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# ON OPTIMAL REPLACEMENT POLICY

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The present paper deals with a system with a single activated unit. We do not assume (as is usually done) that the unit is completely effective until it fails. We suppose that the unit can be in $k + 1$ states denoted by $0, 1, \ldots, k$ ($k \geqq 2$ and finite) at any time. The state $i$, $i \in \{0; 1;, \ldots, ;k\}$ can be interpreted as a level of the wear of the unit. The states 0 and $k$ correspond respectively to the full operative ability of the unit, and to the failure of the unit. Let us put $K = \{0; 1; \ldots; k\}$.

Let us suppose that inspections of the system are carried out at discrete time instants $t = 0, 1, 2, \ldots$, and that we have the possibility of replacing the unit used before $t$ by a new one, i.e. by a unit which is in state 0, at $t$, for every $t = 0, 1, 2, \ldots$. Concerning the changes of states of the unit we assume:

A 1. The probability that the unit used in the system during $(t; t + 1]$, $t = 0, 1, 2, \ldots$ is in state $j$ at $t + 1$ under the condition that it is in state $i$ at $t$ depends only on $i$ and $j$, i.e. this probability depends neither on $t$ nor on the changes of states of the units used in the system before $t$ nor on the particular unit used in the system during $(t; t + 1]$. Let us denote this probability by $p_{ij}$.

A 2. We have

$$p_{ij} = 0 \quad \text{for all} \quad i \in K - \{k\}\,, \quad j \in K - \{i; i + 1; k\}\,,$$
$$p_{ii} \neq 1 \quad \text{for all} \quad i \in K - \{k\}\,.$$

If the unit fails during the interval $(t; t + 1]$ between two successive inspections of the system, we must replace it at $t + 1$. On the other hand, if it does not fail during $(t; t + 1]$ then one of two possible actions (replace or do not replace) can be taken. We shall be interested in such replacement strategies according to which the decision at time $t$, $t = 0, 1, 2, \ldots$ depends only on the state of the unit used during $(t - 1; t]$, at $t$ (independently of $t$). Every such replacement strategy is determined by a set $A \subseteq K$ such that $k \in A$, and has the form: The decision is "replace" at time $t$ if and only if the state at $t$ of the unit used in the system during $(t - 1; t]$ is an element of $A$. The assumption A 2, however, implies that we can limit ourselves only to the replacement strategies $\mathscr{S}_n$, $n \in K$, determined by the sets

$$A_n = \{i;\, i \in K,\, i \geqq n\}\,. \tag{1}$$

Let $R$ $(R > 0)$ be the costs for replacement of the unit and let $m_{ij}$, for $i \in K - \{k\}$, $j \in \{i;\, i + 1;\, k\}$, be the income of the system reached during the interval (say $(t;\, t + 1]$) between two successive inspections of the system under the condition that the states of the unit used during this interval are $i$ at $t$ and $j$ at $t + 1$.

The aim of the present paper is to calculate the average income per unit time $C_n$ of the system with the replacement strategy $\mathscr{S}_n$ for all $n \in K$ and to characterize the value of $k^*$ fulfilling

$$k^* = \min\,\{n;\, n \in K,\, C_n \geqq C_i \text{ for all } i \in K\} \tag{2}$$

under some reasonable conditions on $p_{ij}$ and $m_{ij}$, $i \in K - \{k\}$, $j \in \{i;\, i + 1;\, k\}$. Let us note that Derman showed in [2] that the strategy which maximizes the average income of the system per unit time is stationary and deterministic. Hence, we may limit our considerations to the strategies $\mathscr{S}_n$, $n \in K$, only.

Let us further suppose that the inspection of the system at time $t$, for every $t = 0, 1, 2, \ldots$, involves also a preventive maintenance of the unit which will be used during $(t;\, t + 1]$. If the costs for this preventive maintenance $m_i$ depend only on the state $i$ of the unit at $t$, then this much complicated model can be converted into the one described above, i.e. into the model without preventive maintenance, by substitutions $m_{ij} - m_i$ for $m_{ij}$, for all $i \in K - \{k\}$, $j \in \{i;\, i + 1;\, k\}$.

