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SMALL TIME-PERIODIC SOLUTIONS OF EQUATIONS OF MAGNETOHYDRODYNAMICS AS A SINGULARLY PERTURBED PROBLEM

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1. INTRODUCTION

In dealing with the motion of viscous electrically conducting incompressible fluid the following system of equations for the velocity $v = (v_1, v_2, v_3)$ and the magnetic field $B = (B_1, B_2, B_3)$ if often considered as relevant [1], [7]:

$$\varrho(v_t + (v, \nabla) v) - \eta \Delta v = -\nabla p + \varrho F + \frac{1}{\mu} \operatorname{rot} B \times B, \tag{1.1}$$

$$\operatorname{div} v = 0, \tag{1.2}$$

$$\sigma\mu B_t + \operatorname{rot}\operatorname{rot} B = \sigma\mu \operatorname{rot}(v \times B), \tag{1.3}$$

$$\operatorname{div} B = 0. \tag{1.4}$$

Here ϱ, η, μ and σ are constants. When the fluid occupies a region $\Omega \subset R^3$ with perfectly conducting boundary the following boundary conditions are added to the above system of equations:

$$v = 0 \quad \text{on} \quad \partial\Omega, \tag{1.5}$$

$$B_n = 0 \quad \text{on} \quad \partial\Omega, \tag{1.6}$$

$$\operatorname{rot}_\tau B = 0 \quad \text{on} \quad \partial\Omega. \tag{1.7}$$

We shall suppose that Ω is a bounded region with a C^2 boundary. Here and in what follows, the subscripts n and τ denote the normal and tangential components of a vector, i.e., if n denotes the unit outward normal to $\partial\Omega$ and $(\cdot,\cdot)$ the scalar product in R^3, then $B_n = (B, n)$ and $\operatorname{rot}_\tau B = \operatorname{rot} B - (\operatorname{rot} B)_n n$.

The global existence of weak solutions and local existence of regular solutions to the initial-value problem (1.1)–(1.7) have been proved in [2] and [3].

Looking for a more complete system of governing equations we are led to the following system [1], [7]:

$$\varrho(v_t + (v, \nabla) v) - \eta \Delta v = -\nabla p + \varrho F + qE + j \times B, \tag{1.8}$$

$$\operatorname{div} v = 0, \tag{1.9}$$

$$v = 0 \quad \text{on} \quad \partial\Omega, \tag{1.10}$$

$$B_t + \operatorname{rot} E = 0, \tag{1.11}$$

$$\operatorname{div} B = 0, \tag{1.12}$$

$$B_n = 0 \quad \text{on} \quad \partial\Omega, \tag{1.13}$$

$$\varepsilon E_t + j - \frac{1}{\mu} \operatorname{rot} B = 0, \tag{1.14}$$

$$\varepsilon \operatorname{div} E = q, \tag{1.15}$$

$$E_\tau = 0 \quad \text{on} \quad \partial\Omega, \tag{1.16}$$

to which Ohm's law, an equation relating j to the other quantities, ought to be added. This law can take up a form as complicated as the following one:

$$j = \sigma\{E + v \times B + j \times B/\beta_4 + \alpha(j \times B) \times B\} + qv.$$

In our investigation we shall keep only the first two terms on the right-hand side, to obtain Ohm's law in its simplest form, namely,

$$j = \sigma(E + v \times B). \tag{1.17}$$

We reduce the system (1.8)–(1.17) to one for v and B to be able to compare it with (1.1)–(1.7).

We begin by defining an operator φ_ε, $\varepsilon \geqq 0$, assigning to a function $h(t, x)$ the solution $w(t, x)$ of the equation

$$\frac{\varepsilon}{\sigma} w_t + w = h.$$

As we shall deal exclusively with functions periodic in t with a period ω, i.e. both h and w are supposed to be ω-periodic in t, the function $w = \varphi_\varepsilon(h)$ is uniquely defined. For $h = (h_1, h_2, h_3)$ we set $\Phi_\varepsilon(h) = (\varphi_\varepsilon(h_1), \varphi_\varepsilon(h_2), \varphi_\varepsilon(h_3))$. With the help of the operators φ_ε and Φ_ε the system (1.8)–(1.17) can be reduced to

$$\begin{aligned} \varrho v_t - \eta \Delta v = {} & -\nabla p + \varrho F - \varrho(v, \nabla) v + \frac{1}{\mu} \operatorname{rot} B \times B - \\ & - \varepsilon[\Phi_\varepsilon(\operatorname{rot} B)]_t \times B/\sigma\mu + \varepsilon[\Phi_\varepsilon(v \times B)]_t \times B - \\ & - \varepsilon\, \varphi_\varepsilon(\operatorname{div}(v \times B))\, \Phi_\varepsilon\left(\frac{1}{\sigma\mu} \operatorname{rot} B - v \times B\right), \end{aligned} \tag{1.18}$$

$$\operatorname{div} v = 0, \tag{1.19}$$

(1.20) $$v = 0 \quad \text{on} \quad \partial\Omega\,,$$

(1.21) $$\varepsilon\mu B_{tt} + \sigma\mu B_t + \operatorname{rot}\operatorname{rot} B = \sigma\mu \operatorname{rot}(v \times B)\,,$$

(1.22) $$\operatorname{div} B = 0\,,$$

(1.23) $$B_n = 0 \quad \text{on} \quad \partial\Omega\,,$$

(1.24) $$\operatorname{rot}_\tau B = 0 \quad \text{on} \quad \partial\Omega\,.$$

In the case of functions ω-periodic in t, it is easy to see that if v, p and B satisfy (1.18)–(1.24), then v, p, B, $E = \Phi_\varepsilon(\operatorname{rot} B/\sigma\mu - v \times B)$, $j = \sigma(E + v \times B)$ and $q = \varepsilon \operatorname{div} E$ satisfy (1.8)–(1.17).

