ON PERIODIC SOLUTION OF A NONLINEAR BEAM EQUATION

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1. INTRODUCTION

The purpose of the paper is to prove the existence of a periodic solution of the equation

$$(1) \qquad u_{tt} + \alpha u_{xxxx} + \gamma u_{xxxxt} - \tilde{\gamma} u_{xxt} + \delta u_t - u_{xx}\left[\beta + \varkappa \int_0^\pi u_x^2(\cdot, \xi)\,\mathrm{d}\xi + \right.$$
$$\left. + \sigma \int_0^\pi u_{xt}(\cdot, \xi)\, u_x(\cdot, \xi)\,\mathrm{d}\xi\right] = f,$$

satisfying the boundary conditions

$$(2) \qquad u(t, 0) = u(t, \pi) = u_{xx}(t, 0) = u_{xx}(t, \pi), \quad t \geqq 0.$$

This equation governs vibrations of an extensible viscoelastic beam (e.g. [1]). Stability of such equations was studied by J. M. Ball [2] and T. Narazaki [3] for $f = 0$ and for f being a small perturbation (under more general conditions than here). The case $\gamma = \tilde{\gamma} = \sigma = 0$ was solved by V. Lovicar [4] as an example of a more general equation.

A periodic solution of the equation (1) is found by using the estimates of Lemma 1 and the Schauder-Tichonov Fixed Point Theorem.

2. APRIORI ESTIMATE

Let $f \in C([0, T], L_2)$, $\varphi \in H_2 \cap H_1^0$, $\psi \in L_2$. The function $u \in C([0, T], H_2 \cap H_1^0) \cap$ $\cap\, C^1([0, T], L_2)$ is said to be a (generalized) solution of equation (1) with the boundary conditions (2) and initial data

$$(3) \qquad u(0, x) = \varphi(x), \quad u_t(0, x) = \psi(x), \quad x \in (0, \pi)$$

if u satisfies (3) and the equation

(4)
$$(u_t(t), v)_0 + \gamma(u(t), v)_2 + \tilde{\gamma}(u(t), v)_1 + \\ + \frac{\sigma}{2} |u_x(t)|^2 (u(t), v)_1 + \int_0^t [\alpha(u(\tau), v)_2 + \\ + \beta(u(\tau), v)_1 + \delta(u_t(\tau), v)_0 + \varkappa|u_x(\tau, \cdot)|^2 (u(\tau), v)_1] \, d\tau = \\ = \int_0^t (f(\tau), v)_0 \, d\tau + (\psi, v)_0 + \gamma(\varphi, v)_2 + \tilde{\gamma}(\varphi, v)_1$$

for a.e. $t \in (0, T)$ and every $v \in H_2 \cap H_1^0$, where

$$(u(t), v)_k = \int_0^\pi \frac{\partial^k u}{\partial x^k}(t, x) \frac{\partial^k v}{\partial x^k}(x) \, dx, \quad |u(t)|^2 = (u(t), u(t))_0 .$$

We will suppose throughout the paper:

(5)
$$\alpha, \varkappa > 0, \quad \gamma, \sigma \geqq 0, \quad \gamma + \tilde{\gamma} \geqq 0, \quad \gamma + \tilde{\gamma} + \delta > 0, \quad \alpha + \beta > 0, \\ \sigma = 0 \quad \text{for} \quad \gamma + \tilde{\gamma} = 0 .$$

If we denote $\gamma_1 = \gamma - \frac{1}{2}(|\gamma| - \tilde{\gamma})$, $\gamma_2 = \frac{1}{2}(|\tilde{\gamma}| + \gamma)$, the conditions (5) imply $\gamma_i \geqq 0$, $i = 1, 2$, $\gamma_1 = 0$ for $\gamma = 0$, $\gamma_2 = 0$ for $\tilde{\gamma} \leqq 0$ and $\gamma_1 = \gamma_2 = 0$ if $\gamma + \tilde{\gamma} = 0$.