A model very close to that just described is considered by Kolesar in [4]. In Kolesar's model, the matrix of state-transition probabilities is almost fully general and the replacement strategies prescribe replacements of units with the delay equal to a unit of time, i.e. if at an inspection, say at time $t$, a unit is in such a state that its replacement is either necessary or recommended by the applied strategy then this replacement is carried out at time $t + 1$. Corollary 1 of [4] and Theorem 2 of the present paper have similar assertions – the ditonic property of the sequence of average costs (incomes) per unit time of the system with control limit rules. Corollary 1 of [4] is, however, false as we can find out in [7] where a counter-example is given. The paper [8] shows, moreover, that the delay of replacements, i.e. the main difference of the two models in question, is much more important than one might expect.

## 1. AVERAGE INCOME OF THE SYSTEM PER UNIT TIME

Let the replacement strategy $\mathscr{S}_n$, $n \in K$, be accepted and let the unit used during $(0;\, 1]$ be in state $i$, $i \in B_n = (K - A_n) \cup \{0\}$, at time $t = 0$. Let us denote by $D_n(i)$ and $R_n(i)$, respectively, the expected time to the first replacement of a unit and the expected income of the system up to the first replacement of a unit with the costs for this replacement included. Using the renewal theory, it can be easily verified that the average income of the system with the replacement strategy $\mathscr{S}_n$ per unit time $C_n$ can be expressed as

$$(3) \qquad C_n = \frac{R_n(0)}{D_n(0)} \quad \text{for all} \quad n \in K\,.$$

The values of $D_n(i)$ and $R_n(i)$ satisfy the relations

$$(4) \qquad D_n(h) = 1 + p_{hh}\, D_n(h) + p_{h,h+1}\, D_n(h+1) \quad \text{for} \quad n \in K - \{0\}\,, \quad h \in B_n - \{n-1\}\,,$$

$$(5) \qquad D_n(n-1) = 1 + p_{n-1,n-1}\, D_n(n-1) \quad \text{for} \quad n \in K - \{0\}\,,$$

$$(6) \qquad D_0(0) = 1\,,$$

$$(7) \qquad R_n(h) = p_{hh}[m_{hh} + R_n(h)] + p_{h,h+1}[m_{h,h+1} + R_n(h+1)] + p_{hk}[m_{hk} - R] \quad \text{for} \quad n \in K - \{0\}\,, \quad h \in B_n - \{n-1\}\,,$$

$$(8) \qquad R_n(n-1) = p_{n-1,n-1}[m_{n-1,n-1} + R_n(n-1)] + p_{n-1,n}[m_{n-1,n} - R] + p_{n-1,k}[m_{n-1,k} - R] \quad \text{for} \quad n \in K - \{0; k\}\,,$$

$$(9) \qquad R_k(k-1) = p_{k-1,k-1}[m_{k-1,k-1} + R_k(k-1)] + p_{k-1,k}[m_{k-1,k} - R]\,,$$

$$(10) \qquad R_0(0) = p_{00}m_{00} + p_{01}m_{01} + p_{0k}m_{0k} - R\,.$$

Solving these difference equations we obtain the following theorem.

**Theorem 1.** *The values of $C_n$ for $n \in K$ are*

$$(11) \qquad C_0 = m(0) - R\,,$$

$$(12) \qquad C_n = \frac{-R + \sum_{j=0}^{n-1} m(j)\, P_j}{\sum_{j=0}^{n-1} P_j} \quad \textit{for} \quad n \in K - \{0\}\,,$$

*where*

$$(13) \qquad m(i) = p_{ii}m_{ii} + p_{i,i+1}m_{i,i+1} + p_{ik}m_{ik} \quad \text{for} \quad i \in K - \{k-1; k\}\,,$$

$$(14) \qquad m(k-1) = p_{k-1,k-1}m_{k-1,k-1} + p_{k-1,k}m_{k-1,k}\,,$$

$$(15) \qquad P_j = \frac{1}{1 - p_{jj}} \cdot \prod_{i=0}^{j-1} \frac{p_{i,i+1}}{1 - p_{ii}} \quad \text{for} \quad j \in K - \{k\}\,.$$

