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BUCKLING OF ANISOTROPIC SHELLS I

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1. INTRODUCTION

The solution of buckling of cylindrical shells in case of isotropic material is known from the literature on shells, e.g. Flügge [1]. Singer and Fershcher [3] solved the buckling of the orthotropic conical shell under external pressure. Singer [2] solved the buckling of orthotropic and stiffened conical shells.

The object of this paper is to investigate the differential equations of the buckling problem for anisotropic cylindrical shells under the most general homogeneous stress action. The corresponding equations for isotropic shells are obtained as a special case.

The solution of the differential equations of the buckling problem for anisotropic shells without shear load in case of two way compression is found.

Solution for isotropic shells is deduced as a special case, the results being identical with known results, cf. Flügge [1].

2. THEORY

The equations of equilibrium in case of buckling of a circular cylindrical shells, see Flügge [1], are given by

$$aN'_x + aN^{\cdot}_{\varphi x} - pa(u^{\cdot\cdot} - w') - Pu'' - 2Tu'^{\cdot} = 0\,, \tag{1a}$$

$$aN^{\cdot}_{\varphi} + aN'_{x\varphi} - M^{\cdot}_{\varphi} - M'_{x\varphi} - pa(v^{\cdot\cdot} + w^{\cdot}) - Pv'' - 2T(v'^{\cdot} + w') = 0\,, \tag{1b}$$

$$M^{\cdot\cdot}_{\varphi} + M'^{\cdot}_{x\varphi} + M''_x + aN_{\varphi} + pa(u' - v^{\cdot} + w^{\cdot\cdot}) + M'^{\cdot}_{\varphi x} + Pw'' - 2T(v' - w'^{\cdot}) = 0\,, \tag{1c}$$

where $(\ )'$ and $(\ )^{\cdot}$ indicate $a(\partial/\partial x)(\ )$ and $(\partial/\partial\varphi)(\ )$, respectively.

The shell is simultaneously subjected to three simple loads (Fig. 1):

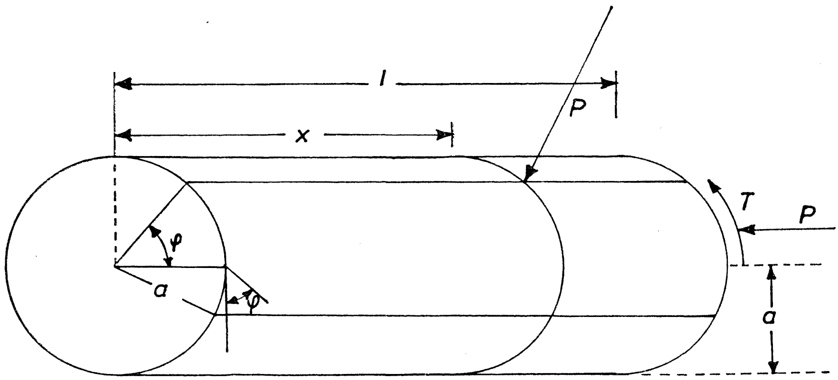

Fig. 1.

(i) a uniform normal pressure on its wall, $p_r = -p$;

(ii) an axial compression applied at the edges, the force per unit circumference being P;

(iii) a shear load applied at the edges so as to produce a torque in the cylinder; the shearing force (shear flow) is T.

The forces N and the moments M in terms of displacements u, v, w in anisotropic plywood shell, see Flügge [1], are given by

(2)
$$N_\varphi = \frac{D_\varphi}{a}(v^\bullet + w) + \frac{D_\nu}{a}u' + \frac{K_\varphi}{a^3}(w + w^{\bullet\bullet}),$$

$$N_x = \frac{D_x}{a}u' + \frac{D_\nu}{a}(v^\bullet + w) - \frac{K_x}{a^3}w'',$$

$$N_{\varphi x} = \frac{D_{x\varphi}}{a}(u^\bullet + v') + \frac{K_{x\varphi}}{a^3}(u^\bullet + w'^\bullet),$$

$$N_{x\varphi} = \frac{D_{x\varphi}}{a}(u^\bullet + v') + \frac{K_{x\varphi}}{a^3}(v' - w'^\bullet),$$