Lemma 1. *If u is a solution of (1)–(3) (in the above sense) and $\gamma u_{xxt}, \tilde{\gamma} u_{xt} \in L_2$, then there are positive constants a, c_0, c such that the following estimates hold:*

(6)
$$S(u(t), u_t(t)) \leqq S(\varphi, \psi) e^{-c_0 t} + \int_0^t |f(\tau, \cdot)| e^{-c_0(t-\tau)} \, d\tau ,$$

(7)
$$\gamma_1 \int_0^T |u_{xxt}(t, \cdot)|^2 \, dt + \gamma_2 \int_0^T |u_{xt}(t, \cdot)|^2 \, dt \leqq \\ \leqq 4S^2(\varphi, \psi) + c \int_0^T |f(t, \cdot)|^2 \, dt ,$$

where

$$S^2(\varphi, \psi) = |\psi + a\varphi|^2 + \alpha|\varphi_{xx}|^2 + \beta|\varphi_x|^2 + \frac{\varkappa + a\sigma}{2} |\varphi_x|^4 .$$

Remarks. $S^2(\varphi, \psi)$ is nonnegative, as $\alpha|\varphi_{xx}|^2 + \beta|\varphi_x|^2 \geqq (\alpha + \beta) |\varphi_x|^2 \geqq 0$ for $\beta < 0$.

Proof. Let u be the solution from Lemma 1 and $u \in C^2([0, T], L_2) \cap C^1([0, T], H_2 \cap H_1^0)$ (for general solution an approximation process is to be used). u satisfies the equation

$$(u_{tt}(t), v)_0 + \alpha(u(t), v)_2 + \gamma(u_t(t), v)_2 + \tilde{\gamma}(u_t(t), v)_1 + \delta(u_t(t), v)_0 + [\beta + N(u)]\,(u_x, v)_1 = (f(t), v)_0$$

for every $t \in (0, T)$ and every $v \in H_2 \cap H_1^0$, where

$$N(u)\,(t) = \varkappa|u_x(t, \cdot)|^2 + \sigma(u_t(t), u(t))_1 = \left(\varkappa + \frac{\sigma}{2}\frac{d}{dt}\right)|u_x(t, \cdot)|^2 .$$

Substituting $v = 2(u_t + au)$ (a being a positive constant) into this identity we get

$$\begin{aligned}(8)\qquad &\frac{d}{dt}\left(|u_t + au|^2 + \alpha|u_{xx}|^2 + \beta|u_x|^2 + \tfrac{1}{2}(\varkappa + a\sigma)\,|u_x|^4\right) + \\ &+ \gamma|(u_t + au)_{xx}|^2 + \tilde{\gamma}|(u_t + au)_x|^2 + (\delta - a)\,|u_t + au|^2 + \\ &+ \gamma|u_{txx}|^2 + \tilde{\gamma}|u_{xt}|^2 + (\delta - a)\,|u_t|^2 - a^2(\delta - a)\,|u|^2 + \\ &+ a(2\alpha - a\gamma)\,|u_{xx}|^2 + a(2\beta - a\tilde{\gamma})\,|u_x|^2 + \tfrac{1}{2}\,\sigma|u_x t|^2 + 2a\,\varkappa|u_x|^4 = 2(f, u_t + au) .\end{aligned}$$

As

$$\gamma|v_{xx}|^2 + \tilde{\gamma}|v_x|^2 + (\delta - a)\,|v|^2 \geqq (\gamma + \tilde{\gamma} + \delta - a)\,|v|^2$$

and

$$\gamma|v_{xx}|^2 + \tilde{\gamma}|v_x|^2 \geqq \gamma_1|v_{xx}|^2 + \gamma_2|v_x|^2$$

for $v \in H_2 \cap H_1^0$, the left hand side of (8) is greater than

$$\frac{d}{dt} S^2(u, u_t) + (\gamma + \tilde{\gamma} + \delta - a)\,|u_t + au|^2 + a[(2\alpha - a\gamma)\,|u_{xx}|^2 + (2\beta - a\tilde{\gamma})\,|u_x|^2 - a(\delta - a)\,|u|^2 + 2\varkappa|u_x|^4] + \tfrac{1}{2}(\gamma_1|u_{xLt}|^2 + \gamma_2|u_{xt}|^2) .$$