Proof. The values of $D_0(0)$ and $R_0(0)$ are given in (6) and (10), respectively. Further, the unique solutions of the systems of difference equations (4), (5) and (7), (8), (9) have, respectively, the forms

$$D_n(h) = \sum_{j=h}^{n-1} P_j^h \quad \text{for} \quad n \in K - \{0\}, \quad h \in B_n, \tag{16}$$

$$R_n(h) = -R + \sum_{j=h}^{n-1} m(j) P_j^h \quad \text{for} \quad n \in K - \{0\}, \quad h \in B_n, \tag{17}$$

where

$$P_j^h = \frac{1}{1 - p_{jj}} \cdot \prod_{i=h}^{j-1} \frac{p_{i,i+1}}{1 - p_{ii}} \quad \text{for} \quad h, j \in K - \{k\}, \quad h \leqq j.$$

Substituting (16) and (17) for $h = 0$ into (3) we obtain (12).

In the next section we shall need the following relation, based on (16) and (17), of the average incomes of the system per unit time corresponding to different replacement strategies $\mathscr{S}_n$.

**Lemma 1.** *Let $n, n' \in K - \{0\}$ be such that $n' > n$. Then*

$$C_{n'} = \frac{C_n D_n(0) + \sum_{j=n}^{n'-1} m(j) P_j}{D_n(0) + \sum_{j=n}^{n'-1} P_j}. \tag{18}$$

## 2. OPTIMAL REPLACEMENT POLICY

In this section we introduce an algorithm for finding the value of $k^*$ without calculating $C_n$ for all $n \in K$. The following theorem characterizes the structure of the sequence $\{C_n\}_{n=0}^k$.

**Theorem 2.** *We have*

$$C_1 > C_0 \quad \text{if} \quad p_{00} > 0, \tag{19}$$

$$C_1 = C_0 \quad \text{if} \quad p_{00} = 0. \tag{20}$$

*Let the sequence $\{m(n)\}_{n=0}^{k-1}$ be decreasing and let*

$$p_{n,n+1} \neq 0 \quad \textit{for every} \quad n \in K - \{k\}. \tag{21}$$

*Let us put*

$$z = \max\{k^*; 1\}. \tag{22}$$

*Then*

$$\{C_n\}_{n=1}^{k^*} \quad \textit{is increasing}, \tag{23}$$

(24) $$\{C_n\}_{n=z+1}^{k} \quad \textit{is decreasing},$$

*and the following implications are true provided* $z \neq k$:

a) *if* $C_z \neq m(z)$ then $C_z > C_{z+1}$;

b) if $C_z = m(z)$ then $C_z = C_{z+1}$.

Proof. According to Theorem 1 we have

$$C_0 = m(0) - R$$

and

(25) $$C_1 = m(0) - R(1 - p_{00}),$$

so that (19) and (20) are obviously true. Concerning the relation (23) it is sufficient to prove it only for $k^* \geqq 2$. We show that

(26) $$\text{if} \quad k^* \geqq 2 \quad \text{then} \quad m(k^* - 1) > C_{k^*}.$$

Indeed, if $k^* \geqq 2$ and $m(k^* - 1) \leqq C_{k^*}$ then we obtain from Lemma 1 and from the definition of $k^*$ the following impossible relation:

(27) $$C_{k^*} \leqq \frac{C_{k^*-1}\, D_{k^*-1}(0) + C_{k^*}P_{k^*-1}}{D_{k^*-1}(0) + P_{k^*-1}} < C_{k^*}.$$

The sequence $\{m(n)\}_{n=0}^{k-1}$ is decreasing so that

(28) $$m(n) > C_{k^*} \quad \text{for every} \quad n \in K - A_{k^*}$$

and according to Lemma 1, the definition of $k^*$ and (21) (which secures that $P_n > 0$ for every $n \in K - \{0\}$),

(29) $$C_{n+1} > \frac{C_n\, D_n(0) + C_{k^*}P_n}{D_n(0) + P_n} > \frac{C_n\, D_n(0) + C_nP_n}{D_n(0) + P_n} = C_n$$

for every $n \in \{1; \ldots; k^* - 1\}$ so that the sequence $\{C_n\}_{n=1}^{k^*}$ is increasing.

