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BIVARIATE GAMMA DISTRIBUTION AS A LIFE TEST MODEL

G. S. Lingappaiah

This paper is dedicated to my father on his eightieth birthday

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1. INTRODUCTION

This paper deals with a bivariate model in relation to life tests and in particular, to a series system with two dependent components. Univariate exponential and gamma distributions have been extensively used as life test models. For example, in Lingappaiah [6], [7] and Lawless [5], univariate exponential and gamma models are used to predict the future lives in a life test with available lives and the approach in these works is based on the classical sampling distributions of statistics. Similarly in Lingappaiah [8], [9], these two models are used for the same purpose of prediction, but here, the approach is Bayesian. A vast amount of literature is available regarding the applications of univariate and gama distributions in life tests as suitable models. Recently, bivariate exponential and gamma distributions have been getting more attention as suitable models in life tests. Works by Al-Saadi, Scrimshaw and Young [1] and also Al-Saadi and Young [2] deal with the bivariate exponential and its properties in much detail. Downton [3] derives a bivariate exponential distribution from a simple failure model and uses it in reliability context by considering that the shocks a component receives, are independently distributed and the number of shocks itself is a random variable. Mukherjee and Sasmal [10] use a bivariate exponential model for life distribution of coherent dependent systems and treat three different cases of the parallel system, standby system and series system. Moran [11] and Vere-Jones [12], on the other hand, give various properties of the bivariate gamma distribution. From the above references, it is clear that a bivariate distribution does indeed serve as a suitable model for life tests. Since the bivariate exponential may either turn out to be too simplistic or inadequate, in this paper, the bivariate gamma distribution, based on Gumbel's [4] model, is considered in the life test context. The object of this paper is threefold. Firstly, to obtain the distribution of a function of x and y, where the joint distribution of the couple of random variables x and y is the bivariate exponential. Minimum (x, y), denoted by the variable U is chosen for this purpose. Secondly, the reliability function is being evaluated and is also tabulated for various values of the parameter and those of u. Finally estimates of the parameters are also obtained using Bayesian method. The paper also includes a separate table for the values of the mean and the variance of U corresponding to various values of the parameter. Since, in life tests, both these quantities U and the reliability are of considerable importance, their analysis is undertaken here.

2. RELIABILITY

According to Gumbel [4] the joint distribution of (x, y) has the density

(1) $$f(x, y, \theta, \alpha) = g(x, \alpha)\, g(y, \alpha)\,[1 + \theta\{2G(x, \alpha) - 1\}\{2G(y, \alpha) - 1\}]$$

where $g(x, \alpha) = e^{-x}x^{\alpha-1}/\Gamma(\alpha)$

(1a) $$G(x, \alpha) = 1 - \sum_{k=0}^{\alpha-1} e^{-x}x^k/k!$$

$x, y, > 0$, α a positive integer and $-1 \leqq \theta \leqq 1$.

From (1) and (1a), we get

(2) $$\begin{aligned} f(x, y, \theta, \alpha) &= (1 + \theta)\, e^{-(x+y)}(xy)^{\alpha-1}/\Gamma^2(\alpha) + \\ &+ \Big[4\theta \sum_k \sum_t e^{-(2x+2y)} x^{k+\alpha-1} y^{t+\alpha-1}/k!\, t!\Big] \left[\frac{1}{\Gamma^2(\alpha)}\right] - \\ &- 2\theta \sum_k (e^{-2x}x^{\alpha+k-1}/k!)\,(e^{-y}y^{\alpha-1}/\Gamma(\alpha))\,(1/\Gamma(\alpha)) - \\ &- 2\theta \sum_t (e^{-2y}y^{t+\alpha-1}/t!)\,(e^{-x}x^{\alpha-1}/\Gamma(\alpha))\,(1/\Gamma(\alpha))\,. \end{aligned}$$

Throughout this paper, the upper limit of $\sum$ is $\alpha - 1$ unless otherwise specified and the lower limit is 0. Now min (x, y) has the distribution

(3) $$f(u, \theta, \alpha) = \int_u^\infty f(x, u, \theta, \alpha)\, dx + \int_u^\infty f(u, y, \theta, \alpha)\, dy$$

from (2) and (3), we get the first integral as

(4) $$\begin{aligned} \int_u^\infty f(x, u, \theta, \alpha)\, du &= (1 + \theta) \sum_k A(k, \alpha)\,[e^{-2u}u^{k+\alpha-1}/\Gamma(k + \alpha)] + \\ &+ 4\theta \sum_k \sum_t \sum_{r=0}^{k+\alpha-1} A(k, \alpha)\, A(t, \alpha)\,[e^{-4u}u^{t+r+\alpha-1}/r!\, 2^{k+\alpha-r}\Gamma(\alpha + t)] - \\ &- 2\theta \sum_k \sum_t A(t, \alpha + k)\, A(k, \alpha)\,[e^{-3u}u^{t+\alpha+k-1}/\Gamma(t + \alpha + k)] - \\ &- 2\theta \sum_k \sum_{r=0}^{k+\alpha-1} A(k, \alpha)\, A(r, \alpha)\,[e^{-3u}u^{r+\alpha-1}\, 2^r/2^{k+\alpha}\Gamma(r + \alpha)]\,. \end{aligned}$$