Due to the assumptions (5) positive constants a, c_0 may be chosen so that the second and the third member of the above expression is bounded from below by $2c_0S^2(u, u_t)$ a, c_0 may be found to satisfy the inequalities:

$$a \leqq \min\left(\frac{\gamma + \tilde{\gamma} + \delta}{2},\ \frac{2\alpha}{\gamma},\ \frac{\alpha + \beta}{\gamma + \tilde{\gamma} + \delta},\ c_0 \leqq \frac{\varkappa + \sigma a}{4\varkappa a}\right),$$

$$a - c_0 \geqq \frac{a^2}{2}\min\left(\frac{\gamma + \tilde{\gamma} + \delta}{\alpha + \beta},\ \frac{\gamma}{\alpha}\right),\quad c_0 \leqq \tfrac{1}{2}(\gamma + \tilde{\gamma} + \delta - a) .$$

Then both sides of (8) may be estimated and we get

$$(9)\qquad \frac{d}{dt} S^2(u, u_t) + 2c_0S^2(u, u_t) + \tfrac{1}{2}(\gamma_1|u_{xxt}|^2 + \gamma_2|u_{xt}|^2) \leqq 2|f(t)|\,S(u, u_t) ,$$

which implies

$$\frac{d}{dt} S^2(u, u_t) + 2c_0 S^2(u, u_t) \leqq 2|f(t, \cdot)|\, S(u, u_t) \quad \text{for} \quad t \in [0, T],$$

$$\int_0^T (\gamma_1 |u_{xxt}|^2 + \gamma_2 |{}_x u_t|^2)\, dt \leqq 2 \int_0^T |f(t, \cdot)|\, S(u, u_t)\, dt.$$

The estimates (6), (7) easily follow from the last two inequalities.

3. A SOLUTION OF THE INITIAL-BOUNDARY VALUE PROBLEM

The solution u of (1), (2) and (3) may be written in the form

$$u(t, x) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} u_n(t) \sin nx$$

and $u_n(t)$ satisfy the system of equations

$$u_n'' + b_n u_n' + (a_n + n^2 N(u))\, u_n = f_n, \quad t \in (0, T) \tag{10}$$

and the initial data $u_n(0) = \varphi_n$, $u_n'(0) = \psi_n$, $n = 1, 2, \ldots$, where $b_n = \gamma n^4 + \tilde{\gamma} n^2 + \delta$, $a_n = \alpha n^4 + \beta n^2$; φ_n, ψ_n and f_n are the Fourier coefficients of φ, ψ and f, respectively. The equation (10) with the initial data is equivalent to the system

$$u_n(t) = \tilde{K}_n(t)\, \varphi_n + K_n(t)\, \psi_n + \int_0^t K_n(t - \tau)\, [f_n(\tau) - n^2 N(u)(\tau)\, u_n(\tau)]\, d\tau,$$

$$t \in (0, T), \quad n = 1, 2, \ldots, \tag{11}$$

where

$$K_n(t) = \frac{1}{(\lambda_n)_1 - (\lambda_n)_2} (e^{(\lambda_n)_1 t} - e^{(\lambda_n)_2 t}), \quad \tilde{K}_n(t) = K_n'(t) + b_n K_n(t),$$

$$(\lambda_n)_{1,2} = \tfrac{1}{2}(-b_n \pm \sqrt{(b_n^2 - 4a_n)}).$$

Denoting the right hand side of (11) by $A_n u(t)$ and setting

$$A\, u(t, x) = \sum_{n=1}^{\infty} (A_n u)(t) \sin nx,$$

we infer that the solution u is a fixed point of the mapping A.