If we put $\varepsilon = 0$ in the system (1.18)–(1.24), we get (1.1)–(1.7). The question arises whether for $\varepsilon \searrow 0$ the time-periodic solutions of (1.18)–(1.24), say $(v^\varepsilon, \nabla p^\varepsilon, B^\varepsilon)$, tend to $(v^0, \nabla p^0, B^0)$, a solution of (1.1)–(1.7). The answer is affirmative at least if we deal with a small forcing term F and therefore with small solutions. The result formulated in the spaces defined in the next section is given in Theorem 1.1 below. We recall that all the functions involved depend on t in the ω-periodic manner.

Theorem 1.1. *Given $\varepsilon_0 > 0$, there exist positive numbers r_0 and $\hat{r}$ such that the following three assertions hold:*

(1) *If $F \in G^3$, $\|F\|_{G^3} \leqq \hat{r}$, then for every ε, $0 < \varepsilon \leqq \varepsilon_0$, there is a unique solution $(v^\varepsilon, \nabla p^\varepsilon, B^\varepsilon) \in X^3 \times G^3 \times Y^2$ of (1.18)–(1.24) satisfying $\|v^\varepsilon\|_{X^3} \leqq r_0$ and $\|B^\varepsilon\|_{Y^2} \leqq$ $\leqq r_0$.*

(2) *If $F \in G^3$, $\|F\|_{G^3} \leqq \hat{r}$, then there is a unique solution $(v^0, \nabla p^0, B^0) \in X^3 \times$ $\times\, G^3 \times X^3$ of (1.1)–(1.7) satisfying $\|v^0\|_{X^3} \leqq r_0$ and $\|B^0\|_{X^3} \leqq r_0$.*

(3) *Finally, we have $\|v - v^\varepsilon\|_{X^2} + \|\nabla(p^\varepsilon - p^0)\|_{G^2} + \|B^\varepsilon - B^0\|_{X^2} = O(\varepsilon)$.*

Proof will be given in Section 4.

Various questions arising in the study of the system consisting of (1.18) taken for $\varepsilon = 0$ and (1.19)–(1.24) have been investigated by L. Stupjalis [8], [9] and [10]. In these papers no attention has been paid to either the existence of time-periodic solutions or to the behaviour of solutions for $\varepsilon \searrow 0$. It is the approach of [9] which has been modified for the purpose of this paper. Some aspects of the singular perturbation problem for Maxwell's equations have been investigated in [5] and [6].

In the next section, Section 2, the spaces will be defined and basic auxiliary results concerning the linearized equations will be formulated. In Section 3, we establish some lemmas needed when treating nonlinear terms in the equations. In Section 4, the proof of Theorem 1.1 will be given.

2. SPACES AND AUXILIARY RESULTS FOR THE LINEAR PART OF THE PROBLEM

We shall make no difference in notation between spaces of functions and vectors. The same symbols will be used for both of them. Essentially, we shall keep the notations from [3] and [4]. It is well-known that [4]

$$L^2(\Omega) = \dot{J}(\Omega) \oplus G(\Omega),$$

where $\dot{J}(\Omega)$ is the closure in $L^2(\Omega)$ of all solenoidal vectors from $\mathscr{D}(\Omega)$ and $G(\Omega)$ is the space of all vectors $u = \nabla\varphi$, $\varphi \in H^1(\Omega)$. By P we denote the orthogonal projector on $\dot{J}(\Omega)$.

We shall frequently use the following two basic spaces:

$$\dot{J}^2(\Omega) = \{u \in H^2(\Omega);\ \operatorname{div} u = 0,\ u = 0 \text{ on } \partial\Omega\}$$

$$\mathscr{J}^2(\Omega) = \{u \in H^2(\Omega);\ \operatorname{div} u = 0,\ u_n = 0 \text{ and } \operatorname{rot}_\tau u = 0 \text{ on } \partial\Omega\}.$$

By [4], for $u \in \dot{J}^2(\Omega)$ we have

$$\alpha^{-1}\|u\|_{H^2(\Omega)} \leqq \|P\,\Delta u\|_{L^2(\Omega)} \leqq \alpha\|u\|_{H^2(\Omega)},$$

and by [3], for $u \in \mathscr{J}^2(\Omega)$ we have

$$\alpha^{-1}\|u\|_{H^2(\Omega)} \leqq \|\operatorname{rot}\operatorname{rot} u\|_{L^2(\Omega)} \leqq \alpha\|u\|_{H^2(\Omega)}, \tag{2.1}$$

$$\alpha^{-1}\|u\|_{H^1(\Omega)} \leqq \|\operatorname{rot} u\|_{L^2(\Omega)} \leqq \alpha\|u\|_{H^1(\Omega)} \tag{2.2}$$

with a constant α independent of u.

By [3] and [4] the following result holds:

Lemma 2.1. *The operators* $-P\,\Delta$ *mapping* $\dot{J}^2(\Omega)$ *onto* $\dot{J}(\Omega)$ *and* rot rot *mapping* $\mathscr{J}^2(\Omega)$ *onto* $\dot{J}(\Omega)$ *are positive definite, selfadjoint operators with compact inverses.*

We now introduce the spaces of functions depending on t. In what follows functions will be supposed to be ω-periodic in t without any particular reference. We set

$$Q = [0, \omega] \times \Omega.$$

By $\dot{J}(Q)$, $\dot{J}^2(Q)$ and $\mathscr{J}^2(Q)$ we shall denote the spaces of functions $u \in L^2(Q)$ which, respectively, satisfy $u(t, \cdot) \in \dot{J}(\Omega)$, $\dot{J}^2(\Omega)$ and $\mathscr{J}^2(\Omega)$ for almost every t.

