulses is studied in [15, 24, 28] and the exponentially distributed impulse times are discussed in [14, 15, 25, 27, 28].

The general case has been taken into account by Pollaczek [14] who has given the solution in the form of complicated contour integrals. Takács [24] and Pyke [15] obtained only integral equations without their solutions. Similar results are obtained by using the multiplicative processes by Smith [21]. In the present authors' paper [9], explicit computable formulae for the discrete primary process and discrete lengths of impulses are given. Many other problems of the theory of Type II counters have been investigated, for example, in [6, 20].

This problem is important not only for the counter theory. The same problems (from the mathematical point of view) are studied in the film or filmless measurements of the particle track ionization in the so called bubble and streamer chambers (for details see, for example, [7, 8, 12]). The description of the queueing systems with infinitely many servers leads to analogous problems, in general.

## 3. TYPE II COUNTER

As has been noticed by several authors [5, 15, 21] the determination of $G$ or $\gamma$ is an extremely difficult problem. However, there are integral equations which formally, but not always in practice, determine $G$ or $\gamma$. Takács [24] obtained an integral equation for $M(t)$, the expected number of registered particles in a time interval $(0, t)$ for all $t \geqq 0$.

**Lemma 1.** (*Takács* [24]) *For all* $t \geqq 0$,

$$(1)\quad M(t) = \int_0^t H(y)\,\mathrm{d}F(y) + H(t)\int_0^t M(t-y)\,\mathrm{d}F(y) - \int_0^t\int_0^t M(z-y)\,\mathrm{d}H(z)\,\mathrm{d}F(y)\,.$$

If we know $M(t)$, then $\gamma(s)$ may be determined from

$$(2)\qquad \gamma(s) = \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}M(t)\Big/\left(1 + \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}M(t)\right), \quad s \geqq 0\,.$$

Barlow [5] generalized the equation (1) to the case of the semi-Markov primary process.

Pyke [15] obtained an integral equation for $G$.

**Lemma 2.** (*Pyke* [15]) *For all* $x \geqq 0$,

$$(3)\qquad G(x) = \int_0^x\int_0^{x-y} (1 - G(x-y-t))\,H(y+t)\,\mathrm{d}N(t)\,\mathrm{d}F(y)\,,$$

*and for all* $s \geqq 0$,

(4) $$\gamma(s) = \lambda(s)(1 + \lambda(s))^{-1},$$

*where*

$$\lambda(s) = \int_0^\infty \int_0^\infty e^{-s(x+t)} H(x + t)\, dF(x)\, dN(t).$$

These two representations are equivalent in the sense that $G$ and $N$ are uniquely determined by each other. Below we give an explicit form of the Laplace transform $\gamma$ of the distribution function $G$ which determines the solutions of the equations (1) or (3).

Let us denote by $q_n$ the number of the impulses preceding the arrival of the $n$th particle ($n = 0, 1, 2, \ldots$). The sequence of events $A_n$ defined by

$$A_n = \{q_n = 0\}, \quad n = 0, 1, \ldots,$$

is a sequence of recurrent events in the sense of Feller [10], i.e.,

$$P(A_{i_n}/A_{i_1} \ldots A_{i_{n-1}}) = P(A_{i_n - i_{n-1}})$$

for any finite system of indices

$$i_1 < i_2 < \ldots < i_n, \quad n = 2, 3, \ldots.$$

Hence we have

(5) $$P(A_n) = \int_0^\infty \ldots \int_0^\infty H(x_1) \ldots H(x_1 + \ldots + x_n)\, dF(x_1) \ldots dF(x_n), \quad n \geqq 1,$$

$$P(A_0) = 1.$$

We suppose that $P(A_1) > 0$ (the case $P(A_1) = 0$ corresponds to the case when during the dead time there arrive infinitely many particles). If we denote by $\nu$ the number of particles which arrive at the counter during the dead time, then $P(\nu = n) = P(\bar{A}_1 \ldots \bar{A}_{n-1} A_n)$ and for $P_n = P(\nu = n)$ we have

