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THE EXISTENCE OF A PERIODIC SOLUTION OF A PARABOLIC EQUATION WITH THE BESSEL OPERATOR

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In this paper we prove the existence of a weak solution of a problem stated (in a little more special form) by R. S. Minasjan [1]. This problem is given by the equations

$$u_t(t, r) = a(t)\left[u_{rr}(t, r) + \frac{1}{r} u_r(t, r)\right] + f(t, r), (t, r) \in \mathbb{R} \times (0, 1), \tag{1.1}$$

$$u_r(t, 1) + h(t)\, u(t, 1) = 0\,, \quad t \in \mathbb{R}\,, \tag{1.2}$$

$$u_r(t, 0) = 0\,, \quad t \in \mathbb{R}\,, \tag{1.3}$$

where the functions $f(t, r)$, $h(t)$, $a(t)$ are assumed to be real, ω-periodic in t, $0 < h_0 \leqq \leqq h(t)$, $0 < a_0 \leqq a(t)$. (The other properties of these functions needed for our purpose will be specified later.) The physical interpretation of the problem is that of a periodic heat flow in an infinite cylinder with the assumption that the cylinder is subjected to a convective heat transfer (periodic in time) at the boundary surface ($r = 1$) with a medium at zero temperature. Inside the cylinder there are circular symmetric sources of heat that change periodically. Further, we admit that the thermal diffusivity is a periodic function.

In the paper [1], the author looks for a classical solution of this problem using the Fourier transform. This method leads to an infinite pseudoregular system of linear algebraic equations. However, the solvability of this system is not proved in detail in [1]. In this paper this problem is treated by a different method.

Denote by H the real Hilbert space $L_{2,\sqrt{r}}((0, 1))$ with the scalar product

$$(u, v) = \int_0^1 r\, u(r)\, v(r)\, \mathrm{d}r\,,$$

and denote by V the real Hilbert space $W^1_{2,\sqrt{r}}((0, 1))$ with the scalar product

$$(u, v)_V = \int_0^1 r[u(r)\, v(r) + u'(r)\, v'(r)]\, \mathrm{d}r$$

(derivatives in the sense of distributions).

The norms in H and V induced by the corresponding scalar products are denoted by $\| \|$ and $\| \|_V$, respectively.

V is continuously and densely embedded in H. Identifying H with H^* we have $V \hookrightarrow H \hookrightarrow V^*$.

Further, denote $X = L_2(S, V)$, $X^* = L_2(S, V^*)$, where $S = (0, \omega)$. For the dual pairing between X and X^* we use the notation $\langle g, v \rangle$, $g \in X^*$, $v \in X$.

The scalar product on the Hilbert space $L_2(S, H)$ is denoted by

$$(g, v)_S = \int_0^\omega (g(t), v(t))\, \mathrm{d}t\,, \quad g, v \in L_2(S, H)\,.$$

In particular, $\langle g, v \rangle = (g, v)_S$ for $g, v \in L_2(S, H)$.

Define the operator $A : X \to X^*$ by

$$(2) \qquad \langle Au, v \rangle = \int_0^\omega \int_0^1 a(t)\, r\, u_r(t, r)\, v_r(t, r)\, \mathrm{d}r\, \mathrm{d}t + \\ + \int_0^\omega a(t)\, h(t)\, u(t, 1)\, v(t, 1)\, \mathrm{d}t\,.$$

Theorem. *Let functions $f(t, r)$, $h(t)$, $a(t)$ be real, ω-periodic in t, and assume that $f \in L_2(S, H)$, $a' \in L_\infty(\mathbb{R})$, $h' \in L_\infty(\mathbb{R})$, and*

$$(3) \qquad h(t) \geqq h_0 > 0\,, \quad a(t) \geqq a_0 > 0\,.$$

Then there exists a function $u \in L_2(S, V) \cap L_\infty(S, H)$ with the derivative $u' \in L_2(S, H)$, satisfying the equations

$$(4.1) \qquad (u', v)_S + \langle Au, v \rangle = (f, v)_S\,, \quad v \in X\,,$$

$$(4.2) \qquad u(0) = u(\omega)\,.$$

Proof (the Faedo-Galerkin method). Denote by $\{w_j,\ j = 1, 2, \ldots\}$ the infinite orthonormal base in the separable Hilbert space $V = W^1_{2,\sqrt{r}}((0, 1))$.