$$M_\varphi = \frac{K_\varphi}{a^2}(w + w^{\bullet\bullet}) + \frac{K_\nu}{a^2}w'',$$

$$M_x = \frac{K_x}{a^2}(w'' - u') + \frac{K_\nu}{a^2}(w^{\bullet\bullet} - v^\bullet),$$

$$M_{\varphi x} = \frac{K_{x\varphi}}{a^2}(2w'^\bullet + u^\bullet - v'),$$

$$M_{x\varphi} = \frac{2K_{x\varphi}}{a^2}(w'^\bullet - v'),$$

where the rigidities are given by

(i) extensional rigidities:

(3a)
$$D_x = E_1 t_1 + 2E_2 t_2 ,$$
$$D_\varphi = E_2 t_1 + 2E_1 t_2 ,$$
$$D_v = E_v t ;$$

(ii) shear rigidity:

(3b)
$$D_{x\varphi} = Gt ;$$

(iii) bending rigidities:

(3c)
$$K_x = \tfrac{1}{12}[E_2(t^3 - t_1^3) + E_1 t_1^3] ,$$
$$K_\varphi = \tfrac{1}{12}[E_1(t^3 - t_1^3) + E_2 t_1^3] ,$$
$$K_v = \tfrac{1}{12} E_v t^3 ;$$

(iv) twisting rigidity:

(3d)
$$K_{x\varphi} = \tfrac{1}{12} G t^3 ,$$

in which E_1, E_2, E_v and G are four moduli of elasticity and $t = t_1 + 2t_2$ (Fig. 2) is the thickness of the shell.

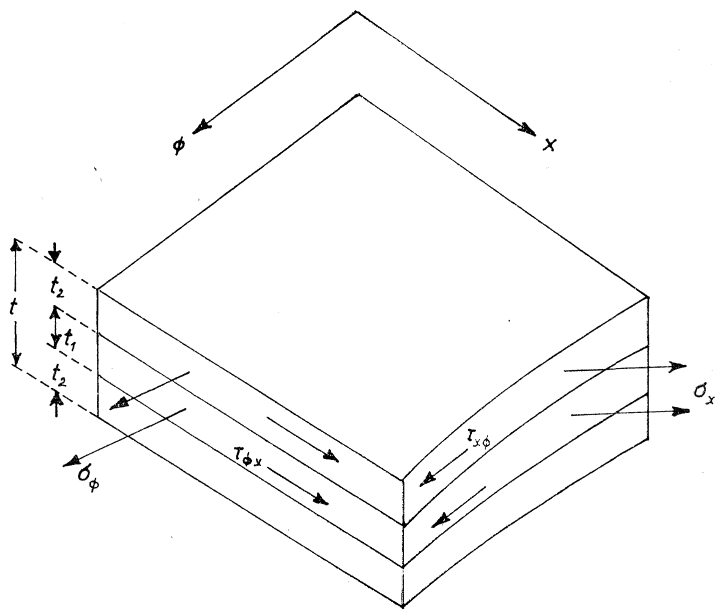

Fig. 2.

Substituting (2) in (1), the differential equations for the buckling problem of an anisotropic shell appear in the following form after proper simplification:

(4a) $$u'' + A_1 u^{\bullet\bullet} + A_2 v'^{\bullet} + A_3 w' + k_1\{A_4(u^{\bullet\bullet} + w'^{\bullet\bullet}) - w'''\} - \\ - q_1(u^{\bullet\bullet} - w') - q_2 u'' - 2q_3 u'^{\bullet} = 0\,,$$

(4b) $$A_5 u'^{\bullet} + v^{\bullet\bullet} + A_6 v'' + w^{\bullet} + k_1[3A_7 v'' - A_8 w''^{\bullet}] - \\ - A_9[q_1(v^{\bullet\bullet} + w^{\bullet}) + q_2 v'' + 2q_3(v'^{\bullet} + w')] = 0\,,$$

