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ON MEASURABLE SOLUTIONS OF A FUNCTIONAL EQUATION AND THEIR APPLICATION TO INFORMATION THEORY

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1. INTRODUCTION

Let $\Gamma_n = \{P = (p_1, \ldots, p_n);\ p_i \geqq 0,\ i = 1, \ldots, n; \sum p_i = 1\}$ for $n \geqq 1$ be a set of all n-complete probability distributions. Let R be the set of all real numbers and let $I = [0, 1]$.

Consider measurable functions $f, g, h, k: I \to R$ satisfying the system of functional equations

$$\sum_i \sum_j h(x_i y_j) = \sum_j \sum_i f(x_i)\, g(y_j) + \sum_i k(x_i)\,, \tag{1.1}$$

where $X = (x_1, \ldots, x_n) \in \Gamma_n$, $Y = (y_1, \ldots, y_m) \in \Gamma_m$.

The continuous solutions of (1.1) were given by Taneja [4].

The objective of this paper is to find the measurable solutions of (1.1). As an application, a joint characterization of Shannon's entropy and entropy of type β is given.

2. MEASURABLE SOLUTIONS OF (1.1)

In this section we will find the measurable solutions of system (1.1). This is done in the following theorem.

Theorem 1. *If f, g, h and k are measurable solutions of (1.1) for $X \in \Gamma_n$, $Y \in \Gamma_m$ where $n, m = 2, 3$, then they are given by one of the following set of solutions for $x \in [0, 1]$. 1st set of solutions:*

$$h(x) = Bx + Ax \log x\,, \quad f(x) = Cx\,, \tag{2.1}$$

$$g(x) = Dx + A/Cx \log x\,, \quad k(x) = (B - CD)\, x + Ax \log x\,; \tag{2.2}$$

2nd set of solutions:

$$h(x) = Bx + A(x^\beta - x)\,, \quad f(x) = Cx^\beta\,, \tag{2.3}$$

$$g(x) = Dx + (A/C)\,(x^\beta - x)\,, \quad k(x) = (B - A)\, x + (A - CD)\, x^\beta\,, \tag{2.4}$$

where A, B, C and D are arbitrary constants and $\beta > 0$ $(1 \neq \beta > 0)$ is a parameter.

In addition to the above two sets of solutions, we also get the trivial solution:

$$h(x) = Bx\,, \quad f(x) = \text{arbitrary}\,, \quad g(x) = Dx \quad \text{and}$$

$$k(x) = Bx - D\,f(x)\,.$$

Proof. Substituting $Y = (y, u, l - y - u) \in \Gamma_3$ and $Y = (y + u, l - y - u) \in \Gamma_2$ in (1.1), we get respectively

$$\sum_i (h(x_i y) + h(x_i u) + h(x_i(1 - y - u)) = \tag{2.5}$$

$$= \sum_i f(x_i)\,(g(y) + g(u) + g(1 - y - u)) + \sum_i k(x_i)$$

$$\sum_i (h(x_i(y + u) + h(x_i(1 - y - u))) = \sum_i f(x_i)\,(g(y + u) + \tag{2.6}$$

$$+ g(1 - y - u) + \sum_i k(x_i)\,.$$

Subtracting (2.6) from (2.5), we obtain

$$\sum_i (h(x_i y) + h(x_i u) - h(x_i(y + u)) = \sum_i f(x_i)\,(g(y) + g(u) - g(y + u))\,. \tag{2.7}$$

Let us define for $X \in \Gamma_n$, $n = 2, 3$, $t \in I = [0, 1]$:

$$A_X(t) = \sum_i h(x_i t) - \sum_i f(x_i)\,g(t)\,. \tag{2.8}$$

By virtue of (2.8) it is easy to see that $A_X(\cdot)$ is additive on I, i.e.

$$A_X(y + u) = A_x(y) + A_x(u)\,. \tag{2.9}$$

It now follows from a result of Daroczy and Losonczi [3] that

$$A_X(t) = t\,A_x(1)\,, \quad t \in I \tag{2.10}$$

is a measurable solution.