For the proof of (24) we need to verify

(30) $$\text{if} \quad k^* < k \quad \text{then} \quad m(z) \leqq C_z.$$

The proof of (30) will be divided into two parts:

1) If $k^* = 0$, i.e. $z = 1$, and $m(1) > C_1$ then we obtain from (19), (20), (21), Lemma 1 and from the definition of $k^*$ the following impossible relation:

$$C_0 \geqq C_2 > \frac{C_1\, D_1(0) + C_1P_1}{D_1(0) + P_1} = C_1 \geqq C_0.$$

2) If $k^* > 0$, i.e. $z = k^*$, and $m(z) > C_z$ then we similarly have

$$C_{k^*+1} = C_{z+1} > \frac{C_z\, D_z(0) + C_z P_z}{D_z(0) + P_z} = C_z = C_{k^*}\,.$$

According to (30) and Lemma 1 and by virtue of the fact that the sequence $\{m(n)\}_{n=0}^{k-1}$ is decreasing the following relation holds for every $n \in A_{z+1} - \{k\}$:

$$(31) \qquad C_n = \frac{C_z\, D_z(0) + \sum_{j=z}^{n-1} m(j)\, P_j}{D_z(0) + \sum_{j=z}^{n-1} P_j} > \frac{m(n)\, D_z(0) + \sum_{j=z}^{n-1} m(n)\, P_j}{D_z(0) + \sum_{j=z}^{n-1} P_j} = m(n)\,.$$

Thus for every $n \in A_{z+1} - \{k\}$ the inequality

$$C_{n+1} = \frac{C_n\, D_n(0) + m(n)\, P_n}{D_n(0) + P_n} < C_n$$

is fulfilled and the sequence $\{C_n\}_{n=z+1}^{k}$ is decreasing. The two last statements of Theorem 2 are easy consequences of Lemma 1 and of (30) because we know that $m(z) \neq C_z$ is equivalent to $m(z) < C_z$.

Theorem 2 can be applied in the following way: If we want to find k*, i.e. the least subscript of the elements of $\{C_i\}_{i=0}^{k}$ which maximize the values of $C_n$ for $n \in K$, we need not calculate $C_n$ for all $n \in K$. The complexity of the expressions for $C_n$ given in Theorem 1 increases with increasing $n$. Therefore is seems to be convenient to calculate the values of $C_n$ in the natural order: $C_0, C_1, \ldots, C_k$. Theorem 2 guarantees that for finding $k^*$ it is sufficient to start with $C_1$ and after calculating $C_n$ $(n \geqq 2)$ to compare $C_n$ with $C_{n-1}$ and to proceed to $C_{n+1}$ in the case $C_n > C_{n-1}$. On the other hand, if $C_n \leqq C_{n-1}$ then Theorem 2 states:

$$\begin{aligned} &\text{if} \quad n \geqq 3 \quad \text{then} \quad k^* = n - 1\,, \\ &\text{if} \quad n = 2 \quad \text{and} \quad p_{00} \neq 0 \quad \text{then} \quad k^* = 1\,, \\ &\text{if} \quad n = 2 \quad \text{and} \quad p_{00} = 0 \quad \text{then} \quad k^* = 0\,, \end{aligned}$$

and we need not know the values of $C_i$ for $i > n$.