(6) $$P_1 = P(A_1),$$

$$P_n = P(A_n) - \sum_{i=1}^{n-1} P(A_i) P_{n-i}, \quad n \geqq 2.$$

It is clear that $P(A_n) \geqq P(A_{n+1})$, and the limit $\lim_n P(A) = P_\infty$ exists and $M(\nu) = 1/P_\infty$. In [3] sufficient conditions are given guaranteeing $P_\infty > 0$.

Let us put

$$a_n = P(A_n) - P(A_{+1}), \quad n = 0, 1, \ldots.$$

Hence for the generating function $f(z) = M(z^\nu)$ of $\nu$ we have

(7) $$f(z) = z(1 - \sum_{n=0}^\infty a_n z^n)(1 - z \sum_{n=0}^\infty a_n z^n)^{-1}, \quad |z| \leqq 1.$$

Define

$$a(s) = \int_0^\infty e^{-sx}\, dF(x)\,, \quad s \geqq 0\,,$$

$$\mu \;= \int_0^\infty x\, dF(x)\,.$$

With the given recurrent primary process $\{\tau_n\}_{n=0}^\infty$ we define a new recurrent one $\{\tau_n^s\}_{n=0}^\infty$ for any $s \geqq 0$ with the distribution function

$$F_s(x) = P(\tau_n^s - \tau_{n-1}^s < x) = a^{-1}(s) \int_0^x e^{-st}\, dF(t)\,.$$

Let $f_s(z)$ be the generating function of the number $\nu_s$ of the particles arriving during the dead time according to the primary process $\{\tau_n^s\}_{n=0}$ and the lengths of impulses $\{\chi_n\}_{n=0}^\infty$. Then we obtain the following forms for $\Phi(s, z) = M(e^{-sZ_1} z^\nu)$ and $\gamma(s)$, respectively.

**Theorem 1.** *For any* $s \geqq 0$, $|z| \leqq 1$

$$\Phi(s, z) = f_s(a(s)\, z)\,, \tag{8}$$

$$\gamma(s) \;= f_s(a(s))\,, \tag{9}$$

$$M(Z_1) = \mu\, M(\nu)\,. \tag{10}$$

Proof. Since $Z_1 = \tau_\nu$, we have

$$\Phi(s, z) = \sum_{n=1}^\infty \int_{\{\nu=n\}} e^{-s\tau_\nu} z^\nu\, dP =$$

$$= \sum_{n=1}^\infty \int \cdots \int_{C_n} e^{-s(t_1 + \ldots + t_n)} z^n\, dF(t_1) \ldots dF(t_n)\, dH(x_1) \ldots dH(x_n)\,,$$

where the integration area $C_n$ is of the form

$$(x_1 < t_1)^c\,, \quad \begin{pmatrix} x_1 < t_1 + t_2 \\ x_2 < t_2 \end{pmatrix}^c, \ldots, \begin{pmatrix} x_1 < t_1 + \ldots + t_{n-1} \\ \vdots \\ x_{n-1} < t_{n-1} \end{pmatrix}^c,$$

$$\begin{pmatrix} x_1 < t_1 + \ldots + t_n \\ \vdots \\ x_n < t_n \end{pmatrix}$$

(here the superscript "$c$" denotes the complement of the set indicated in the parentheses).

Hence

$$\Phi(s, z) = \sum_{n=1}^{\infty} a^n(s)\, z^n\, P(v_s = n) = f_s(a(s)\, z)\,.$$

Analogously we proceed for $\gamma(s)$. The mean value of $Z_1$ is obtained from the Wald identity. Q.E.D.

