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FURTHER CONVERGENCE RESULTS FOR TWO QUADRATURE RULES FOR CAUCHY TYPE PRINCIPAL VALUE INTEGRALS

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1. INTRODUCTION

Several quadrature rules are available for the evaluation of Cauchy type principal value integrals of the form

$$I(f;\lambda) = \int_{-1}^{1} w(x)\,\frac{f(x)}{x-\lambda}\,dx\,,\quad \lambda \in (-1,1)\,, \tag{1.1}$$

where the integration interval is assumed finite and, without loss of generality, it is further assumed to coincide with $[-1, 1]$. In the sequel, we will also assume that the weight function $w(x)$ is a non-negative integrable function and, moreover, that the integrand $f(x)$ is at least a continuous function along $[-1, 1]$ ($f \in C[-1, 1]$). Finally, we will assume that the integral $I(f; \lambda)$ exists in the principal value sense.

Among these rules, the most useful is the Gauss quadrature rule, based on the corresponding quadrature rule for ordinary integrals

$$\int_{-1}^{1} w(x)\,g(x)\,dx = \sum_{i=1}^{n} \mu_{i,n}\,g(x_{i,n}) + E_n(g)\,, \tag{1.2}$$

where $x_{i,n}$ are the roots of the polynomial $p_n(x)$ (the polynomial of degree n of the system of orthonormal polynomials associated with the given weight function $w(x)$ and the interval $[-1, 1]$), $\mu_{i,n}$ are the corresponding weights (or Christoffel numbers) and $E_n(g)$ is the error term. For the approximation of $I(f; \lambda)$, the Gauss quadrature rule (1.2) takes the form

$$Q_n^{\dagger}(f;\lambda) = \sum_{i=1}^{n} \mu_{i,n}\,\frac{f(x_{i,n})}{x_{i,n}-\lambda} + \frac{q_n(\lambda)}{p_n(\lambda)}\,f(\lambda)\,,\quad \lambda \neq x_{i,n}\,,\quad i = 1(1)\,n\,, \tag{1.3}$$

where $q_n(\lambda)$ is defined by

$$q_n(\lambda) = \int_{-1}^{1} w(x)\,\frac{p_n(x)}{x-\lambda}\,dx\,. \tag{1.4}$$

This quadrature rule is due to Hunter [7], who first derived it (generalizing a series of previous results) for the case when $w(x) = 1$. In the general case, this rule is due to the author, whose results are reported in [8, 9]. In the case when λ coincides with some of the nodes $x_{i,n}$, suppose $x_{j,n}$, then $Q_n^\dagger(f; \lambda)$ is given by [9]

$$Q_n^\dagger(f; x_{j,n}) = \sum_{\substack{i=1 \\ i \neq j}}^{n} \mu_{i,n} \frac{f(x_{i,n})}{x_{i,n} - x_{j,n}} + \mu_{j,n} f'(x_{j,n}) + \nu_{j,n} f(x_{j,n}), \tag{1.5}$$

where

$$\nu_{j,n} = [q_n'(x_{j,n}) - \tfrac{1}{2}\mu_{j,n}\, p_n''(x_{j,n})]/p_n'(x_{j,n}). \tag{1.6}$$

A more complicated expression for $Q_n^\dagger(f; x_{j,n})$ was derived by Elliott and Paget [5], whose notation is used here.

Another method of approximating $I(f; \lambda)$ is to replace $f(x)$ by the corresponding Lagrange interpolation polynomial $L_n(f; x)$ (of degree $n - 1$), based on the nodes $x_{i,n}$. Then $I(f; \lambda)$ is approximated by

$$Q_n(f; \lambda) = \sum_{i=1}^{n} A_{i,n}(\lambda) f(x_{i,n}), \tag{1.7}$$

where

$$A_{i,n}(\lambda) = \begin{cases} \dfrac{q_n(x_{i,n}) - q_n(\lambda)}{p_n'(x_{i,n})(x_{i,n} - \lambda)} & \text{if } \lambda \neq x_{i,n}, \quad i = 1(1)\,n, \\[2ex] q_n'(x_{i,n})/p_n'(x_{i,n}) & \text{if } \lambda = x_{i,n}, \quad i = 1(1)\,n. \end{cases} \tag{1.8}$$

This quadrature rule was suggested by Korneichuk [11] and, independently, by Paget and Elliott [12].

Both quadrature rules (1.3) and (1.7) present the disadvantage that the quantity $x_{i,n} - \lambda$ in the denominators tends to infinity as $\lambda \to x_{i,n}$ (although neither $Q_n^\dagger(f; \lambda)$ nor $Q_n(f; \lambda)$ tend to infinity in this case). Moreover, it can be mentioned that $Q_n^\dagger(f; \lambda)$ is exact whenever $f \in P^{2n}$ (that is, it is a polynomial of degree up to $2n$), whereas $Q_n(f; \lambda)$ is exact whenever $f \in P^{n-1}$.