**Corollary 1.** *Let the assumptions of Theorem 2 be fulfilled. Then*

$$(32) \qquad z = \min\left[\{n;\, n \in K - \{0; k\},\ m(n) \leqq C_n\} \cup \{k\}\right].$$

Proof. Let us put

$$Z = \{n;\, n \in K - \{0; k\},\ m(n) \leqq C_n\}\,.$$

If $z = 1$ then $k^* < k$ and according to (30), $1 \in Z$. Thus $\min\left[Z \cup \{k\}\right] = 1$. If $z = k^* = k$ then according to (26) and to the definition of $k^*$

$$m(n) \geqq m(k^* - 1) > C_{k^*} > C_n \quad \text{for every} \quad n \in K - \{k\}$$

and thus $Z = \emptyset$ and $\min [Z \cup \{k\}] = k$. Finally, if $z \in K - \{0; 1; k\}$ then $z = k^*$, according to (30) $z \in Z$ and using (26) we obtain

$$m(n) \geqq m(k^* - 1) > C_{k^*} > C_n \quad \text{for every} \quad n \in K\,, \quad n < z\,.$$

In the algorithm suggested above for finding the value of $k^*$ we can use the following comparison of $m(n)$ and $C_n$ based on (32) before calculating $C_{n+1}$:

$$\begin{aligned}
&\text{if} \quad n = 1, \quad m(1) \leqq C_1 \quad \text{and} \quad p_{00} = 0 \quad \text{then} \quad k^* = 0\,,\\
&\text{if} \quad n = 1, \quad m(1) \leqq C_1 \quad \text{and} \quad p_{00} \neq 0 \quad \text{then} \quad k^* = 1\,,\\
&\text{if} \quad n \geqq 2 \quad \text{and} \quad m(n) \leqq C_n \quad \text{then} \quad k^* = n\,,\\
&\text{if} \quad n \geqq 1 \quad \text{and} \quad m(n) > C_n \quad \text{then} \quad k^* > n \quad \text{and} \quad C_{n+1} > C_n\,.
\end{aligned}$$

Using the criteria just determined instead of the comparison of $C_n$ and $C_{n-1}$ we do not calculate the superfluous value of $C_{z+1}$.

Remarks. 1) If the sequence $\{m(n)\}_{n=0}^{k-1}$ is only non-increasing then the results similar to Theorem 2 are true.

2) If the relation (21) is not fulfilled and we put

$$n_0 = \min \{n; n \in K - \{k\}, p_{n,n+1} = 0\} \tag{33}$$

then from Theorem 1 it is evident that

$$C_{n_0+1} = C_{n_0+2} = \ldots = C_k\,.$$

On the other hand, it is obvious that the unit in state 0 can by no means enter any of states $n \in \{i; i \in K - \{k\}, i > n_0\}$, so we can pass to the model including only the states of the unit $0, 1, \ldots, n_0$ and $k$. In this model the condition (21) is fulfilled and we can use Theorem 2.

## 3. A MORE EFFECTIVE ALGORITHM FOR FINDING THE VALUE OF k*

The procedure for finding the value of $k^*$ suggested in the preceding section is very suitable if $k^*$ is small enough. For example, if $k^* = 1$ then, evidently, there exists no better one. On the other hand, if $k^* = k$ we have to calculate all the values of $C_n$, $n \in K - \{0; k\}$. Thus we can state that this procedure is very weak in this case. Our aim is to minimize the number of those $C_n$, $n \in K$, which are to be calculated for the least favourable value of $k^*$. For this purpose, we introduce the following algorithm.

Let the preceding considerations (at the beginning we can make use e.g. of the results of Section 4 of the present paper) imply that

$$a < z < b\,, \tag{34}$$

where $a$ and $b$ are certain elements of the set $K \cup \{k + 1\}$ such that $b - a \geqq 2$. If at the beginning we know nothing concerning our task we obviously start with $a = 0$ and $b = k + 1$. The case $b = a + 2$ is trivial and may be omitted, i.e. we may suppose that

$$b - a > 2 . \tag{35}$$

Let us calculate the value of $C_d$, where $d$ is the whole part of $(a + b + 1)/2$. It is easy to see that

$$a + 2 \leqq d \leqq b - 1 . \tag{36}$$

Thus

$$d \in K - \{0; 1\} . \tag{37}$$

There are four possibilities:

1) $C_d \geqq m(d - 1)$ – in this case we put $a' = a$ and $b' = d$;
2) $d \neq k$ and $C_d < m(d)$ – in this case we put $a' = d$ and $b' = b$;
3) $d \neq k$ and $m(d) \leqq C_d < m(d - 1)$;
4) $d = k$ and $C_k < m(k - 1)$.