(4c) $$A_{10} u' + v^{\bullet} + w + k_1[A_7 u'^{\bullet\bullet} - A_9 u''' - A_8 v''^{\bullet} + A_9 w'''' + \\ + 2A_{11}\omega''^{\bullet\bullet} + A_{12}(w^{\bullet\bullet\bullet\bullet} + 2w^{\bullet\bullet} + w)] + \\ + A_9[q_1(u' - v^{\bullet} + w^{\bullet\bullet}) + q_2 w'' - 2q_3(v' - w'^{\bullet})] = 0\,,$$

where

(5) $$A_1 = \frac{D_{x\varphi}}{D_x}\,,\quad A_2 = \frac{D_v + D_{x\varphi}}{D_x}\,,\quad A_3 = \frac{D_v}{D_x}\,,$$

$$A_4 = \frac{K_{x\varphi}}{K_x}\,,\quad A_5 = \frac{D_v + D_{x\varphi}}{D_\varphi}\,,\quad A_6 = \frac{D_{x\varphi}}{D_\varphi}\,,$$

$$A_7 = \frac{D_x K_{x\varphi}}{D_\varphi K_x}\,,\quad A_8 = \frac{D_x(3K_{x\varphi} + K_v)}{D_\varphi K_x}\,,$$

$$A_9 = \frac{D_x}{D_\varphi}\,,\quad A_{10} = \frac{D_v}{D_\varphi}\,,\quad A_{11} = \frac{D_x(2K_{x\varphi} + K_v)}{D_\varphi K_x}\,.$$

$$A_{12} = \frac{D_x K_\varphi}{D_\varphi K_x}$$

and

(6) $$k_1 = \frac{K_x}{a^2 D_x}\,,\quad q_1 = \frac{pa}{D_x}\,,\quad q_2 = \frac{P}{D_x}\,,\quad q_3 = \frac{T}{D_x}\,.$$

The equations (4) describe the buckling of a cylindrical shell under the most general homogeneous stress action in the anisotropic case.

It is easy to observe that the parameters defined by equations (6) are small quantities. For k_1 it is obvious, since we are interested in thin shells where $t \ll a$. The three load parameters q are approximately the elastic strains, in the limiting case, caused by the corresponding basic loads. Since all our theory is based on the assumption that such strains are small as compared with unity, we shall neglect the squares and higher order terms whenever possible.

Substituting

(7) $$t_2 = 0\,,\quad t_1 = t\,,\quad E_1 = E_2 = \frac{E}{1 - \nu^2}\,,\quad E_\nu = \frac{E\nu}{1 - \nu^2}\,,\quad G = \frac{E}{2(1 + \nu)}\,,$$

(ν = Poisson's ratio)

the equations (4) and the dimensionless parameters given by (6) reduce to the corresponding equations (7) and (6) of Flügge [1] for the isotropic case.

(A) SOLUTION FOR SHELLS WITHOUT SHEAR LOAD

Two way compression

When there is no shear load on the shell ($T = 0$, hence $q_3 = 0$) the equations (4) admit a solution of the form

$$
\begin{aligned}
u &= A \cos m\varphi \cos \lambda x/a , \\
v &= B \sin m\varphi \sin \lambda x/a , \\
w &= C \cos m\varphi \sin \lambda x/a ,
\end{aligned} \tag{8}
$$

where

$$\lambda = n\pi a/l, \quad l = \text{length of the shell and } n \text{ is an integer} . \tag{9}$$

The solution (8) describes a buckling mode with n half waves along the length of the cylinder and 2m half waves around its circumference. Although this is far from being the most general solution, it is the one which fulfils reasonable boundary conditions.

It is evident that the solution (8) satisfies the boundary conditions

$$v = w = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = 1 .$$

Also

$$N_x = M_x = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = 1 ,$$

which shows that the solution (8) represents the buckling of a shell whose edges are supported in tangential and radial directions, but are neither restricted in the axial direction nor clamped.