A similar expression is obtained for the second integral in (3). In (4)

(4a) $$A(k, \alpha) = \binom{k + \alpha - 1}{k}.$$

From (4) we have the reliability function in the form

(5) $$(1/2)\, R(u, \theta, \alpha) = (1 + \theta) \sum_{k} \sum_{r=0}^{k+\alpha-1} A(k, \alpha) \left[e^{-2u}(2u)^r / r!\, 2^{k+\alpha}\right] -$$

$$- 4\theta \sum_{k} \sum_{t} \sum_{r=0}^{k+\alpha-1} \sum_{s=0}^{t+r+\alpha-1} \left[A(k, \alpha)\, A(t, \alpha)\, A(r, t + \alpha)\right] \left[e^{-4u}(4u)^s / 2^r\, 8^\alpha\, 2^k\, 4^t s!\right] -$$

System Reliabilities

Table I: Values of $R(u)$

	$\alpha = 2$						$\alpha = 3$				
θ/u	1	2	3	4	5	θ/u	1	2	3	4	5
−1·0	·5035	·1067	·0142	·0015	·0001	−1·0	·8404	·4100	·1195	·0238	·0036
−·8	·5111	·1183	·0193	·0028	·0004	−·8	·8415	·4196	·1314	·0304	·0060
−·6	·5187	·1299	·0244	·0042	·0007	−·6	·8426	·4292	·1433	·0369	·0084
−·4	·5262	·1416	·0295	·0056	·0010	−·4	·8437	·4387	·1553	·0435	·0108
−·2	·5338	·1532	·0346	·0070	·0013	−·2	·8448	·4483	·1672	·0501	·0132
0·0	·5413	·1648	·0397	·0084	·0016	0·0	·8458	·4579	·1791	·0567	·0155
·2	·5489	·1765	·0447	·0098	·0019	·2	·8469	·4675	·1910	·0633	·0179
·4	·5565	·1881	·0498	·0112	·0022	·4	·8480	·4770	·2029	·0699	·0203
·6	·5640	·1997	·0549	·0125	·0025	·6	·8491	·4866	·2148	·0764	·0227
·8	·5716	·2114	·0600	·0139	·0028	·8	·8502	·4962	·2268	·0830	·0251
1·0	·5791	·2230	·0651	·0153	·0031	1·0	·8513	·5058	·2387	·0896	·0274

	$\alpha = 4$						$\alpha = 5$				
θ/u	1	2	3	4	5	θ/u	1	2	3	4	5
−1·0	·9620	·7197	·3668	·1276	·0323	−1·0	·9927	·8950	·6420	·3410	·1333
−·8	·9621	·7227	·3772	·1397	·0399	−·8	·9927	·8955	·6465	·3519	·1454
−·6	·9622	·7257	·3876	·1517	·0475	−·6	·9927	·8960	·6510	·3628	·1576
−·4	·9622	·7287	·3981	·1638	·0551	−·4	·9927	·8965	·6556	·3736	·1697
−·2	·9623	·7317	·4085	·1758	·0627	−·2	·9927	·8970	·6601	·3845	·1819
0·0	·9624	·7347	·4189	·1879	·0702	0·0	·9927	·8975	·6647	·3954	·1940
·2	·9625	·7377	·4293	·2000	·0778	·2	·9927	·8980	·6692	·4063	·2062
·4	·9625	·7407	·4398	·2120	·0854	·4	·9927	·8985	·6737	·4172	·2183
·6	·9626	·7437	·4502	·2241	·0930	·6	·9927	·8990	·6783	·4281	·2305
·8	·9627	·7467	·4606	·2361	·1006	·8	·9927	·8995	·6828	·4390	·2426
1·0	·9627	·7497	·4710	·2482	·1082	1·0	·9927	·9000	·6873	·4499	·2548

$$- 2\theta \sum_t \sum_k \sum_{r=0}^{t+k+\alpha-1} [A(t, \alpha + k)\, A(k, \alpha)] \, [e^{-3u}(3u)^r/r!\, 3^{t+k+\alpha}] -$$

$$- 2\theta \sum_k \sum_{r=0}^{k+\alpha-1} \sum_{s=0}^{r+\alpha-1} [A(k, \alpha)\, A(r, \alpha)] \, [e^{-3u}(3u)^s\, (2/3)^r/s!\, 2^k\, 6^\alpha] .$$

Table I gives the values of $R(u)$ for various values of θ and α, and for $u = 1, 2, 3, 4$ and 5 (though $u > 0$, integer values are chosen just for the table).