By (34), (37) and Corollary 1 of the present paper, and by Theorem 4 of the paper [6] (stating that the inequalities $C_d \geqq m(d- 1)$ and $z < d$ are equivalent if $d \in K - \{0; 1\}$) we have

$$a' < z < b' , \tag{38}$$

where

$$a', b' \in K \cup \{k + 1\} \tag{39}$$

in the first two cases, and

$$z = d \tag{40}$$

in the cases 3) and 4). We shall deal with the cases 1) and 2) only. The relations (38) and (39) and the fact that $z \in K - \{0\}$ imply that $b' - a' \geqq 2$. It may happen that $b' - a' = 2$. Then evidently $z = a' + 1$. On the other hand, if

$$b' - a' > 2 \tag{41}$$

we repeat this construction starting with the new parameters $a = a'$ and $b = b'$. The relations (37), (38), (41) and the assumption that the original parameters $a$ and $b$ are from the set $K \cup \{k + 1\}$ guarantee that the new ones meet all the demands.

So the procedure of finding the value of $z$ is divided into several steps each of which has the form just described. The set of possible values of $z$ is reduced approximately to one half in every step.

**Lemma 2.** *It is necessary to carry out not more than* $\log_2 k$ *steps of the algorithm to find the value of* $z$.

Proof. Let $q$ be the natural number such that

(42) $$2^{q-1} \leqq k < 2^q .$$

Let exactly $r$ steps of the algorithm have to be carried out and let $a_s$, $b_s$ and $d_s$ be the corresponding parameters $a$, $b$ and $d$ of the $s$-th step, $s = 1, \ldots, r$. By the mathematical induction we shall prove that

(43) $$2 \leqq b_s - a_s - 1 < 2^{q-s+1} \quad \text{for every} \quad s = 1, \ldots, r .$$

We have $b_1 - a_1 \leqq k + 1$ so that (43) is true for $s = 1$. Let $r > 1$ and let (43) hold for some $s \in \{1; \ldots; r - 1\}$. We know that

(44) $$b_{s+1} - a_{s+1} = b'_s - a'_s > 2 ,$$

because the $s$-th step is not the last one which is to be carried out. Further, the realization of the $(s + 1)$-st step of the algorithm implies either $C_{d_s} \geqq m(d_s - 1)$ or $d_s \neq k$ and $C_{d_s} < m(d_s)$, i.e. either

(45) $$b_{s+1} - a_{s+1} = d_s - a_s$$

or

(46) $$b_{s+1} - a_{s+1} = b_s - d_s .$$

Let (45) be true. If $a_s + b_s + 1$ is even then we obtain

$$b_{s+1} - a_{s+1} - 1 = \frac{a_s + b_s + 1}{2} - a_s - 1 = \frac{b_s - a_s - 1}{2} < 2^{q-s} .$$

If $a_s + b_s + 1$ is odd then

$$b_{s+1} - a_{s+1} - 1 = \frac{a_s + b_s}{2} - a_s - 1 < \frac{b_s - a_s - 1}{2} < 2^{q-s} .$$

On the other hand, if (46) is fulfilled then

$$b_{s+1} - a_{s+1} - 1 \leqq b_s - \frac{a_s + b_s}{2} - 1 < \frac{b_s - a_s - 1}{2} < 2^{q-s} .$$

In this way, the relation (43) is verified. In particular, for $s = r$ we have $2 < 2^{q-r+1}$ so that

$$r \leqq q - 1 \leqq \log_2 k .$$

**Theorem 3.** *The number of those* $C_n$, $n \in K$, *the values of which it is necessary to calculate for finding the value of* $k^*$, *if the algorithm considered in this section is used, is less than or equal to* $\log_2 k$.