We shall look for the functions $u^m(t) \in V$ in the form

$$u^m(t) = \sum_{j=1}^m d_j^m(t)\, w_j$$

and so that they satisfy the equations

$$(5) \qquad (u^{m\prime}(t), w_j) + a(t)\,(u_r^m(t), w_{jr}) + a(t)\, h(t)\, u^m(t, 1)\, w_j(1) = \\ = (f(t), w_j)\,, \quad j = 1, 2, \ldots, m\,,$$

$$(6) \qquad u^m(0) = u^m(\omega)\,.$$

If instead of this problem (5), (6), we consider an initial value problem given by (5) and

$$(6') \qquad u^m(0) = u_{0m}\,,$$

where u_{0m} is (for every m) a given function from the space V, then we obtain a system of m ordinary linear differential equations for the unknowns $d_j^m(t)$, $j = 1, \ldots, m$, with the initial conditions (6'). This system has a unique solution which is defined on $\langle 0, \omega \rangle$ and the operator $\mathscr{T}_m \colon u_{0m} \to u^m(\omega)$ is continuous.

We shall prove that there exists $R > 0$ such that $\mathscr{T}_m(B_m(0, R)) \subseteq B_m(0, R)$, where $B_m(0, R)$ denotes the closed ball in the space of linear combinations of the functions w_j, $j = 1, \ldots, m$, with the norm $\| \; \|$.

Multiplying the j-th equation of the system (5) by $d_j^m(t)$ and adding these equations for $j = 1, \ldots, m$, we have

$$(7) \qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^m(t)\|^2 + a(t)\|\,u_r^m(t)\|^2 + a(t)\,h(t)\,[u^m(t, 1)]^2 = (f(t), u^m(t))\,.$$

Since $V \subset\!\!\!\subset H$ (cf. [2], p. 354) it can be proved by (3) that there exists a constant independent of t such that the inequality

$$a(t)\int_0^1 r|w_r(r)|^2\,\mathrm{d}r + a(t)\,h(t)\,w^2(1) \geqq c\|w\|_V^2$$

holds for every $w \in V$ and every $t \in \mathbb{R}$.

Hence, from (7) we obtain

$$(8) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\|u^m(t)\|^2 + c\|u^m(t)\|_V^2 \leqq K\|f(t)\|^2\,,$$

where K is a suitable constant (depending on c).

Multiplying the last inequality by e^{ct} and integrating it, we find that if

$$\|u^m(0)\| \leqq R \quad \text{for} \quad R^2 \geqq \frac{K\displaystyle\int_0^\omega \mathrm{e}^{ct}\|f(t)\|^2\,\mathrm{d}t}{\mathrm{e}^{c\omega} - 1}\,, \text{ then } \|u^m(\omega)\| \leqq R.$$

It follows from the fixed point theorem that there exists (for every m) a function $u_{0m} \in B_m(0, R)$ such that the solution of the initial problem (5) and (6') is a periodic solution of the system (5).

This solution satisfies the inequality

$$(9) \qquad \|u^m(0)\| \leqq R \quad (R \text{ independent of } m)\,.$$

Further, integrating (7) we get

$$(10) \qquad \|u^m(t)\|^2 \leqq \|u^m(0)\|^2 + \int_0^t \|f(\sigma)\|^2\,\mathrm{d}\sigma + \int_0^t \|u^m(\sigma)\|^2\,\mathrm{d}\sigma\,.$$

Using Gronwall's lemma and the inequality (9), we obtain from (10) that

(11) $\|u^m(t)\| \leqq M$ (M a suitable constant independent of m, t) and, consequently, (because of $a(t) \geqq a_0 > 0$),

$$\int_0^t \|u_r^m(\sigma)\|^2 \, d\sigma \leqq M' . \tag{12}$$

Hence the set $\{u^m, m = 1, 2, \ldots\}$ is bounded in the space $L_\infty(S, H) \cap L_2(S, V)$.