Substituting the solution (8) into the differential equation (4) [$q_3 = 0$], the trigonometric function drop out entirely and we are left with the following equations:

$$
\begin{aligned}
& A[\lambda^2 + (A_1 + k_1 A_4) m^2 - q_1 m^2 - q_2 \lambda^2] + B[-A_2 \lambda m] + \\
& \quad + C[-A_3 - k_1(\lambda^3 - A_4 \lambda m^2) - q_1 \lambda] = 0 ,
\end{aligned} \tag{10a}
$$

$$
\begin{aligned}
& A[-A_5 \lambda m] + B[m^2 + (A_6 + 3k_1 A_7) \lambda^2 - q_1 A_9 m^2 - q_2 A_9 \lambda^2] + \\
& \quad + C[m + k_1 A_8 \lambda^2 m - q_1 A_9 m] = 0 ,
\end{aligned} \tag{10b}
$$

$$
\begin{aligned}
& A[-A_{10}\lambda - k_1(A_9 \lambda^3 - A_7 \lambda m^2) - q_1 A_9 \lambda] + \\
& \quad + B[m + k_1 A_8 \lambda^2 m - q_1 A_9 m] + \\
& \quad + C[1 + k_1\{A_9 \lambda^4 + 2A_{11}\lambda^2 m^2 + A_{12}(m^2 - 1)^2 - A_9(q_1 m^2 + q_2 \lambda^2)] = 0 .
\end{aligned} \tag{10c}
$$

The equations (10) are three linear equations with buckling amplitudes A, B, C, as unknowns and with the brackets as coefficients. Since the equations are homogeneous, they admit, in general, only the solution $A = B = C = 0$, which shows that the shell is not in neutral equilibrium. The non-vanishing solution A, B, C is possible if and only if the determinant of the nine coefficients of the equations (10) is equal to zero. Thus the vanishing of this determinant is the buckling condition of the shell. Whenever the buckling condition is fulfilled, any two of the three equations (10) determine the ratios A/C and B/C and thus the buckling mode according to equation (8). As in all cases of neutral equilibrium, the magnitude of the possible deformation remains arbitrary.

The buckling condition contains four unknowns: the dimensionless loads q_1 and q_2 and the modal parameters m and λ. Also we know that m must be an integer (0, 1, 2, 3, 4, ...) and λ must be an integer multiple of $\pi a/l$ ($n = 1, 2, 3, 4, \ldots$)..

Thus we can write the buckling condition separately for every pair m, λ fulfilling these requirements, and consider it as a relation between q_1 and q_2 which describes those conditions of the two loads for which the shell is in neutral equilibrium. When we plot these equations as a curve in the q_1 q_2-plane, we obtain the diagram like Fig. 3, which can be interpreted as follows: The origin $q_1 = q_2 = 0$ represents the

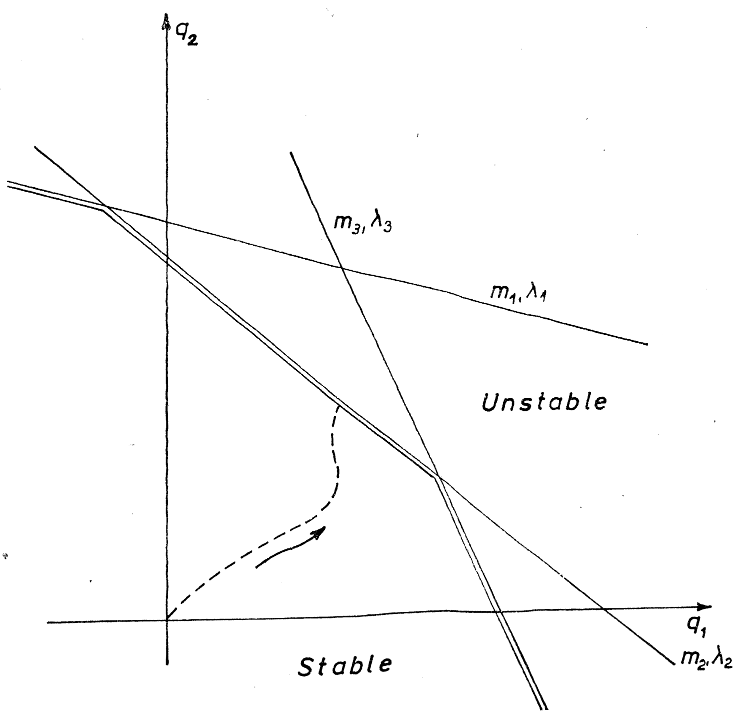

Fig. 3.

unloaded shell. When the load is gradually applied the corresponding diagram point moves along a certain path, as shown by the dotted line. As long as it does not meet any of the curves, the shell is in stable equilibrium, but as soon as one of the curves is reached equilibrium becomes neutral with the buckling mode defined by the parameters m, λ of this curve. The stable domain in the q_1q_2-plane is, therefore, bounded by the envelope of all the curves, which is shown by a heavy line in Fig. 3.