Proof. It is easy to see that exactly one $C_n$, $n \in K - \{0\}$, is enumerated with every step of the algorithm in question. Let the value of $z$ be found. If $z \neq 1$ we know that $k^* = z$. On the other hand, if $z = 1$ we obtain according to (19) and (20) that

$$\text{if} \quad p_{00} = 0 \quad \text{then} \quad k^* = 0 \quad \text{and}$$
$$\text{if} \quad p_{00} > 0 \quad \text{then} \quad k^* = 1$$

so that it is not necessary to calculate any further value of $C_n$.

Remark. In the case of $k = 2^n$, where $n$ is a natural number, it may happen that the number of values of $C_n$ which have to be calculated is equal to $n = \log_2 k$. Indeed, if $z = k^* = 1$ then $a_s = 0$, $b_s = 2^{n-s+1} + 1$, $d_s = 2^{n-s} + 1$ and $C_{d_s} \geqq m(d_s - 1)$ for every $s = 1, \ldots, n$, so that in the $s$-th step we obtain the results $a'_s = 0 < z < b'_s = d_s$, $b'_s - a'_s = d_s > 2$ for every $s = 1, \ldots, n - 1$ and $b'_n - a'_n = d_n = 2$.

We find that the number of those $C_n$, $n \in K$, which are to be calculated in the least favourable case when using the algorithm of the present section is much less than that when using the procedure considered in Section 2. It ought to be mentioned that if $1 + \log_2 k < k^* < \frac{1}{2}(k + 1)$ the former method need not be quicker although it may require the enumeration of a smaller number of values of $C_n$, $n \in K$. Namely, we should calculate $C_{[k+1/2]}$ by the former one while the latter one stops with $C_{k^*-1}$. The former may be essentially more difficult to obtain than the latter due to the increasing complexity of the formula (12) when $n$ increases.

## 4. ADDITIONAL COMMENTS

The following theorems serve for further decrease of the number of necessary calculations of the values of $C_n$ and for the apriori upper estimate of this number. We suppose throughout the present section that the assumptions of Theorem 2 are fulfilled.

**Theorem 4.** *If $n \in K - \{k\}$ and $m(n) \leqq m(0) - R$ then $n \geqq k^*$.*

Proof. It is obvious that $n \geqq 1$. If $n < k^*$ then $k^* = z$ and from Theorems 1 and 2 we obtain

$$C_n \geqq C_1 = m(0) - R(1 - p_{00}) \geqq m(0) - R \geqq m(n),$$

but this relation contradicts (32).

**Lemma 3.** *Let the sequence $\{p_{jk}\}_{j=0}^{k-1}$ be increasing and let $n \in K - \{0; k\}$. Then*

$$m(n - 1) - R(1 - p_{n-1,n-1}) \leqq C_n \leqq m(0) - R p_{0k}. \tag{47}$$

Proof. Let us put $E_j = m(j) - p_{jk}R$ for all $j \in K - \{k\}$. The sequence $\{E_j\}_{j=0}^{k-1}$ is obviously decreasing and

$$-R + \sum_{j=0}^{n-1} m(j)\, P_j = -R + \sum_{j=0}^{n-1} E_j P_j + R \sum_{j=0}^{n-1} (1 - p_{jj})\, P_j - R \sum_{j=0}^{n-1} p_{j,j+1} P_j =$$

$$= \sum_{j=0}^{n-1} E_j P_j + R \sum_{j=1}^{n-1} p_{j-1,j} P_{j-1} - R \sum_{j=0}^{n-1} p_{j,j+1} P_j = \sum_{j=0}^{n-1} E_j P_j - R p_{n-1,n} P_{n-1} \,.$$

By the relation (12), we have

$$C_n \geqq \frac{E_{n-1} \sum_{j=0}^{n-1} P_j - R p_{n-1,n} P_{n-1}}{\sum_{j=0}^{n-1} P_j} \geqq E_{n-1} - R p_{n-1,n} = m(n-1) - R(1 - p_{n-1,n-1})$$

and similarly

$$C_n \leqq E_0 = m(0) - R p_{0k} \,.$$

**Theorem 5.** *Let the sequence* $\{p_{jk}\}_{j=0}^{k-1}$ *be increasing and let* $n \in K - \{0; k\}$. *Then*
1) *if* $m(n) \leqq m(n-1) - R(1 - p_{n-1,n-1})$ then $n \geqq k^*$;
2) *if* $m(n) > m(0) - R p_{0k}$ *then* $n < k^*$.