## 4. THE PROPERTIES OF $\nu$

The integer-valued random variable $\nu - 1$ determines the number of the non-registered particles between two successive registrations. Although in the physical practice it is hardly observable during the registration process by counters, it is of importance in other practical applications. For example, in the film handling of track information, $\nu$ means the number of the streamers with constant diameters with constant diameters in the blobs [7, 12]. Some limit properties of $\nu$ when $P_\infty \to 0$ are investigated in [3]. It was proved that $P(\nu > n) \approx \mathrm{e}^{-P_\infty n}$.

Now we examine the dependence of $P(\nu = n)$ on $n$. To this end we need the following notion.

A distribution function $F$ concentrated on $\langle 0, \infty\rangle$ is a Cramer distribution function if

$$\sup\left\{\lambda \geqq 0 : \int_0^\infty \mathrm{e}^{\lambda x}\, \mathrm{d}F(x) < \infty\right\} = \infty\,.$$

It is clear that the set $\mathbf{C}$ of all Cramer distribution functions is convex. Moreover, if $F(x_0) = 1$ for some $x_0 > 0$, then $F \in \mathbf{C}$, and if $\mathrm{d}F(x) = a \exp(-bx^c)$, $x \geqq 0$, for some $a > 0$, $b > 0$, $c \geqq 2$, then $F \in \mathbf{C}$.

**Theorem 2.** *Let the durations of impulses have the Cramer distribution function $H$. and $P(A_1) > 0$. Then*

$$P_n = (\beta - 1)\, \beta_1 \beta^{-n-1} + r_n\,, \tag{11}$$

*where*

$$\beta = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} [\psi^k(z)]\Big|_{z=1}\,, \tag{12}$$

$$\beta_1 = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\mathrm{d}^{k}}{\mathrm{d}z^{k}} [\psi^k(z)]\Big|_{z=1}\,, \tag{13}$$

*and*

$$|r_n| \leqq CR^{-n}$$

*(the constant $C$ does not depend on $n$, and $R > 1$).*

*Here* $\psi(z) = P(A_1)\, z + \sum_{n=2}^{\infty} (P(A_n) - P(A_{n-1}))\, z^n$.

Proof. According to the Cauchy formula [13] we have

$$P_n = \frac{1}{2\pi i}\oint_{|z|=1} \frac{f(z)\,dz}{z^{n+1}} = \frac{1}{2\pi i}\oint_{|z|=1} \frac{\psi(z)\,dz}{(1 - z + \psi(z))\, z^{n+1}}.$$

From the conditions of the theorem we have

$$0 \leqq a_n = P(A_n) - P(A_{n+1}) = P(\chi_1 < \tau_1, \ldots, \chi_n < \tau_n, \chi_{n+1} \geqq \chi_{n+1}) \leqq$$
$$\leqq P(\chi_{n+1} \geqq \tau_{n+1}).$$

Since $F \in \mathbf{C}$ we obtain

$$P(\chi_{n+1} \geqq \tau_{n+1}) \leqq M(e^{\lambda\chi_1})\, M(e^{-\lambda\tau_1})^{n+1}$$

for any $\lambda \geqq 0$. Hence the series $\sum_{n=0}^{\infty} a_n z^n$ has the radius of convergence

$$R \geqq 1/M(e^{-\lambda\tau_1}) > 1\,;$$

when $\lambda \to \infty$, then $R \to \infty$. Therefore the equation $1 - z + \psi(z) = 0$ has a unique (simple) positive root $z = \beta > 1$ with the minimal module. This follows from the following relations:

$$1 - z + \psi(z) = 1 - z\sum_{n=0}^{\infty} a_n z^n, \quad a_n \geqq 0.$$

Let $R > 1$ be the radius of a circle in which the function $1 - z + \psi(z)$ has a unique zero $z = \beta$. Then

$$r_n = \frac{1}{2\pi i}\oint_{|z|=R} \frac{f(z)\,dz}{z^{n+1}} = \frac{\psi(\beta)}{(\psi'(\beta) - 1)\,\beta^{n+1}} + P_n. \tag{14}$$

The integral on the left-hand side of (14) may be estimated by the maximum module $|r_n| \leqq CR^{-n}$. Putting $\beta_1 = 1/(1 - \psi'(\beta))$ we obtain the formula (11).