The coefficients of the equations (10) are linear functions of k_1, q_1, q_2. The expanded determinant is, therefore, a polynomial of the third degree in these parameters. Since they are very small quantities it is sufficient to keep only the linear terms and to write the buckling condition in the following form:

$$C_1 + C_2 k_1 = C_3 q_1 + C_4 q_2 \,. \tag{11}$$

The equation (11) describes a straight line in the q_1q_2-plane and the limit of the stable domain as shown in Fig. 3 is a polygon consisting of the sections of straight lines for various pairs m, λ.

The coefficients C_1, C_2, C_3 and C_4 of the equation (11) can be found by really expanding the determinant and putting it equal to zero. Since C_1 turns out to be proportional to λ, we may drop the term within all other coefficients, thus obtaining (see Flügge [1])

$$C_1 = A_6(1 - A_3A_{10})\,\lambda^4 \,, \tag{12a}$$

$$\begin{aligned} C_2 = {} & [A_9\lambda^4 + 2A_{11}\lambda^2m^2 + A_{12}m^4]\,[A_6\lambda^4 + 2A_{13}\lambda^2m^2 + A_1m^4] - \\ & - A_6(A_3A_9 + A_{10})\,\lambda^6 - 2\lambda^4m^2[A_8 + A_{10} - A_5 - A_3(A_5A_8 + A_6A_7)] - \\ & - \lambda^2m^4[2A_1A_8 + 4A_{12}A_{13} + A_4(A_5 + A_6 - A_{10})] - 2A_1A_{12}m^6 + \\ & + [3A_1A_7 + A_4A_6 + 2A_{12}A_{13}]\,\lambda^2m^2 + A_1A_{12}m^4 \,, \end{aligned} \tag{12b}$$

$$\begin{aligned} C_3 = {} & m^2[A_9\{A_1m^4 + A_6\lambda^4 + (1 + A_1A_6 - A_5^2)\,\lambda^2m^2\}] + \\ & + \lambda^2m^2[2(A_5 + A_{10}) + A_{10}(2A_5 - A_{10}) + A_6 - A_9] - A_1A_9m^4 \,, \end{aligned} \tag{12c}$$

$$C_4 = \lambda^2[A_9\{A_6\lambda^4 + A_1m^4 + 2A_{13}\lambda^2m^2\} + A_1m^2] \,, \tag{12d}$$

where $A_1, A_2, A_3, \ldots, A_{12}$ are given by (5) and

$$A_{13} = 1 + A_1A_6 - A_2A_5 \,. \tag{13}$$

From the formulas (11) and (12) the stability curve may easily be constructed when l and k_1 are given.

3. PARTICULAR CASE

In particular, substituting (7) in the equations (10), we get the corresponding equations for the isotropic case which are identical with the known results, see Flügge [1] (equation (10)).

By the same substitution the equations (12) give the coefficients C_1, C_2, C_3, C_4 as follows:

(14a) $$C_1 = \frac{1 - \nu}{2} [(1 - \nu^2) \lambda^4] ,$$

(14b) $$C_2 = \frac{1 - \nu}{2} [(\lambda^2 + m^2)^4 - 2(\nu\lambda^6 + 3\lambda^4 m^2 + (4 - \nu) \lambda^2 m^4 + m^6) + $$
$$+ 2(2 - \nu) \lambda^2 m^2 + m^4] ,$$

(14c) $$C_3 = \frac{1 - \nu}{2} [m^2(\lambda^2 + m^2)^2 - m^2(3\lambda^2 + m^2)] ,$$

(14d) $$C_4 = \frac{1 - \nu}{2} [\lambda\{(\lambda^2 + m^2)^2 + m^2\}] .$$

Except for the common factor $(\frac{1}{2}(1 - \nu))$ which can be cancelled throughout from (11), the equations (14) are exactly the same as in Flügge [1] (equation (12)).