Lemma 2. *Let $f \in C([0, T], L_2)$, $\varphi \in H_2 \cap H_1^0$, $\psi \in L_2$. Then there exists a unique solution u of (1), (2) and (3) on $(0, T) \times (0, \pi)$.*

Proof. Denote by $X_T = C^1([0, T], L_2) \cap C([0, T], H_2 \cap H_1^0)$ the Banach space with the norm $\|u\|_{X_T}^2 = (\sup_{t \in [0,T]} |u(t, \cdot)|)^2 + (\sup_{t \in [0,T]} |u_{xx}(t, \cdot)|)^2$.

The proof of Lemma 2 consists of three parts:

(a) the operator A maps X_T into X_T,

(b) there exists $\delta \in (0, T]$ such that A is a contractive mapping in X_δ. Then A has a fixed point, which solves (1)–(3) on $[0, \delta] \times [0, \pi]$,

(c) this solution may be continued onto $[0, T] \times [0, \pi]$.

The function $N(v)(t)$ is continuous on $[0, T]$ and

(12) $$|N(v)(t) - N(w)(t)| \leqq (\varkappa + \sigma)(\|v\| + \|w\|)\|v - w\|, \quad \text{for} \quad t \in [0, T], \quad v, \quad w \in X_T.$$

It is easy to calculate the following estimates:

(13) $$|K_n'(t)|, \quad |n^2 K_n(t)|, \quad \left|\frac{1}{n^2}\tilde{K}_n'(t)\right|, \quad |b_n K_n(t)| \leqq c,$$

where the constant c does not depend on n. Then

$$[k^2 A_k v(t)]^2 \leqq [\tilde{K}_k(t)]^2 k^4 \varphi_k^2 + [k^2 K_k(t)]^2 \psi_k^2 +$$
$$+ \int_0^t K_k^2(\tau)\,d\tau \int_0^t [f_k^2(\tau) + k^4 N^2(v)(\tau)\, v_k^2(\tau)]\,d\tau \leqq$$
$$\leqq c\left((k^4\varphi_k^2 + \psi_k^2 + t\int_0^t f_k^2(\tau)\,d\tau + t\|v\|^4 \int_0^t k^4 v_k^2(\tau)\,d\tau\right),$$

$$[A_k' v(t)]^2 \leqq \left[\frac{1}{k^2}\tilde{K}_k'(t)\right]^2 k^4\varphi_k^2 + [K_k'(t)]^2 \psi_k^2 +$$
$$+ \int_0^t [K_k'(\tau)]^2\,d\tau \int_0^t [f_k^2(\tau) + k^4 N^2(v)(\tau)\, v_k^2(\tau)]\,d\tau \leqq$$
$$\leqq c\left((k^4\varphi_k^2 + \psi_k^2 + t\int_0^t f_k^2(\tau)\,d\tau + t\|v\|^4 \int_0^t k^4 v_k^2(\tau)\,d\tau\right),$$

which implies

$$\left\|\sum_{k=1}^m \sqrt{\left(\frac{2}{\pi}\right)} A_k(t)\sin kx - \sum_{k=1}^n \sqrt{\left(\frac{2}{\pi}\right)} A_k(t)\sin kx\right\|_{X_T}^2 = (\sup_t \sqrt{\sum_{k=n+1}^m A_k^2(t)\,k^4})^2 +$$
$$(\sup_t \sqrt{(\sum_{k=n+1}^m [A_k'(t)]^2})^2 \leqq c\sum_{k=n+1}^m (k^4\varphi_k^2 + \psi_k^2) + ct^2(\sup_t \sqrt{(\sum_{k=n+1}^m f_k^2(t))})^2 +$$
$$+ c\,t^2\|v\|^4 (\sup_t \sqrt{(\sum_{k=n+1}^m k^4 v_k^2(t))})^2,$$

which tends to zero as $m, n \to \infty$.