To obtain explicit expressions for $\beta$ and $\beta_1$, we consider the function $w = z - \psi(z)$ which conformly transforms some neighbourhood of the point $w = 1$ to another one. Therefore $w = w(z)$ has its inverse function $z = z(w)$. It is clear that $\beta = z(1)$ and $\beta_1 = z'(1)$. Using the Lagrange expansion formula we obtain (11) and (13). Q.E.D.

## 5. TYPE III COUNTER

G. E. Albert and L. Nelson [4] used a more general form of the counter model which includes both the above mentioned Type I and Type II counters as special cases.

They supposed that if a particle arrives at the counter, then the impulse (of the length $\chi_n$) starts with probability $p$ if at the moment $\tau_n$ an impulse occurs, and with probability 1 otherwise. If $p = 0$, then we obtain the Type *I* counter while $p = 1$ leads to the Type II counter discussed above.

This model has been studied by several authors [4, 24, 26–28, 15, 17–19]. The distribution function related to the secondary process for $p > 0$ may be easily deduced from the one of the Type II counter as was mentioned by Takács [28].

Indeed, let us suppose that $0 < p < 1$. We define the primary process with the distribution function

$$\hat{F}(x) = p \sum_{n=1}^{\infty} q^{n-1} F_n(x),$$

where $q = 1 - p$, and $F_n(x)$ denotes the $n$th iterated convolution of the distribution function with itself. It is easy to see that the only difference between the secondary process of the Type III counter determined by $F(x)$, $H(x)$ and $p$, and that of the Type II counter determined by $\hat{F}(x)$, $H(x)$ is that the latter contains an additional interval spent in the state $E_0$ immediately before every transformation $E_0 \to E_1$. The lengths of these intervals are identically distributed, independent random variables with the distribution function

$$Q(x) = p \sum_{n=0}^{\infty} q^n F_n(x),$$

and these random variables are independent of any other random variables involved.

If $\gamma(s)$ and $\hat{\gamma}(s)$ denote the Laplace transforms of the distance between successive registrations by the Type III counter and by the Type II counter mentioned above, then we have

$$\hat{\gamma}(s) = \gamma(s)\, p(1 - qa(s))^{-1}.$$

where $a(s)$ is the Laplace transform of the distribution function $F(x)$. Hence we have

$$\gamma(s) = \hat{\gamma}(s)\,(1 - q\, a(s))\, p^{-1}. \tag{15}$$

Here $\hat{\gamma}(s)$ is determined by (9) in which we replace $F(x)$ by $\hat{F}(x)$.

## 6. MARKOV RENEWAL PRIMARY PROCESS OF ZERO ORDER

In this section we consider a generalization of the problem of the registration of particles with one type of particles to the problem with $m$ types of particles ($1 \leqq m < \infty$) which arrive at the Type II counter.

Let us suppose that there are $m$ types of radioactive materials which emit $m$ types of particles according to the Markov renewal primary process of zero order. These processes have been introduced and studied by Pyke [16].

Thus, we suppose that the relationships between different types of particles and their impulses are as follows. Let the $n$th particle ($n \geqq 0$) arriving at the counter be of the type $J_n$ and let us suppose that the type of the particle does not depend on the previous types of particles, and

$$P(J_n = k) = p_k, \quad k = 1, \ldots, m, \sum_{k=1}^{m} p_k = 1.$$

Let $F_i(x)$ and $H_i(x)$, $i = 1, \ldots, m$, be distribution functions with $F_i(0) = H_i(0) = 0$ for each $i$. Then we define the process of interarrival times $\{T_n\}_{n=1}^{\infty}$, where $T_n$ is the time between the arrivals of the $(n - 1)$ th and $n$th particles, specifying $T_0 = 0$ and

$$(16) \qquad P(T_n < x/T_0, \ldots, T_{n-1}, J_0, \ldots, J_n) = F_{J_n}(x) \quad \text{a.s.}$$

for all $x \geqq 0$ and $n > 0$.