Further, we set

$$|||u||| = \max\{\|D_t^j D_x^\alpha u\|_{L^2(Q)};\ 2j + |\alpha| \leqq 2\}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, and α_i, j are nonnegative integers. Finally, we denote

$$H^{1,2}(Q) = \{u;\ |||u||| < +\infty\}$$

and

$$X^p = \{u;\ u,\ D_t^p u \in H^{1,2}(Q)\},$$

$$Y^p = \{u;\, u,\, D_t^p u \in H^2(Q)\}\,,$$
$$Z^p = \{u;\, u,\, D_t^p u \in H^1(Q)\}\,,$$
$$G^p = \{u;\, u,\, D_t^p u \in L^2(Q)\}$$

with norms given by

$$\|u\|_{X^p} = \max\{\|u\|_{H^{1,2}(Q)},\ \|D_t^p u\|_{H^{1,2}(Q)}\}\,,$$

etc. We now give some lemmas about the linearized equations.

Lemma 2.2. *For every $f \in G^p \cap \dot{J}(Q)$ there is a unique $v \in \dot{J}^2(Q) \cap X^p$ satisfying $\varrho v_t - \eta P\, \Delta v = f$, $\operatorname{div} v = 0$ and $v(t, \cdot) = 0$ on $\partial\Omega$. Moreover, $\|v\|_{X^p} \leqq c\|f\|_{G^p}$.*

Lemma 2.3. *Let σ, μ, ε_0 and $g \in G^{p+1} \cap \dot{J}(Q)$ be given. For every ε, $0 < \varepsilon \leqq \varepsilon_0$, there is a unique $B^\varepsilon \in Y^p \cap \mathscr{J}^2(Q)$ such that $\varepsilon\mu B^\varepsilon_{tt} + \sigma\mu B^\varepsilon_t + \operatorname{rot}\operatorname{rot} B^\varepsilon = g$, $\operatorname{div} B^\varepsilon = 0$, $B^\varepsilon_n(t, \cdot) = 0$ and $\operatorname{rot}_\tau B^\varepsilon(t, \cdot) = 0$ on $\partial\Omega$. Moreover, $\|B^\varepsilon\|_{Y^p} \leqq c\|g\|_{G^{p+1}}$, where c does not depend on ε and g.*

Lemma 2.4. *For every $g \in G^p \cap \dot{J}(Q)$ there is a unique $B \in X^p \cap \mathscr{J}^2(Q)$ such that $\sigma\mu B_t + \operatorname{rot}\operatorname{rot} B = g$, $\operatorname{div} B = 0$, $B_n(t, \cdot) = 0$ and $\operatorname{rot}_\tau B(t, \cdot) = 0$ on $\partial\Omega$. Moreover, $\|B\|_{X^p} \leqq c\|g\|_{G^p}$.*

Lemma 2.5. *Let $\varepsilon > 0$ and $h \in Z_p$. Then $\varphi_\varepsilon(h)$, the ω-periodic solution of $\varepsilon\sigma^{-1} w_t + w = h$, satisfies*

$$\varepsilon\|\varphi_\varepsilon(h)\|_{Z^{p+1}} + \|\varphi_\varepsilon(h)\|_{Z^p} \leqq c\|h\|_{Z^p}$$

with c independent of ε.

Proofs of these lemmas are all alike. We give a brief account of the proof of Lemma 2.3. By Lemma 2.1, there is a sequence of vectors $\psi_k \in \mathscr{J}^2(\Omega) \cap \dot{J}(\Omega)$ satisfying $\operatorname{rot}\operatorname{rot}\psi_k = \lambda_k\psi_k$, $\lambda_k > 0$, $k = 1, 2, \ldots$ such that $\{\psi_k\}_{k=1}^\infty$ forms an orthonormal base in $\dot{J}(\Omega)$. Let

$$M_m = \operatorname{lin}\left\{\frac{1}{\sqrt{\omega}}\, e^{i2\pi jt/\omega}\,\psi_k;\ |j| \leqq m,\ 1 \leqq k \leqq m\right\}.$$

For $g \in \dot{J}(\Omega)$ we set

$$g_{jk} = \frac{1}{\sqrt{\omega}}\int_0^\omega\int_\Omega g(t, x)\, e^{-i2\pi jt/\omega}\,\psi_k(x)\,dx\,dt\,,$$
$$B_{jk} = \left(-\varepsilon\mu\left(\frac{2\pi j}{\omega}\right)^2 + \sigma\mu\, i\,\frac{2\pi j}{\omega} + \lambda_k\right)^{-1} g_{jk}$$

and

$$B^m = \sum_{\substack{|j|\leqq m\\ k\leqq m}} B_{jk}\, e^{i2\pi jt/\omega}\psi_k\,.$$

Obviously B^m is a real-valued function from M_m which, for any $w \in M_m$, satisfies

$$(2.3) \qquad (\varepsilon\mu B^m_{tt} + \sigma\mu B^m_t + \operatorname{rot}\operatorname{rot} B^m, w)_{L^2(Q)} = (g, w)_{L^2(Q)} .$$

For brevity we denote $\|\cdot\|_{L^2(Q)}$ simply by $\|\cdot\|$. Taking $w = \operatorname{rot}\operatorname{rot} B^m_t$ in (2.3) we have, in virtue of ω-periodicity in t,

$$(2.4) \qquad \sigma\mu\|\operatorname{rot} B^m_t\|^2 = -(\operatorname{rot}\operatorname{rot} B^m, g_t)_{L^2(Q)} \leqq \|\operatorname{rot}\operatorname{rot} B^m\| \, \|g_t\| .$$