One more obvious fact, which nevertheless seems not to be noticed up to now, is that $Q_n(f; \lambda)$ results from $Q_n^\dagger(f; \lambda)$ simply by approximating $f(\lambda)$ (only in the second term of the right hand side of (1.3)) by $L_n(f; \lambda)$ or $f'(x_{j,n})$ (in the second term of the right hand side of (1.5)) by $L_n'(f; x_{j,n})$. Thus, the quadrature rule $Q_n(f; \lambda)$ is in a sense an "approximation" of $Q_n^\dagger(f; \lambda)$. This can be considered as an explanation of its low accuracy.

Several convergence results are available for the quadrature rules $Q_n^\dagger(f; \lambda)$ and $Q_n(f; \lambda)$ for the approximation of $I(f; \lambda)$. These will be reported in the next two sections and rates of convergence will be established. Particularly, the convergence results for $Q_n(f; \lambda)$ will be seen to require less restrictive assumptions than those already assumed in the literature.

2. THE GAUSS QUADRATURE RULE

We consider first the Gauss quadrature rule $Q_n^\dagger(f;\lambda)$. Elliott and Paget [5] proved the convergence of this rule provided that $f' \in C[-1, 1]$. This result cannot in general be improved. On the basis of this result of [5], the results of Tsamasphyros and Theocaris [17] are seen not to be correct. This can also be verified directly. Moreover, Elliott [2] established sufficient conditions for the convergence of $Q_n^\dagger(f;\lambda)$ to $I(f;\lambda)$ if $f(x)$ satisfies the Hölder (or, equivalently, the Lipschitz) condition along $[-1, 1]$ ($f \in H^\mu[-1, 1]$), that is

$$(2.1)\qquad |f(x_2) - f(x_1)| \leqq A|x_2 - x_1|^\mu . \quad \forall (x_1, x_2) \in [-1, 1], \quad \mu \in (0, 1],$$

where A is a constant independent of x_1, x_2. The conditions established in [2] are rather restrictive and useful only in special cases.

Moreover, Elliott and Paget [5] proved that the error

$$(2.2)\qquad R_n^\dagger(f;\lambda) = I(f;\lambda) - Q_n^\dagger(f;\lambda)$$

is given by

$$(2.3)\qquad R_n^\dagger(f;\lambda) = E_n(g),$$

where

$$(2.4)\qquad E_n(g) = \int_{-1}^{1} w(x)\, g(x, \lambda)\, \mathrm{d}x - \sum_{i=1}^{n} \mu_{i,n}\, g(x_{i,n}, \lambda)$$

with

$$(2.5)\qquad g(x, \lambda) = \begin{cases} [f(x) - f(\lambda)]/(x - \lambda) & \text{if} \quad x \neq \lambda, \\ f'(\lambda) \quad \text{if} \quad x = \lambda, \end{cases}$$

and hence, $R_n^\dagger(f;\lambda) \to 0$ as $n \to \infty$ provided that $f' \in C[-1, 1]$. Clearly, it is sufficient that $f' \in C$ only in a neighbourhood $[c, d]$ of λ, provided that $f \in C[-1, 1]$.

For the rate of convergence of $Q_n^\dagger(f;\lambda)$ to $I(f;\lambda)$, (2.4) yields

Theorem 1. *If $f \in C^{p_1}[-1, 1]$ $(p_1 \geqq 0)$, $f^{(p_1)} \in H^{\mu_1}[-1, 1]$ and, moreover, $f \in C^{p_2}[c, d]$, where $[c, d] \subseteq [-1, 1]$, with $p_2 \geqq 1$, and $f^{(p_2)} \in H^{\mu_2}[c, d]$, then*

$$(2.6)\qquad R_n^\dagger(f;\lambda) \leqq A_1 n^{-\gamma},$$

where A_1 is independent of n and

$$(2.7)\qquad \gamma = \min(p_1 + \mu_1,\ p_2 + \mu_2 - 1)$$

for all $\lambda \in (c, d)$.

Proof. Because of the definition (2.5) of $g(x, \lambda)$ and the fact that $f \in C^{p_1}[-1, 1]$ $(p_1 \geqq 0)$, $f^{(p_1)} \in H^{\mu_1}[-1, 1]$, we conclude first of all that $g \in C^{p_1}[-1, c'] \cup [d', 1]$, $g^{(p_1)} \in H^{\mu_1}[-1, c'] \cup [d', 1]$ where $\lambda \in (c', d')$. Clearly, for a given $\lambda \in (c, d)$, it is possible to find a subinterval $[c', d']$ of $[c, d]$ such that $\lambda \in (c', d')$. By taking into account the Taylor series of $f(x)$ at the point λ

$$(2.8) \qquad f(x) = f(\lambda) + (x - \lambda) f'(\lambda) + \dots + \frac{(x - \lambda)^{p_2 - 1}}{(p_2 - 1)!} f^{(p_2 - 1)}(\lambda) + \\ + \frac{(x - \lambda)^{p_2}}{p_2!} h(x, \lambda), \quad x, \lambda \in [c, d],$$

where $h(x, \lambda)$ should be a Hölder-continuous function, $h \in H^{\mu_2}[c, d]$, as well as the definition (2.5) of $g(x, \lambda)$, we conclude that

$$(2.9) \quad g(x, \lambda) = f'(\lambda) + \dots + \frac{(x - \lambda)^{p_2 - 2}}{(p_2 - 1)!} f^{(p_2 - 1)}(\lambda) + \frac{(x - \lambda)^{p_2 - 1}}{p_2!} h(x, \lambda),$$

and $g \in C^{p_2 - 1}[c, d]$, $g^{(p_2 - 1)} \in H^{\mu_2}[c, d]$.