Further, from (5) we obtain

$$\|u^{m\prime}(t)\|^2 + a(t) \frac{1}{2} \frac{d}{dt} \|u_r^m(t)\|^2 + a(t)\, h(t)\, u^m(t, 1)\, u^{m\prime}(t, 1) \leqq$$

$$\leqq \tfrac{1}{2}\|f(t)\|^2 + \tfrac{1}{2}\|u^{m\prime}(t)\|^2 .$$

Now, via integration by parts, the following inequalities can be derived:

$$\int_0^\omega \|u^{m\prime}(t)\|^2 \, dt \leqq \int_0^\omega \|f(t)\|^2 \, dt + \int_0^\omega a'(t) \, \|u_r^m(t)\|^2 \, dt + \int_0^\omega (a(t)\, h(t))' \, [u^m(t, 1)]^2 \, dt \leqq$$

$$\leqq K_1 + \operatorname*{vraisup}_{t \in (0,\omega)} \left(|a'(t)| \, \|u^m\|^2_{L_2(S,V)}\right) +$$

$$+ \operatorname*{vraisup}_{t \in (0,\omega)} \left(|a'(t)\, h(t) + a(t)\, h'(t)|\right) \int_0^\omega K_2 \|u^m(t)\|_V^2 \, dt \leqq K_1 + K_3 \|u^m\|^2_{L_2(S,V)} \leqq K$$

(K_1, K_2, K_3, K suitable constants independent of m).

Hence the elements of the sequence $\{u^{m\prime}, m = 1, 2, \ldots\}$ form a bounded set in the space $L_2(S, H)$. Consequently, the set $\{u^m, m = 1, 2, \ldots\}$ contains a convergent subsequence with the limit $u \in L_2(S, V) \cap L_\infty(S, H)$ such that the subsequence is convergent weakly in $L_2(S, V)$ and $*$-weakly in $L_\infty(S, H)$, and such that the derivatives of this subsequence converge weakly in $L_2(S, H)$ to an element $v \in L_2(S, H)$, v being necessarily equal to the derivative of u.

Particularly, $u^j(0)$ and $u^j(\omega)$ converge (for $j \to \infty$) weakly in H to $u(0)$ and $u(\omega)$, respectively. (Here and in what follows we understand under u^j the convergent subsequence.)

From (6) we get that the identity

$$u(0) = u(\omega) \tag{13}$$

holds.

Denote by $\{g_i, i = 1, 2, \ldots\}$ the orthonormal base in the real Hilbert space $L_2(S)$. The set $\{g_i w_j, i = 1, 2, \ldots, j = 1, 2, \ldots\}$ forms an orthonormal base in $L_2(S, V)$.

From (5) we have

$$(u^{m\prime}, g_i w_j)_S + \langle A u^m, g_i w_j \rangle = (f, g_i w_j)_S . \tag{14}$$

For i, j fixed, the elements $(u^{m\prime}, g_i w_j)_S$ converge for $m \to \infty$ to $(u', g_i w_j)_S$ and the elements $\langle Au^m, g_i w_j \rangle = \langle Ag_i w_j, u^m \rangle$ converge to $\langle Au, g_i w_j \rangle$, $m \to \infty$.

This implies that the equation

$$(15) \qquad (u', g_i w_j)_S + \langle Au, g_i w_j \rangle = (f, g_i w_j)_S$$

holds for every $i, j \in \mathbb{N}$ and this immediately yields that the equation (4.1) is fulfilled. Together with (13) this completes the proof.

References

[1] *R. S. Minasjan:* On one problem of the periodic heat flow in the infinite cylinder. Dokl. Akad. Nauk Arm. SSR 48 (1969).

[2] *H. Triebel:* Höhere Analysis. VEB Berlin 1972.

Souhrn

EXISTENCE PERIODICKÉHO ŘEŠENÍ PARABOLICKÉ ROVNICE S BESSELOVÝM OPERÁTOREM

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V tomto článku je dokázána existence slabého ω-periodického řešení parabolické rovnice (1.1) s okrajovými podmínkami (1.2) a (1.3) za předpokladu, že reálné funkce $f(t, r)$, $a(t)$, $h(t)$ jsou ω-periodické v proměnné t, $f \in L_2(S, H)$, a, h takové, že $a' \in L_\infty(\mathbb{R})$, $h' \in L_\infty(\mathbb{R})$ a splňují (3).

Získané ω-periodické řešení u leží v prostotu $L_2(S, V) \cap L_\infty(S, H)$, má derivaci $u' \in L_2(S, H)$, splňuje rovnice (4.1) a (4.2). Důkaz je proveden užitím Faedo-Galerkinovy metody.

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