In order to obtain the expression for A_x (1), we will find the expression for the function

$$\sum_i h(x_i) - \sum_i f(x_i)\,g(1)\,. \tag{2.11}$$

Substituting $Y = (1, 0)$ and $Y = (1, 0, 0)$ in (1.1) we get respectively

$$\sum_i h(x_i) + n\,h(0) = \sum_i f(x_i)\,(g(1) + g(0)) + \sum_i k(x_i)\,, \tag{2.12}$$

$$\sum_i h(x_i) + 2n\,h(0) = \sum_i f(x_i)\,(g(1) + 2\,g(0)) + \sum_i k(x_i)\,. \tag{2.13}$$

Subtracting (2.12) from (2.13) we obtain

$$n\,h(0) = \sum_i f(x_i)\,g(0)\,. \tag{2.14}$$

Using (2.14), we transform (2.12) into

$$(2.15)\qquad \sum_i h(x_i) = \sum_i f(x_i)\, g(1) + \sum_i k(x_i)$$

From (2.15) and (2.10) we get

$$(2.16)\qquad \sum_i h(x_i t) - \sum_i f(x_i)\, g(t) = t \sum_i k(x_i)$$

for all $X \in \Gamma_n$, $n = 2, 3$ and $t \in I$.

Let us substitute $X = (x, v, 1 - x - v) \in \Gamma_3$ and $X = (x + v, 1 - x - v) \in \Gamma_2$ in (2.16). We obtain respectively

$$(2.17)\qquad h(xt) + h(vt) + h((1 - x - v)\, t) - (f(x) + f(v) + f(1 - x - v))\, g(t) = \\ = t(k(x) + k(v) - k(1 - x - v)),$$

$$(2.18)\qquad h((x + v)\, t) + h((1 - x - v)\, t) - (f(x + v) + f(1 - x - v))\, g(t) = \\ = t(k(x + v) - k(1 - x - v)).$$

From (2.18) and (2.17), we get

$$(2.19)\qquad h(xt) + h(vt) - h((x + v)\, t) = (f(x) + f(v) - f(x + v))\, g(t) + \\ + t(k(x) + k(v) - k(x + v)).$$

For $t \in I$, let us define

$$(2.20)\qquad B_t(w) = h(wt) - f(w)\, g(t) - t\, h(w).$$

Then using (2.20), we can write (2.19) in the form

$$(2.21)\qquad B_t(x + v) = B_t(x) + B_t(v), \quad \text{for} \quad x, v, x + v \in [0, 1].$$

Again using the result of Daroczy and Losonczi [3], we have

$$(2.22)\qquad B_t(x) = x\, B_t(1).$$

By substituting $X = (1, 0)$ and $X = (1, 0, 0)$ in (2.16) we get the relation

$$(2.23)\qquad h(t) = f(1)\, g(t) + t\, k(1).$$

Using (2.23), (2.22) becomes

$$(2.24)\qquad h(xt) = f(x)\, g(t) + t\, k(x), \quad \text{for all} \quad x, t \in I.$$

Dividing (2.24) by xt $(x \neq 0,\ t \neq 0)$, we get

$$\frac{h(xt)}{xt} = \frac{f(x)}{x}\frac{g(t)}{t} + \frac{k(x)}{x}.$$

Let $h_1(x) = h(x)/x$, $f_1(x) = f(x)/x$, $g_1(t) = g(t)/t$ and $k_1(x) = k(x)/x$.