Hence $\sum_{k=1}^n \sqrt{(2\pi^{-1})}\, A_k(t)\sin kx$ converges in X_T to Av, and $Av \in X_T$. Now, we have to find $\delta \in (0, T]$ such that A is a contractive mapping on X_δ. First, summing the above estimates for $k^2 A_k v$ and $A_k' v$ over $k = 1, 2, \ldots$ we get

$$\|Av\|_{X_\delta} \leqq c(|\varphi''|^2 + |\psi|^2 + \delta^2 \|f\|^2_{C(L_2)}))^{1/2} + c\,\delta \|v\|^3_{X_\delta}, \quad \delta \in [0, T].$$

The estimates (12) and (13) give

$$\|Av - Aw\|_{X_\delta} \leqq c\,\delta(\|v\|_{X_\delta} + \|w\|_{X_\delta})^2 \|v - w\|_{X_\delta}, \quad t \in [0, T].$$

Let $|\varphi''|^2 + |\psi|^2 \leqq R_0^2$ and $q \in (0, 1)$. Then for

$$R = \frac{2c}{2 - q} (R_0^2 + T^2 \|f\|^2_{C(L_2)})^{1/2} \text{ and } \delta = q(2CR^2)^{-1}$$

the operator A will be contractive in X_δ and

$$\|Av\|_{X_\delta} \leqq R \quad \text{for} \quad \|v\|_{X_\delta} \leqq R.$$

Hence there exists a unique function $u \in X_\delta$ such that $u = Au$.

As

$$\gamma n^4 + \gamma_2 n^2 \leqq b_n \quad \text{and} \quad b_n \int_0^T [a_n K_n^2(t) + (K_n'(t))^2] \, dt \leqq c$$

we get

$$\gamma_1 \int_0^T |(Av)_{xxt}(t, \cdot)|^2 \, dt + \gamma_2 \int_0^T |(Av)_{xt}(t, \cdot)|^2 \, dt \leqq$$
$$\leqq c(|\varphi''|^2 + |\psi|^2 + T\|f\|^2_{L_2} + T^2 \|v\|^3),$$

which means that $\gamma_1 (Av)_{xxt} + \gamma_2 (Av_x)_t \in L_2((0, T) \times (0, \pi))$. Thus, we proved that u is a solution of (1), (2) and (3) on $(0, \delta) \times (0, \pi)$. Estimate (6) gives

$$\|u\|_{X_t} \leqq c(|\varphi_{xx}| + |\psi| + |\varphi_{xx}|^2 + \|f\|_{L_2}) \overset{\text{def}}{=} R_0 \quad \text{for every} \quad t \geqq 0$$

for which the solution exists. Hence, we can continue the above process for $t \geqq \delta$ with the same R, δ and find a solution u on $[0, T] \times [0, \pi]$. As A is contractive on every interval of the length δ, the solution is unique.

4. PERIODIC SOLUTION

Theorem. *Let $f \in C([0, T]. L_2)$ be ω-periodic in t function. Then there exists an ω-periodic solution of equation (1) with boundary conditions (2).*

Proof. Denote $K = \{\Phi = (\varphi, \psi),\ \varphi \in H_2 \cap H_1^0,\ \psi \in L_2,\ S(\varphi, \psi) \leqq r\}$, where

$$r \geqq (1 - e^{-c\omega/2})^{-1} \cdot \omega \|f\|_{C([0,\omega],L_2)} \cdot$$

K is a nonempty closed bounded convex set from $H = H_2 \cap H_1^0 \times L_2$. Now, we define a mapping $T: K \to K$, $T\Phi = (u(\omega, \cdot), u_t(\omega, \cdot))$, where u is a solution from Lemma 2 satisfying the initial data (3), $\Phi = (\varphi, \psi)$. The mapping T will have a fixed point in K, if T is weakly continuous on K (by the Schauder-Tichonov Fixed Point Theorem, see e.g. [5], p. 456). Let $\Phi^k \in K$, $\Phi^k \rightharpoonup \Phi$. By (6) the sequence is bounded, that is, $T\Phi^k = (u^k(\omega, \cdot), u_t^k(\omega, \cdot))$ is bounded in H. In order to prove the weak convergence of $T\Phi^k$, it is sufficient to establish the convergence of $(T\Phi^k, \varphi)$ for every φ from a dense set of H. Thus it is sufficient to prove that the sequences $u_n^k(\omega)$, $(u_n^k)'(\omega)$ converge for $n = 1, 2, \ldots$.