3. MEAN AND VARIANCE OF U

From (4), we get

$$(6) \qquad (1/2)\, E(u^s) = (1 + \theta) \sum_k [A(s, \alpha)\, A(k, s + \alpha)\, s!/2^{k+s+\alpha}] +$$

$$+ 4\theta \sum_k \sum_t \sum_{r=0}^{k+\alpha-1} [A(t, \alpha)\, A(k, \alpha)\, A(r, t + \alpha + s)\, A(s, t + \alpha)] \, [s!/8^\alpha\, 2^r\, 4^{t+s}\, 2^k] -$$

$$- 2\theta \sum_k \sum_t [A(t, \alpha + k + s)\, A(k, \alpha + s)\, A(s, \alpha)] \, [s!/3^{t+\alpha+k+s}] -$$

$$- 2\theta \sum_k \sum_{r=0}^{k+\alpha-1} [A(k, \alpha)\, A(s, \alpha)\, A(r, s + \alpha)\, (2/3)^r] \, [s!/3^s\, 6^\alpha\, 2^k] .$$

If $s = 0$ in (6), obviously we have $R(0)$ in (5) which is equal to 1. From (6) we can get the mean and the variance of U. They are tabulated for certain values of α and θ in Table II.

Table II: Expectation and Variances of u

	$\alpha = 2$		$\alpha = 3$		$\alpha = 4$		$\alpha = 6$		$\alpha = 8$		$\alpha = 10$	
θ	E(u)	Var u	E(u)	Var u	E(u)	Var u	E(u)	Var u	E(u)	Var u	E(u)	Var u
−1·0	1·1212	·4578	1·9002	·8284	2·7162	1·2296	4·4105	2·0849	6·1546	2·9829	7·9300	3·9078
−·8	1·1470	·5064	1·9327	·9036	2·7542	1·3315	4·4577	2·2398	6·2094	3·1905	7·9916	4·1682
−·6	1·1727	·5536	1·9651	·9768	2·7922	1·4304	4·5049	2·3901	6·2643	3·3921	8·0532	4·4209
−·4	1·1985	·5996	1·9976	1·0478	2·8302	1·5265	4·5521	2·5360	6·3192	3·5877	8·1148	4·6660
−·2	1·2242	·6442	2·0300	1·1168	2·8682	1·6197	4·5993	2·6775	6·3741	3·7772	8·1764	4·9035
0·0	1·2500	·6875	2·0625	1·1836	2·9063	1·7100	4·6465	2·8145	6·4290	3·9608	8·2380	5·1335
·2	1·2758	·7295	2·0950	1·2483	2·9443	1·7974	4·6937	2·9470	6·4838	4·1383	8·2996	5·3558
·4	1·3015	·7701	2·1274	1·3109	2·9823	1·8819	4·7409	3·0751	6·5387	4·3098	8·3612	5·5706
·6	1·3273	·8094	2·1599	1·3715	3·0203	1·9635	4·7881	3·1987	6·5936	4·4752	8·4229	5·7778
·8	1·3530	·8474	2·1923	1·4299	3·0583	2·0422	4·8353	3·3179	6·6485	4·6347	8·4845	5·9774
1·0	1·3788	·8841	2·2248	1·4862	3·0963	2·1181	4·8825	3·4326	6·7033	4·7881	8·5461	6·1694

4. ESTIMATE $\hat{\theta}$ IN $f(x, y; \theta, \alpha, \delta)$

Now we consider the distribution of x, y involving θ, α and δ where

$$(7) \qquad g(x, \alpha, \delta) = e^{-\delta x}(\delta x)^{\alpha-1}\,\delta/\Gamma(\alpha)\; x, \alpha, \delta > 0$$

and the corresponding distribution function

$$(7a) \qquad G(x; \alpha, \delta) = 1 - \sum_{k=0}^{\alpha-1} e^{-\delta x}(\delta x)^k/k!\,.$$

Estimation of θ (α, δ known):

In view of (1) and (7). the likelihood function with a sample of size n can be written as

$$(8) \qquad L(\mathbf{x}, \mathbf{y}; \alpha, \theta, \delta) = \prod_{i=1}^{n} f(x_i; \alpha, \theta, \delta)\,.$$