Then we have

(2.25) $$h_1(xt) = f_1(x)\, g_1(t) + k_1(x) .$$

Putting first $x = 1$ and then $t = 1$ in (2.25) we get

(2.26) $$h_1(t) = f_1(1)\, g_1(t) + k_1(1) ,$$
$$h_1(1) = f_1(1)\, g_1(1) + k_1(1) .$$

If $f_1(1) = 0$ then (2.26) implies

$$h_1(t) = h_1(1) \quad \text{or} \quad h(t) = t\, h_1(1) = At \quad \text{where} \quad A = h_1(1) = h(1) .$$

In this case h is a homogeneous linear function. Now suppose that $f_1(1) \neq 0$. Then from (2.25) and (2.26) we obtain

$$h_1(xt) = \frac{f_1(x)}{f_1(1)}\, h_1(t) + k_1(x) - \frac{f_1(x)}{f_1(1)}\, k_1(1) .$$

Define $f_2(x) = f_1(x)/f_1(1)$, $k_2(x) = k_1(x) - f_2(x)\, k_1(1)$. Then we have from the above equation that

(2.28) $$h_1(xt) = f_2(x)\, h_1(t) + k_2(x) .$$

Since f, g, h, k are measurable functions, hence h_1, f_2, h_1 and k_2 are also measurable.

The general measurable solution of (2.28) with h_1, f_2, k_2 measurable is given by (see Aczel [1])

(2.29) $$h_1(x) = h_0(x) + \alpha \,; \quad f_2(x) = 1 \,; \quad k_2(x) = h_0(x)$$

and

(2.30) $$h_1(x) = \gamma\, e^{h_0(x)} + \alpha , \quad f_2(x) = e^{h_0(x)} , \quad k_2(x) = \alpha(1 - e^{h_0(x)})$$

with an additional trivial solution

(2.31) $$h_1(x) = \alpha , \quad f_2(x) \quad \text{arbitrary} , \quad k_2(x) = \alpha(1 - f_2(x))$$

where $\gamma \neq 0$ and α are arbitrary constants and h_0 is an arbitrary measurable solution of the equation

(2.32) $$h_0(xt) = h_0(x) + h_0(t) .$$

However the most general measurable solution of (2.32) is

(2.33) $$h_0(x) = A \log x$$

where A is an arbitrary constant.

Thus the solutions (2.29), (2.30) and (2.31) together with (2.33), (2.28) and (2.25) give the required set of solutions.

3. APPLICATIONS TO INFORMATION THEORY

Shannon's measure of information is defined as

$$H(P) = -\sum_i p_i \log p_i\,, \quad P \in \Gamma_n\,. \tag{3.1}$$

A well known generalization of (3.1) is covered by the entropy of type β and is given as (see [2])

$$H^\beta(P) = (2^{1-\beta} - 1)^{-1} (\sum_i p_i^\beta - 1)\,, \quad \beta \neq 1,\ \beta > 0\,, \quad P \in \Gamma_n\,. \tag{3.2}$$

In terms of measurable solution of (1.1), we can define $H(P)$ or $H^\beta(P)$ as

$$H(P) = \sum_i h(p_i) \tag{3.3}$$

under suitable boundary and normalization conditions.

In the following theorem a joint characterization of (3.1) and (3.2) is given.

Theorem 2. *The entropies of distribution P under the conditions $h(1) = h(0)$ and $h(1/2) = 1/2$ corresponding to the measurable solutions are* (3.1) *and* (3.2), *respectively.*

Proof. Putting $x = 0$ in (2.1) and (2.3) we have $h(0) = h(1) = 0$. Using $h(1/2) = 1/2$, the constant A becomes -1 and $(2^{1-\beta} - 1)^{-1}$, respectively. The result follows from (3.3).

References

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Souhrn

MĚŘITELNÁ ŘEŠENÍ JISTÉ FUNKCIONÁLNÍ ROVNICE A JEJICH APLIKACE V TEORII INFORMACE

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V článku jsou nalezena měřitelná řešení jisté funkcionální rovnice se čtyřmi neznámými funkcemi. Jako jejich aplikace je dána společná charakterizace Shannonovy entropie a entropie β.

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