Proof. The proof of part 1) is based on (32) and on Lemma 3. If $n \in K - \{0; k\}$, $n \geqq k^*$ and $m(n) > m(0) - R p_{0k}$ then $k > n \geqq z$, so that according to (30) and (31), $m(n) \leqq C_n$. This result contradicts, however, the relation

$$m(n) > m(0) - R p_{0k} \geqq C_n$$

which can be easily obtained from Lemma 3.

Let us denote

$$n_1 = \max \left[\{n;\ n \in K - \{0; k\},\ m(n) > m(0) - R p_{0k}\} \cup \{0\}\right], \tag{48}$$

$$n_2 = \min \left[\{n;\ n \in K - \{0; k\},\ m(n) \leqq m(0) - R \ \text{ or } \ m(n) \leqq m(n-1) - R(1 - p_{n-1,n-1})\} \cup \{k\}\right]; \tag{49}$$

then

$$n_1 \leqq k^* \leqq n_2 \tag{50}$$

and if $n_1 \neq 0$ then $n_1 < k^*$.

It is worth mentioning that the optimal replacement strategy $\mathscr{S}_{k^*}$ does not involve only the comparison of the mean incomes of the system achieved during the nearest unit of time with the decisions ,,replace" and "do not replace". In other words, the strategy $\mathscr{S}_{k^*}$ is not generally equivalent to the strategy $\mathscr{S}$ determined by the set $A \subseteq K$ with the properties

a) $k \in A$,
b) if $n \in K - \{k\}$ and $m(n) < m(0) - R$ then $n \in A$,
c) if $n \in K - \{k\}$ and $m(n) > m(0) - R$ then $n \notin A$.

Theorem 4 guarantees that $A \subset A_{k^*}$ but the following example shows that generally the sets $A$ and $A_{k^*}$ need not coincide.

Example 1. Let

$$\begin{aligned} &k = 2\,, \\ &m(0) = 2R\,, \quad m(1) = \tfrac{3}{2}R\,, \\ &p_{00} \;= \tfrac{3}{4}\,, \qquad p_{01} = p_{02} = \tfrac{1}{8}\,, \\ &p_{11} \;= p_{12} = \tfrac{1}{2}\,. \end{aligned}$$

From the definition of the set $A$ we see that $A = \{2\}$, particularly $1 \notin A$. On the other hand, $p_{0k} < p_{1k}$ and

$$m(1) = \tfrac{3}{2}R < \tfrac{7}{4}R = m(0) - R(1 - p_{00})\,,$$

so that according to the first part of Theorem 5, $k^* \leqq 1$, i.e. $1 \in A_{k^*}$. Altogether we obtain

$$1 \in A_{k^*} - A\,.$$

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## Souhrn

# O OPTIMÁLNÍ ZAMĚŇOVACÍ STRATEGII

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V článku je uvažován systém s jedním prvkem, který může být v $k + 1$ stavech. Inspekce prvku jsou prováděny v diskrétních časových okamžicích. Proces zhoršování prvku se předpokládá markovovský. Prvek svou činností přináší určitý zisk, který klesá se zhoršujícím se jeho stavem. Výměna prvku je spojena s náklady na pořízení jiného. Článek přináší efektivní algoritmus nalezení takové strategie záměn prvků, která maximalizuje průměrný výnos systému za jednotku času. Použití tohoto postupu vyžaduje zkoumat nanejvýš $\log_2 k$ časově stacionárních strategií.



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