Relation (11) gives

$$(14) \quad u_n^k(\omega) = \tilde{K}_n(\omega)\,\varphi_n^k + K_n(\omega)\,\psi_n^k + \int_0^\omega K_n(t - \tau)\,[f_n(\tau) - n^2 N(u^k)\,u_n^k(\tau)]\,d\tau$$

$$(u_n^k)'(\omega) = \tilde{K}_n'(\omega)\,\varphi_n^k + K_n'(\omega)\,\psi_n^k + \int_0^\omega K_n'(t - \tau)\,[f_n(\tau) - n^2 N(u^k)\,u_n^k(\tau)]\,d\tau\,.$$

By the estimates (6) and (7) the sequences $\{\sigma u_{xt}^k\}$, $\{u_{xx}^k\}$ and $\{u_t^k\}$ are bounded in $L_2((0, \omega) \times (0, \pi))$, $\{u_n^k\}_{k=1}^\infty$, $\{(u_n^k)'\}_{k=1}^\infty$ are compact in $C([0, \omega])$, hence there is a subsequence u^{k_m} such that $u_{x\,t}^{k_m}$ weakly converges in $L_2((0, \omega) \times (0, \pi))$, u_x^k converges in $L_2((0, \omega) \times (0, \pi))$, $u_n^{k_m}$, $(u_n^{k_m})'$ uniformly converge in $C([0, T])$. Then the limits of (14) for $m \to \infty$ exist. It follows from the uniqueness of the solution of (1), (2) and (3) that

$$\lim_{m \to \infty} (u^{k_m}(\omega, x), (u^{k_m})'(\omega, x)) = T\Phi\,.$$

Hence, from every subsequence of $T\Phi^k$ it is possible to choose a sequence weakly converging to $T\Phi$. Then $T\Phi^k \rightharpoonup T\Phi$ in H. There is (by Schauder-Tichonov theorem) a fixed point Φ^0 of T in H, i.e. $(\varphi^0, \psi^0) = \Phi^0 = T\Phi^0 = (u^0(\omega, \cdot), u_t^0(\omega, \cdot))$. The function $u^0(t, x)$ is an ω-periodic solution of (1) and (2).

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Souhrn

O PERIODICKÉM ŘEŠENÍ NELINEÁRNÍ ROVNICE TYČE

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V článku je dokázána existence ω-periodického řešení rovnice

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial^2 u}{\partial x^4} + \gamma \frac{\partial^5 u}{\partial x^4 \partial t} - \tilde{\gamma} \frac{\partial^3 u}{\partial x^2 \partial t} + \delta \frac{\partial u}{\partial t} -$$

$$- \left[\beta + \varkappa \int_0^\pi \left(\frac{\partial u}{\partial x} \right)^2 (\cdot, \xi) \, d\xi + \sigma \int_0^\pi \frac{\partial^2 u}{\partial x \, \partial t} (\cdot, \xi) \frac{\partial u}{\partial x} (\cdot, \xi) \, d\xi \right] \frac{\partial^2 u}{\partial x^2} = f$$

s okrajovými podmínkami

$$u(t, 0) = u(t, \pi) = \frac{\partial^2 u}{\partial x^2} (t, 0) = \frac{\partial^2 u}{\partial x^2} (t, \pi) = 0$$

pro každou ω-periodickou funkci $f \in C([0, \omega], L_2)$.

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