We see that the primary process $\tau_n = \sum_{i=0}^{n} T_i$, $n \geqq 0$, is a recurrent one, determined by the function

$$F(x) = \sum_{j=1}^{m} p_j F_j(x).$$

The lengths of the impulses $\{\chi_n\}_{n=0}^{\infty}$ of particles arriving at the moments $\tau_n$, $n \geqq 0$, are the determined as follows

$$(17) \qquad \begin{aligned} &P(\chi_0 < x/J_0, T_0) = H_{J_0}(x) \quad \text{a.s.} \quad \text{for all} \quad x \geqq 0, \\ &P(\chi_n < x/J_0, \ldots, J_n, \chi_0, \ldots, \chi_{n-1}, T_0, \ldots, T_{n-1}) = H_{J_n}(x) \quad \text{a.s.} \end{aligned}$$

for all $x \geqq 0$ and $n > 1$.

From (16) and (17) we have that the lengths of impulses are independent, positive random variables with the common distribution function

$$H(x) = \sum_{j=1}^{m} p_j H_j(x), \quad x \geqq 0,$$

and they do not depend on $\{\tau_n\}$. Then the above interesting characteristics of the resulting secondary process may be determined by the methods developed in Section 3.

## 7. EXAMPLES

Example 1. Let $F(x) = 1 - e^{-\lambda x}$, $x \geqq 0$, and let $H(x)$ be an arbitrary distribution function. Then

$$P(A_n) = \lambda^{n+1}/n! \int_0^{\infty} \left\{ \int_0^t H(x)\, dx \right\}^n e^{-\lambda t}\, dt, \quad n = 0, 1, \ldots,$$

$$f(z) = 1 - \left( \lambda \int_0^{\infty} \exp\left( -\lambda \int_0^t (1 - zH(x))\, dx \right) dt \right)^{-1}, \quad |z| \leqq 1,$$

$$\Phi(s, z) = 1 - \left( (\lambda + s) \int_0^{\infty} \exp\left( -st - \lambda \int_0^t (1 - z\, H(x))\, dx \right) dt \right)^{-1},$$

$$s \geqq 0, \quad |z| \leqq 1,$$

$$\gamma(s) = 1 - \left( (\lambda + s) \int_0^{\infty} \exp\left( -st - \lambda \int_0^t (1 - H(x))\, dx \right) dt \right)^{-1},$$

$$s > 0.$$

If $h = \int_0^\infty x\,\mathrm{d}H(x) < \infty$, then by [29] $\lim_n P(A_n) = \mathrm{e}^{-\lambda h}$. Hence

$$M(v) = \mathrm{e}^{\lambda h},$$
$$M(Z_1) = \mathrm{e}^{\lambda h}/\lambda .$$

The expression for $\gamma$ agrees with that in [24]. In an analogous way we may obtain the formula for $\gamma$ for the Type III counter since $\hat{F}(x) = 1 - \mathrm{e}^{-\lambda p x}$, $x \geqq 0$.

Example 2. Let $F(x) = 1$ if $x > a > 0$ for some $a$, elsewhere 0, and let $H$ be an arbitrary distribution function with $H(a) \neq 0$. Then

$$P(A_n) = \prod_{i=1}^{n} H(ia), \quad n = 0, 1, \ldots,$$

where the empty product is put equal to 1.

Hence $M(v) < \infty$ iff $\int_0^\infty x\mathrm{d}\,H(x) < \infty$.