For $w = \operatorname{rot}\operatorname{rot} B^m$ we get

$$\|\operatorname{rot}\operatorname{rot} B^m\|^2 \leqq \|g\| \, \|\operatorname{rot}\operatorname{rot} B^m\| + \varepsilon\mu\|\operatorname{rot} B^m_t\|^2 ,$$

which by (2.4) implies

$$(2.5) \qquad \|\operatorname{rot}\operatorname{rot} B^m\| \leqq \|g\| + \frac{\varepsilon}{\sigma}\|g_t\| .$$

This applied to (2.4) gives

$$(2.6) \qquad \|\operatorname{rot} B^m_t\| \leqq c(\|g\| + \|g_t\|) .$$

Taking $w = -D^3_t B^m$ in (2.3), we get $\sigma\mu\|B^m_{tt}\|^2 = (g_t, B^m_{tt}) \leqq \|g_t\| \, \|B^m_{tt}\|$, i.e.,

$$(2.7) \qquad \|B^m_{tt}\| \leqq \frac{1}{\sigma\mu}\|g_t\| .$$

In virtue of (2.1) and (2.2), we get from (2.5), (2.6) and (2.7)

$$\|B^m\|_{H^2(Q)} \leqq c(\|g\| + \|g_t\|) .$$

Similarly we obtain

$$\|D^p_t B^m\|_{H^2(Q)} \leqq c(\|D^p_t g\| + \|D^{p+1}_t g\|) .$$

Letting $m \to \infty$, we complete the proof of Lemma 2.3.

3. AUXILIARY RESULTS FOR NONLINEARITIES

For the purpose of this section we denote

$$\|u\|_{H^{0,s}(Q)} = \Big(\sum_{|\alpha| \leqq s} \|D^\alpha_x u\|^2_{L^2(Q)}\Big)^{1/2} .$$

We shall frequently use the Sobolev inequality

$$\|u\|_{C(\Omega)} \leqq c_s\|u\|_{H^2(\Omega)}$$

and the well-known inequalities

$$\|u\|_{L^6(\Omega)} \leqq c\|u\|_{H^1(\Omega)}$$

and

$$\operatorname{supess}\{\|u(t,\cdot)\|_{H^s(\Omega)};\ t \in R\} \leqq c\{\|u\|_{H^{0,s}(Q)} + \|u_t\|_{H^{0,s}(Q)}\} .$$

The following series of lemmas make it possible to show in a nearly obvious manner that for $p \geqq 1$ the mappings given by the right-hand sides of the equations (1.18) and (1.21) map $v \in X^{p+1} \cap \dot{J}^2(Q)$ and $B \in Y^p \cap \mathscr{J}^2(Q)$ into $G^p \cap \dot{J}(Q)$ and satisfy the assumptions of the next section. The first three lemmas are obvious.

Lemma 3.1. $X^{p+1} \subset Y^p$.

Lemma 3.2. *Let* $|\alpha| \leqq 1$. *Then* $D_x^\alpha : Y^p \to Z^p$ *is a linear and continuous mapping.*

Lemma 3.3. $Z^p \subset G^{p+1}$.

Lemma 3.4. *Let* $p \geqq 1$. *For any* $a_1 \in Z^p$ *and* $a_2 \in Y^p$, *we have* $a_1 a_2 \in G^{p+1}$ *and* $\|a_1 a_2\|_{G^{p+1}} \leqq c\|a_1\|_{Z^p} \|a_2\|_{Y^p}$.

Proof. For $j_1 + j_2 \leqq p + 1$ we must estimate the quantity

$$V = \|(D_t^{j_1} a_1)(D_t^{j_2} a_2)\|^2_{L^2(Q)} = \int_0^\omega \int_\Omega (D_t^{j_1} a_1)^2 (D_t^{j_2} a_2)^2 \, dx \, dt \leqq$$

$$\leqq c \int_0^\omega \|D_t^{j_1} a_1(t, \cdot)\|^2_{H^1(\Omega)} \|D_t^{j_2} a_2(t, \cdot)\|^2_{H^1(\Omega)} \, dt .$$

For $j_1 = 0$ we get

$$V \leqq c\{\|D_t^1 a_1\|^2_{H^{0,1}(Q)} + \|a_1\|^2_{H^{0,1}(Q)}\} \|D_t^{j_2} a_2\|^2_{H^{0,1}(Q)} \leqq c\|a_1\|^2_{Z^p}\|a_2\|^2_{Y^p}$$

and similarly for $j_1 \leqq p, j_2 \leqq p$ we have

$$V \leqq c\|D_t^{j_1} a_1\|^2_{H^{0,1}(Q)} \{\|D_t^{j_2+1} a_2\|^2_{H^{0,1}(Q)} + \|D_t^{j_2} a_2\|^2_{H^{0,1}(Q)}\} \leqq c\|a_1\|^2_{Z^p} \|a_2\|^2_{Y^p} .$$

In the last case when $j_1 = p + 1$ and $j_2 = 0$ we have

$$V = \int_0^\omega \int_\Omega (D_t^{p+1} a_1)^2 a_2^2 \, dx \, dt \leqq c \int_0^\omega \|D_t^{p+1} a_1(t, \cdot)\|^2_{L^2(\Omega)} \|a_2(t, \cdot)\|^2_{H^2(\Omega)} \, dt \leqq$$

$$\leqq c\|D_t^{p+1} a_1\|^2_{L^2(Q)} \{\|D_t^1 a_2\|^2_{H^{0,2}(Q)} + \|a_2\|^2_{H^{0,2}(Q)}\} \leqq c\|a_1\|^2_{Z^p} \|a_2\|^2_{Y^p} .$$

This completes the proof.