Using (1) again, (8) can be written as

$$(8a) \qquad L(\mathbf{x}, \mathbf{y}; \alpha, \theta, \delta) = (B)\prod_{i=1}^{n}(1 + \theta A_i)\,,$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ and similarly $\mathbf{y} = (y_1, y_2, \ldots, y_n)$

$$(B) = \prod_{i=1}^{n}[g(x_i, \alpha, \delta)\, g(y_i, \alpha, \delta)]\,,$$

$$A_i = [1 - 2\sum_{k=0}^{\alpha-1} e^{-\delta x_i}(\delta x_i)^k/k!]\,[1 - 2\sum_{t=0}^{\alpha-1} e^{-\delta y_i}(\delta y_i)^t/t!]\,.$$

Now (8) can be written in the form

$$(8b) \qquad L(x, y; \alpha, \theta, \delta) = (B)\sum_{i=0}^{n}\sum \theta^i \prod_{j=1}^{i} A_{r_j}$$

where $r_j = 1, 2, \ldots, n$ and $\sum$ is the sum over all combinations of $r_1, r_2, \ldots, r_i$. For example if $n = 3$ and $i = 2$ then $\sum$ is the sum $(A_1A_2 + A_1A_3 + A_2A_3)$. Now θ in (8) is between -1 and 1 as seen from (1). Due to this fact, we take the prior for θ e.g. as

$$(9) \qquad g(\theta) = 1/2\,, \quad -1 \leqq \theta \leqq 1\,.$$

One could take $g(\theta)$ as one chooses. However, from all the possible forms of $g(\theta)$, (9) seems to be the simplest in its nature. After integrating with respect to θ we get from (8b) and (9)

$$(10) \qquad L(\mathbf{x}, \mathbf{y}; \alpha, \delta) = \int L(\mathbf{x}, \mathbf{y}; \alpha, \theta, \delta)\, g(\theta)\, d\theta\,.$$

Now (10) becomes

$$L(\mathbf{x}, \mathbf{y}; \alpha, \delta) = (B) \sum_{i=0}^{n} \sum (\varepsilon_i / i + 1) \prod_{j=1}^{i} A_{r_j} \tag{10a}$$

with $\varepsilon_i = 0$ if i is odd

$= 1$ if i is even .

From (8b) and (10a) we get the estimate of θ

$$E(\theta) = \frac{\int \theta L(\mathbf{x}, \mathbf{y}; \alpha, \theta, \delta)\, g(\theta)\, \mathrm{d}\theta}{\int L(\mathbf{x}, \mathbf{y}; \alpha, \theta, \delta)\, g(\theta)\, \mathrm{d}\theta}, \tag{10b}$$

and from (10b) we have the estimate $\hat{\theta}$:

$$\hat{\theta} = \frac{\sum_{i=0}^{n} \sum [(1 - \varepsilon_i)/(i + 2)] \prod_{j=1}^{i} A_{r_j}}{\sum_{i=0}^{n} \sum (\varepsilon_i / i + 1) \prod_{j=1}^{i} A_{r_j}}. \tag{11}$$

For example, if $n = 1$, (11) gives

$$\hat{\theta} = 1/3\, A_1 \tag{11a}$$

and for $n = 2$, we have

$$\hat{\theta} = (1/3)(A_1 + A_2)/[1 + (1/3) A_1 A_2]. \tag{11b}$$

Similarly δ can be estimated as well, though it would require much more work.

5. COMMENTS

a) The bivariate gamma model obviously requires more work and computation as compared to the univariate case. However, the importance of the bivariate model outweighs the problems encountered.

b) The analysis done here can easily be applied to find the distribution of max (x, y) whose density can be written as

$$f(v, \theta, \alpha) = \int_0^v f(x, v, \theta, \alpha)\, \mathrm{d}x + \int_0^v f(v, y, \theta, \alpha)\, \mathrm{d}y, \tag{12}$$

and again the reliability, mean and variance of V can be tabulated. As the variable U, that is min (x, y) represents the length of life of a series system, similarly, variable V, max (x, y) is of equal importance since it represents the length of life of a 2-component parallel redundant system.

c) From the system reliabilities, cf. Table I, it can be seen that $R(u)$ increases as α increases for a given θ and u and so is the case for increasing θ and given α and u.

d) From Table II it can be easily seen that both the mean and the variance increase for the case of increasing α as well as that of increasing θ.

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Souhrn

DVOUROZMĚRNÉ GAMA ROZLOŽENÍ V MODELU DOBY ŽIVOTA

G. S. LINGAPPAIAH

Dvourozměrné gama rozložení je uvažováno v modelu dob bezporuchového provozu x, y dvou závislých prvků. V článku je odvozeno rozložení doby bezporuchového provozu systému vzniklého sériovým zapojením těchto prvků. Pro některé hodnoty parametrů dvourozměrného gama rozložení jsou tabelovány hodnoty funkce spolehlivosti, střední hodnoty a rozptylu doby do poruchy systému. Dále jsou uvedeny bayesovské odhady parametrů.

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