Now let $n_0 + 1$ be the minimal integer $n$ such that $H(na) = 1$, then

$$M(v) = 1/\prod_{i=1}^{n_0} H(ia),$$

and for generating function of $v$ we have

$$f(z) = z(1 - \sum_{n=0}^{n_0-1} a_n z^n)(1 - z\sum_{n=0}^{n_0-1} a_n z^n)^{-1},$$

where

$$a_n = (1 - H((n+1)\,a))\prod_{i=1}^{n} H(ia).$$

Hence we obtain

$$\Phi(s, z) = f(z\,\mathrm{e}^{-as}), \quad s \geqq 0, \quad |z| \leqq 1,$$
$$\gamma(s) = f(\mathrm{e}^{-as}), \quad s \geqq 0.$$

Example 3. Let $F$ be an arbitrary distribution function, and let $H(x) = 1$ if $x > b > 0$ for some $b$, and zero elsewhere. Then

$$P(A_n) = I, \quad n = 1, 2, \ldots,$$
$$P_n = I(1 - I)^{n-1}, \quad n = 1, 2, \ldots,$$

where

$$I = 1 - F(b - 0)$$

(we assume that $F(b - 0) \neq 1$).

Moreover,

$$M(v) = 1/I,$$
$$f(z) = Iz(1 - (1 - I)\,z)^{-1}, \quad |z| \leqq 1,$$
$$\Phi(s, z) = z\int_b^\infty \mathrm{e}^{-sx}\,\mathrm{d}F(x)\Big/\Big(1 - z\int_0^b \mathrm{e}^{-sx}\,\mathrm{d}F(x)\Big), \quad s \geqq 0, \quad |z| \leqq 1,$$

$$\gamma(s) = \int_b^\infty e^{-sx} \, dF(x) \bigg/ \left(1 - \int_0^\infty e^{-sx} \, dF(x)\right), \quad s \geqq 0 .$$

The expression for $\gamma(s)$ is the same as that in [15, 28].

Let us put

$$\bar{F}(x) = P(\tau_1 < x/\tau_1 < b),$$

$$\bar{\bar{F}}(x) = P(\tau_1 < x/\tau_1 \geqq b),$$

and let $\bar{\mu}_r$ and $\bar{\bar{\mu}}_r$ be the $r$th moments with respect to the distribution functions $\bar{F}(x)$ and $\bar{\bar{F}}(x)$, respectively. Then for the dead time $B$ we obtain

$$M(B) = b + (1 + I)\,\bar{\mu}_1/I,$$

$$M(B^r) = b^r + (1 - I)\,I^{-1} \sum_{j=0}^{r-1} M(B^j)\,\bar{\mu}_{r-j}, \quad r = 2, 3, \ldots,$$

and for the moments of $Z_1$ we have

$$M(Z_1) = \mu/I,$$

$$M(Z_1^r) = \sum_{\substack{r_1+r_2+r_3=r \\ r_i \geqq 0}} M(B^{r_1})\,(-b)^{r_2}\,\bar{\bar{\mu}}_{r_3}, \quad r = 2, 3, \ldots,$$

where

$$\mu = \int_0^\infty x \, dH(x) < \infty .$$

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## Súhrn

## POZNÁMKA K PROBLÉMU ČÍTAČA TYPU II

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Vyšetruje sa explicitný tvar združenej Laplaceovej transformácie vzdialenosti medzi dvoma susednými momentami registrácie častíc čítačom typu II (čítač s predlžujúcim sa mŕtvym časom), vo všeobecnom prípade, a vytvárajúcej funkcie počtu

častíc. ktoré prišli na čítač počas mŕtveho času. Tým sa našli explicitné riešenia zložitých integrálnych rovníc, ktoré odvodili L. Takács a R. Pyke. Okrem toho sa študuje geometrické správanie rozdelenia poslednej zmienenej náhodnej veličiny a sú spravené niektoré poznámky o čítači typu III, ako aj registrácia $m$ typov častíc sa uvažuje.

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