Lemma 3.5. *Let* $p \geqq 1$. *For any* $a_1, a_2 \in Y^p$, *we have* $a_1 a_2 \in Y^p$ *and* $\|a_1 a_2\|_{Y^p} \leqq$ $\leqq c\|a_1\|_{Y^p} \|a_2\|_{Y^p}$.

Proof. For $|\alpha_1| + |\alpha_2| \leqq 2$, $j_1 + j_2 + |\alpha_1| + |\alpha_2| \leqq 2 + p$ we must estimate

$$V = \|(D_t^{j_1} D_x^{\alpha_1} a_1)(D_t^{j_2} D_x^{\alpha_2} a_2)\|^2_{L^2(Q)} .$$

We shall distinguish several cases.

(1) Let $|\alpha_1| + |\alpha_2| = 2$. Firstly, we shall suppose $|\alpha_1| = 2$ and $|\alpha_2| = 0$. Then $j_1 + j_2 \leqq p$ and we have

$$V = \int_0^\omega \int_\Omega (D_t^{j_1} D_x^{\alpha_1} a_1)^2 (D_t^{j_2} a_2)^2 \, dx \, dt \leqq$$

$$\leqq \int_0^\omega \| D_t^{j_1} D_x^{\alpha_1} a_1(t, \cdot) \|_{L^2(\Omega)}^2 \| D_t^{j_2} a_2(t, \cdot) \|_{C(\Omega)}^2 \, dt \leqq$$

$$\leqq c_s^2 \int_0^\omega \| D_t^{j_1} a_1(t, \cdot) \|_{H^2(\Omega)}^2 \| D_t^{j_2} a_2(t, \cdot) \|_{H^2(\Omega)}^2 \, dt \, .$$

For at least one j_i we have $j_i \leqq p - 1$. As the last expression is symmetric in j_1 and j_2 we can suppose $j_1 \leqq p - 1$. Then

$$V \leqq c\{ \| D_t^{j_1 + 1} a_1 \|_{H^{0,2}(Q)}^2 + \| D_t^{j_1} a_1 \|_{H^{0,2}(Q)}^2 \} \| D_t^{j_2} a_2 \|_{H^{0,2}(Q)}^2 \leqq$$
$$\leqq c \| a_1 \|_{Y^p}^2 \| a_2 \|_{Y^p}^2 \, .$$

Secondly, we shall suppose $|\alpha_1| = |\alpha_2| = 1$. Then

$$V = \int_0^\omega \int_\Omega (D_t^{j_1} D_x^{\alpha_1} a_1)^2 (D_t^{j_2} D_x^{\alpha_2} a_2)^2 \, dx \, dt \leqq$$

$$\leqq \int_0^\omega \| D_t^{j_1} D_x^{\alpha_1} a_1(t, \cdot) \|_{L^4(\Omega)}^2 \| D_t^{j_2} D_x^{\alpha_2} a_2(t, \cdot) \|_{L^4(\Omega)}^2 \, dt \leqq$$

$$\leqq c \int_0^\omega \| D_t^{j_1} a_1(t, \cdot) \|_{H^2(\Omega)}^2 \| D_t^{j_2} a_2(t, \cdot) \|_{H^2(\Omega)}^2 \, dt \, ,$$

which gives $V \leqq c \| a_1 \|_{Y^p}^2 \| a_2 \|_{Y^p}^2$ as in the preceding case.

(2) Let $|\alpha_1| + |\alpha_2| = 1$. Then $j_1 + j_2 \leqq 1 + p$. With no loss of generality we can assume $|\alpha_1| = 0$, $|\alpha_2| = 1$. Then we have

$$V = \int_0^\omega \int_\Omega (D_t^{j_1} a_1)^2 (D_t^{j_2} D_x^{\alpha_2} a_2)^2 \, dx \, dt \leqq$$

$$\leqq \int_0^\omega \| D_t^{j_1} a_1(t, \cdot) \|_{L^4(\Omega)}^2 \| D_t^{j_2} D_x^{\alpha_2} a_2(t, \cdot) \|_{L^4(\Omega)}^2 \, dt \leqq$$

$$\leqq c \int_0^\omega \| D_t^{j_1} a_1(t, \cdot) \|_{H^1(\Omega)}^2 \| D_t^{j_2} a_2(t, \cdot) \|_{H^2(\Omega)}^2 \, dt \, .$$

If $j_1 \leqq p$ and $j_2 \leqq p$, we have

$$V \leqq c\{ \| D_t^{j_1 + 1} a_1 \|_{H^{0,1}(Q)}^2 + \| D_t^{j_1} a_1 \|_{H^{0,1}(Q)}^2 \} \| a_2 \|_{Y^p}^2 \leqq c \| a_1 \|_{Y^p}^2 \| a_2 \|_{Y^p}^2 \, .$$

If $j_1 = p + 1$, i.e. $j_2 = 0$, we have

$$V \leqq c \| a_1 \|_{Y^p}^2 \{ \| a_2 \|_{H^{0,1}(Q)}^2 + \| D_t^1 a_2 \|_{H^{0,2}(Q)}^2 \} \leqq c \| a_1 \|_{Y^p}^2 \| a_2 \|_{Y^p}^2 \, .$$