4. NUMERICAL RESULT

From the formulas (12) and (11) the stability curve may easily be drawn when l and k_1 are given.

Taking $t_1 = 3$ cm, $t_2 = 2$ cm, $t = t_1 + 2t_2 = 7$ cm, $k_1 = 10^{-5}$, and considering the shell to be made of the same material as that of Gaboon (Okoumme') – 3 ply, so that

$$E_1 = 1.28 \times 10^6 \psi , \quad E_2 = 0.11 \times 10^6 \psi ,$$
$$E_\nu = 0.014 \times 10^6 \psi , \quad G = 0.085 \times 10^6 \psi ,$$

see Timoshenko and Woinowsky-Krieger [4], the buckling diagram of an anisotropic cylindrical shell subject to two way thrust is sketched (Fig. 4) and the following conclusion may be drawn.

Although the load and the basic stress system has axial symmetry, the buckling mode not $(m \neq 0)$ but it develops nodal generators. Their number increases as q_1 does, and is higher for thinner shells.

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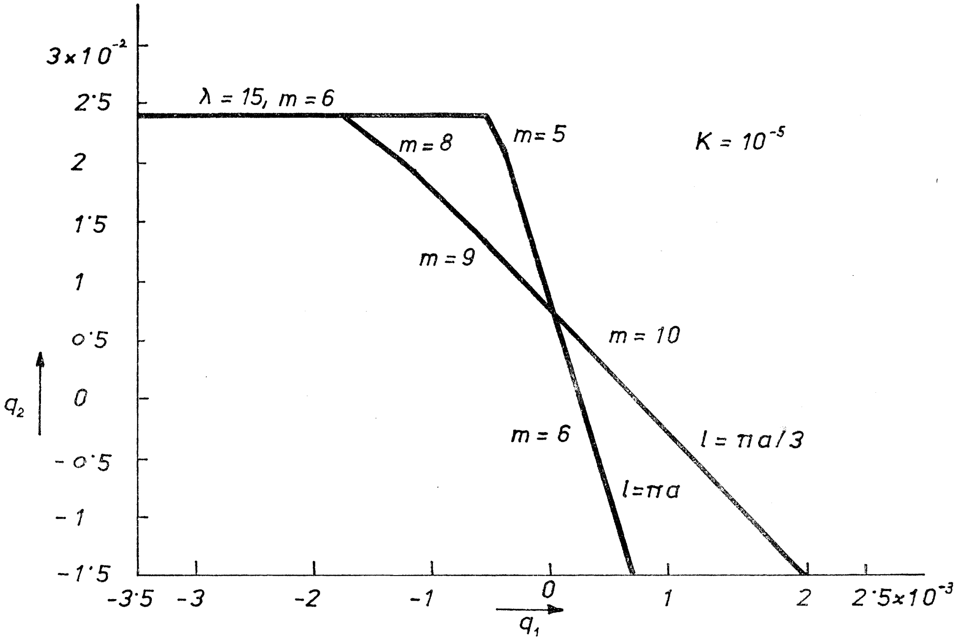

Fig. 4.

References

[1] *W. Flügge:* Stresses in shells. Springer Verlag, New York, 1967.

[2] *J. Singer:* Buckling of orthotropic and stiffened conical shells. Collected papers on instability of shell structures. N. A. S. A. T. N. D — 1510, p. 463.

[3] *J. Singer, R. Fersh-Scher:* Buckling of orthotropic conical shells under external pressure. Aeronautical Quarterly XV (1964), 151—168.

[4] *S. Timoshenko, S. Woinowsky-Krieger:* Theory of plates and shells. McGraw Hill, New York, 1959.

Souhrn

STABILITA ANISOTROPNÍCH SKOŘEPIN

ANUKUL DE

V článku jsou formulovány diferenciální rovnice pro stabilitu anisotropních válcových skořepin. Z těchto rovnic je nalezeno řešení problému pro anisotropní skořepiny bez smykového zatížení v případě současného radiálního a osového tlaku. Odpovídající výsledky pro isotropní problémy jsou odvozeny jako speciální případ.

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