Finally, for $j_1 = 0$ and $j_2 = p + 1$, we get

$$V \leqq \int_0^\omega \|a_1(t, \cdot)\|^2_{C(\Omega)} \|D_t^{p+1} a_2(t, \cdot)\|^2_{H^1(\Omega)} \, dt \leqq$$

$$\leqq c\{\|D_t^1 a_1\|^2_{H^{0,2}(Q)} + \|a_1\|^2_{H^{0,2}(Q)}\} \|a_2\|^2_{Y^p} \leqq c\|a_1\|^2_{Y^p} \|a_2\|^2_{Y^p} .$$

(3) In this case we have $|\alpha_1| + |\alpha_2| = 0$, hence, $j_1 + j_2 \leqq p + 2$. Firstly, we shall assume that $j_1 j_2 \neq 0$. Then at least one of j_1, j_2 is smaller or equal to p. Let us suppose that $j_1 \leqq p$ and $j_2 \leqq p + 1$. Then

$$V = \int_0^\omega \int_\Omega (D_t^{j_1} a_1)^2 (D_t^{j_2} a_2)^2 \, dx \, dt \leqq$$

$$\leqq c \int_0^\omega \|D_t^{j_1} a_1(t, \cdot)\|^2_{L^4(\Omega)} \|D_t^{j_2} a_2(t, \cdot)\|^2_{L^4(\Omega)} dt \leqq$$

$$\leqq c \int_0^\omega \|D_t^{j_1} a_1(t, \cdot)\|^2_{H^1(\Omega)} \|D_t^{j_2} a_2(t, \cdot)\|^2_{H^1(\Omega)} \, dt \leqq$$

$$\leqq c\{\|D_t^{j_1+1} a_1\|^2_{H^{0,1}(Q)} + \|D_t^{j_1} a_1\|^2_{H^{0,1}(Q)}\} \|D_t^{j_2} a_2\|^2_{H^{0,1}(Q)} \leqq c\|a_1\|^2_{Y^p} \|a_2\|^2_{Y^p} .$$

To complete the proof we investigate the case when j_1 or j_2 is equal to 0. Let us suppose that $j_1 = 0$. Then $j_2 \leqq p + 2$ and we have

$$V = \int_0^\omega \int_\Omega a_1^2 (D_t^{j_2} a_2)^2 \, dx \, dt \leqq \int_0^\omega \|a_1(t, \cdot)\|^2_{C(\Omega)} \|D_t^{j_2} a_2(t, \cdot)\|^2_{L^2(\Omega)} \, dt \leqq$$

$$\leqq c\{\|D_t^1 a_1\|^2_{H^{0,2}(Q)} + \|a_1\|^2_{H^{0,2}(Q)}\} \|D_t^{j_2} a_2\|^2_{L^2(Q)} \leqq c\|a_1\|^2_{Y^p} \|a_2\|^2_{Y^p} .$$

This completes the proof.

Lemma 3.6. *For any $a_1, a_2 \in Z^p$ we have*

$$\varepsilon \, \varphi_\varepsilon(a_1) \, \varphi_\varepsilon(a_2) \in G^{p+1}$$

and

$$\|\varepsilon \, \varphi_\varepsilon(a_1) \, \varphi_\varepsilon(a_2)\|_{G^{p+1}} \leqq c\|a_1\|_{Z^p} \|a_2\|_{Z^p}$$

with c independent of ε.

Proof. We set $b_i = \varphi_\varepsilon(a_i)$, $i = 1, 2$. By Lemma 2.5 we have $\|\varepsilon b_i\|_{Z^{p+1}} + \|b_i\|_{Z^p} \leqq \leqq c\|a_i\|_{Z^p}$. We must estimate, for $j_1 + j_2 \leqq p + 1$,

$$V = \|\varepsilon(D_t^{j_1} b_1)(D_t^{j_2} b_2)\|^2 = \varepsilon^2 \int_0^\omega \int_\Omega (D_t^{j_1} b_1)^2 (D_t^{j_2} b_2)^2 \, dx \, dt \leqq$$

$$\leqq c\varepsilon^2 \int_0^\omega \|D_t^{j_1} b_1(t, \cdot)\|^2_{H^1(\Omega)} \|D_t^{j_2} b_2(t, \cdot)\|^2_{H^1(\Omega)} \, dt \leqq$$

$$\leqq c\{\|\varepsilon D_t^{j_1+1} b_1\|^2_{H^{0,1}(Q)} + \|\varepsilon D_t^{j_1} b_1\|^2_{H^{0,1}(Q)}\} \|D_t^{j_2} b_2\|^2_{H^{0,1}(Q)} \leqq$$
$$\leqq c\|\varepsilon b_1\|^2_{Z^{p+1}} \|b_2\|^2_{Z^p} \leqq c\|a_1\|^2_{Z^p} \|a_2\|^2_{Z^p},$$

since with no loss of generality we can assume $j_1 \leqq p$. This completes the proof.

Lemma 3.7. *Let $p \geqq 1$. For any $a_1, a_2 \in Z^p$ we have $a_1 a_2 \in G^p$ and $\|a_1 a_2\|_{G^p} \leqq \leqq c\|a_1\|_{Z^p} \|a_2\|_{Z^p}$.*

Proof. For $j_1 + j_2 \leqq p$ we must estimate $V = \|(D_t^{j_1} a_1)(D_t^{j_2} a_2)\|^2_{L^2(Q)}$. At least one of j_1, j_2 is less or equal to $p - 1$. We can suppose that $j_1 \leqq p - 1$. Then we have

$$V \leqq c \int_0^\omega \int_\Omega \|D_t^{j_1} a_1(t, \cdot)\|^2_{H^1(\Omega)} \|D_t^{j_2} a_2(t, \cdot)\|^2_{H^1(\Omega)}\, dt \leqq$$
$$\leqq c\{\|D_t^{j_1+1} a_1\|^2_{H^{0,1}(Q)} + \|D_t^{j_1} a_1\|^2_{H^{0,1}(Q)}\} \|D_t^{j_2} a_2\|^2_{H^{0,1}(Q)} \leqq c\|a_1\|^2_{Z^p} \|a_2\|^2_{Z^p}.$$

This completes the proof.

4. PROOF OF THEOREM 1.1.

We denote by K_1 the inverse operator to $\varrho D_t - \eta P \Delta$ described in Lemma 2.2, by K_2^ε the inverse operator to $\varepsilon\mu D_t^2 + \sigma\mu D_t + \operatorname{rot}\operatorname{rot}$ described in Lemma 2.3 and by K_3 the inverse operator to $\sigma\mu D_t + \operatorname{rot}\operatorname{rot}$ described in Lemma 2.4. Writing v^ε and B^ε instead of v and B in (1.18)–(1.24) and applying P to (1.18) we get with the help of K_1 and K_2^ε the following two equations for v^ε and B^ε:

$$v^\varepsilon = K_1 P\{\varrho F + \Psi_1(v^\varepsilon, B^\varepsilon) + \varepsilon\, \Psi_3(v^\varepsilon, B^\varepsilon, \varepsilon)\} \tag{4.1}$$

$$B^\varepsilon = \sigma\mu K_2^\varepsilon\, \Psi_2(v^\varepsilon, B^\varepsilon), \tag{4.2}$$

where

$$\Psi_1(v, B) = -\varrho(v, \nabla)\, v + \frac{1}{\mu} \operatorname{rot} B \times B,$$

$$\Psi_2(v, B) = \sigma\mu \operatorname{rot}(v \times B),$$

$$\Psi_3(v, B, \varepsilon) = [\Phi_\varepsilon(v \times B)]_t \times B - \frac{1}{\sigma\mu}[\Phi_\varepsilon(\operatorname{rot} B)]_t \times B -$$
$$- \varphi_\varepsilon(\operatorname{div}(v \times B))\, \Phi_\varepsilon\left(\frac{1}{\sigma\mu} \operatorname{rot} B - v \times B\right).$$

Similarly, from (1.1)–(1.7) we get

$$v^0 = K_1 P\{\varrho F + \Psi_1(v^0, B^0)\}, \tag{4.3}$$

$$B^0 = \sigma\mu K_3\, \Psi_2(v^0, B^0). \tag{4.4}$$

For a Banach space X we shall denote

$$\mathscr{B}(0, r, X) = \{u \in X;\ \|u\| \leqq r\}.$$

By using the lemmas of the preceding section it is easy to see that for any $\bar{r}$ positive there is b such that for every $v, \bar{v} \in \mathscr{B}(0, r, X^3)$, $B, \bar{B} \in \mathscr{B}(0, r, Y^2)$, $r \leqq \bar{r}$, $0 < \varepsilon \leqq \leqq \varepsilon_0$, and $i = 1, 2$ we have

$$(4.5) \qquad \|\Psi_i(v, B)\|_{G^3} \leqq br^2,$$

$$\|\Psi_i(v, B) - \Psi_i(\bar{v}, \bar{B})\|_{G^3} \leqq br(\|v - \bar{v}\|_{X^3} + \|B - \bar{B}\|_{Y^2}),$$

$$(4.6) \qquad \|\Psi_i(v, B) - \Psi_i(\bar{v}, \bar{B})\|_{G^2} \leqq br(\|v - \bar{v}\|_{X^2} + \|B - \bar{B}\|_{X^2}),$$

$$(4.7) \qquad \|\varepsilon\, \Psi_3(v, B, \varepsilon)\|_{G^3} \leqq br^2,$$

$$(4.8) \qquad \|\Psi_3(v, B, \varepsilon)\|_{G^2} \leqq br^2,$$

$$(4.9) \qquad \|\varepsilon\, \Psi_3(v, B, \varepsilon) - \varepsilon\, \Psi_3(\bar{v}, \bar{B}, \varepsilon)\|_{G^3} \leqq br(\|v - \bar{v}\|_{X^3} + \|B - \bar{B}\|_{Y^2}).$$

To get (4.5) we must, for example, estimate the term $vD_x^\alpha\bar{v}$, $|\alpha| \leqq 1$, in G^3 for $v, \bar{v} \in X^3$. By Lemma 3.1, $v \in Y^2$, by Lemma 3.2, $D_x^\alpha\bar{v} \in Z^2$. Applying Lemma 3.4, we have $vD_x^\alpha\bar{v} \in G^3$ and the corresponding estimate. The other terms in Ψ_i, $i = 1, 2$, can be treated along the same lines with the help of Lemmas 3.5 and 3.3. Similarly for (4.6). To show (4.7) and (4.9) the following terms must be estimated in G^3:

$$(4.10) \qquad \varepsilon[\varphi_\varepsilon(a)]_t\, b, \quad a \in Z^2; \quad b \in Y^2,$$

$$(4.11) \qquad \varepsilon\, \varphi_\varepsilon(a)\, \varphi_\varepsilon(b), \quad a, b \in Z^2.$$

By Lemma 2.5, $\|\varepsilon[\varphi_\varepsilon(a)]_t\|_{Z^2} \leqq c\|a\|_{Z^2}$. Using Lemmas 3.4 and 3.6, we can estimate (4.10) and (4.11), respectively. To prove (4.8) we must estimate in G^2 the terms

$$(4.12) \qquad [\varphi_\varepsilon(a)]_t\, b, \quad a \in Z^2, \quad b \in Y^2,$$

$$(4.13) \qquad \varphi_\varepsilon(a)\, \varphi_\varepsilon(b), \quad a, b \in Z^2.$$

By Lemma 2.5, $\|[\varphi_\varepsilon(a)]_t\|_{Z^1} \leqq c\|a\|_{Z^2}$. Hence using Lemma 3.4, we deal with (4.12) and with the help of Lemma 3.7 the term (4.13) is estimated.

For $(x, y) \in X \times Y$, X, Y Banach spaces, we set

$$\|(x, y)\|_{X \times Y} = \|x\|_X + \|y\|_Y.$$

By (4.5)–(4.9) we find two positive numbers $\hat{r}$ and r_0 such that for $\|F\|_{G^3} \leqq \hat{r}$ the right hand sides of (4.1) and (4.2) form a contractive mapping of $\mathscr{B}(0, r_0, X^3 \cap \cap \dot{J}^2(Q) \times Y^2 \cap \mathscr{J}^2(Q))$ into itself. Similarly, the right hand sides of (4.3) and (4.4) form a contractive mapping of $\mathscr{B}(0, r_0, X^3 \cap \dot{J}^2(Q) \times X^3 \cap \mathscr{J}^2(Q))$ into itself as well as a contractive mapping of $\mathscr{B}(0, r_0, X^2 \cap \dot{J}^2(Q) \times X^2 \cap \mathscr{J}^2(Q))$ into itself with the contractivity constant α.

This shows that for every ε, $0 < \varepsilon \leqq \varepsilon_0$ there is a unique $(v^\varepsilon, B^\varepsilon) \in \mathscr{B}(0, r_0, X^3 \cap \cap \, \hat{J}^2(Q) \times Y^2 \cap \mathscr{J}^2(Q))$ satisfying (4.1) and (4.2). Furthermore, there is a unique $(v^0, B^0) \in \mathscr{B}(0, r_0, X^3 \cap \hat{J}^2(Q) \times X^3 \cap \mathscr{J}^2(Q))$ satisfying (4.3) and (4.4). Hence the existence part of Theorem 1.1 is proved as ∇p^ε is uniquely defined when v^ε, ∇p^ε and B^ε are to satisfy (1.18). Similarly for ∇p^0.

Denoting $w^\varepsilon = v^\varepsilon - v^0$ and $b^\varepsilon = B^\varepsilon - B^0$, find $\varrho(w^\varepsilon_t - \eta P \, \Delta w^\varepsilon) = P\{\Psi_1(v^\varepsilon, B^\varepsilon) - - \Psi_1(v^0, B^0) + \Psi_3(v^\varepsilon, B^\varepsilon, \varepsilon)\}$, $\sigma\mu b^\varepsilon_t + \text{rot rot } b^\varepsilon = \sigma\mu(\Psi_2(v^\varepsilon, B^\varepsilon) - \Psi_2(v^0, B^0) - - \varepsilon\mu B^\varepsilon_{tt})$. If these two equations are written in the form

$$w^\varepsilon = K_1 P\{\Psi_1(v^\varepsilon, B^\varepsilon) - \Psi_1(v^0, B^0) + \Psi_3(v^\varepsilon, B^\varepsilon, \varepsilon)\}\,,$$
$$b^\varepsilon = K_3\{\sigma\mu(\Psi_2(v^\varepsilon, B^\varepsilon) - \Psi_2(v^0, B^0)) - \varepsilon\mu B^\varepsilon_{tt}\}\,,$$

we immediately obtain

$$\|(w^\varepsilon, b^\varepsilon)\|_{X^2 \times X^2} \leqq \alpha\|(w^\varepsilon, b^\varepsilon)\|_{X^2 \times X^2} + \varepsilon\beta(v^\varepsilon, B^\varepsilon, \varepsilon)\,,$$

where

$$\beta(v^\varepsilon, B^\varepsilon, \varepsilon) = \|K_1 P \, \Psi_3(v^\varepsilon, B^\varepsilon, \varepsilon)\|_{X^2} + \mu\|K_3 B^\varepsilon_{tt}\|_{X^2} \leqq$$
$$\leqq c(\|\Psi_3(v^\varepsilon, B^\varepsilon, \varepsilon)\|_{G^2} + \|B^\varepsilon\|_{Y^2})\,.$$

As $\|B^\varepsilon\|_{Y^2} \leqq r_0$ and, by (4.8), $\|\Psi_3(v^\varepsilon, B^\varepsilon, \varepsilon)\|_{G^2}$ is bounded, we have the estimates for $\|v^\varepsilon - v^0\|_{X^2}$ and $\|B^\varepsilon - B^0\|_{X^2}$. The estimate of $\|\nabla(p^\varepsilon - p^0)\|_{G^2}$ is a simple consequence. This completes the proof.

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Souhrn

MALÁ ČASOVĚ PERIODICKÁ ŘEŠENÍ ROVNIC MAGNETOHYDRODYNAMIKY JAKO SINGULÁRNĚ PORUŠENÝ PROBLÉM

MILAN ŠTĚDRÝ, OTTO VEJVODA

V článku je vyšetřován systém rovnic popisujících pohyb viskósní, nestlačitelné a vodivé tekutiny v omezené třírozměrné oblasti, jejíž hranice je ideálně vodivá. Posuvný proud v Maxwellových rovnicích, εE_t, není zanedbán. Je dokázáno, že pro malé periodické síly a malé kladné ε existuje lokálně jediné periodické řešení vyšetřovaného problému. Je ukázáno, že pro $\varepsilon \searrow 0$ toto řešení konverguje k řešení zjednodušeného (a obvykle uvažovaného) systému rovnic magnetohydrodynamiky.

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