Another proof of this statement can be made as follows. We consider a polynomial $p_n^*(f; x)$ (of degree n) for which we have

$$(2.10) \qquad \max_{x \in [c,d]} |f(x) - p_n^*(f; x)| \leqq A_2 n^{-(p_2 + \mu_2)}$$

and

$$(2.11) \qquad \max_{x \in [c,d]} |f'(x) - p_n^{*\prime}(f; x)| \leqq A_2 n^{1 - (p_2 + \mu_2)}.$$

(Here and in the sequel A_i denote constants independent of n.) The existence of the polynomial $p_n^*(f; x)$ is assured by a theorem reported by Kalandiya [10, p. 108]. We define also a polynomial $q_n^*(f; x, \lambda)$ (of degree $n - 1$) by

$$(2.12) \qquad q_n^*(f; x, \lambda) = \begin{cases} [p_n^*(f; x) - p_n^*(f; \lambda)]/(x - \lambda) & \text{if} \quad x \neq \lambda, \\ p_n^{*\prime}(f; \lambda) \quad \text{if} \quad x = \lambda. \end{cases}$$

Then we have

(2.13)

$$g(x; \lambda) - q_n^*(f; x, \lambda) = \begin{cases} \{[f(x) - p_n^*(f; x)] - [f(\lambda) - p_n^*(f; \lambda)]\}/(x - \lambda) & \text{if} \quad x \neq \lambda, \\ f'(\lambda) - p_n^{*\prime}(f; \lambda) \quad \text{if} \quad x = \lambda. \end{cases}$$

By applying the mean value theorem, we find further that

$$(2.14) \qquad g(x; \lambda) - q_n^*(f; x, \lambda) = f'(\xi(x, \lambda)) - p_n^{*\prime}(f; \xi(x, \lambda)),$$

where $\xi \in [c, d]$ provided that $x, \lambda \in [c, d]$. Then, because of (2.11), it is concluded that

$$(2.15) \qquad \max_{x \in [c,d]} |g(x, \lambda) - q_n^*(f; x, \lambda)| \leqq A_2 n^{-(p_2 - 1) - \mu_2}.$$

Now, from Bernstein's theorems of the approximation theory [1, p. 201] it follows that $g \in C^{p_2 - 1}[c, d]$, $g^{(p_2 - 1)} \in H^{\mu_2}[c, d]$.

These properties of $g(x, \lambda)$ along $[c, d]$ together with the corresponding properties of the same function along $[-1, c'] \cup [d', 1]$ (with $[c', d'] \subset [c, d]$) reveal that $g \in C^{p_3}[-1, 1]$, where $p_3 = \min(p_1, p_2 - 1)$, and $g^{(p_3)} \in H^{\mu_3}[-1, 1]$, where $\mu_3 = \mu_1$ if $p_1 < p_2 - 1$, $\mu_3 = \mu_2$ if $p_1 > p_2 - 1$ and $\mu_3 = \min(\mu_1, \mu_2)$ if $p_1 = p_2 - 1$. Hence, for the polynomial $p_n^*(g; x, \lambda)$ (of degree $n - 1$) of the best uniform approximation of $g(x)$ along $[-1, 1]$ we will have from a corollary of Jackson's theorem [10, p. 108; 13, pp. 22–23]:

$$G_n(g) \equiv \max_{x \in [-1,1]} |g(x, \lambda) - p_n^*(g; x, \lambda)| \leqq A_3 n^{-\gamma}, \tag{2.16}$$

where γ was defined by (2.7).

Now, since the error $R_n^\dagger(f; \lambda)$ in (2.2) is equal in the case of Gaussian quadrature rules (because of (2.3)) to

$$E_n(g) = \int_{-1}^{1} w(x) [g(x, \lambda) - L_n(g; x, \lambda)] \, dx, \tag{2.17}$$

where $L_n(g; x, \lambda)$ denotes the polynomial of degree $n - 1$ interpolating $g(x, \lambda)$ at the nodes $x_{i,n}$ (with respect to the variable x) and $w(x)$ is assumed a non-negative weight function, we have (because of (2.3))

$$R_n^\dagger(f; \lambda) \leqq \int_{-1}^{1} w(x) |g(x, \lambda) - L_n(g; x, \lambda)| \, dx. \tag{2.18}$$

Furthermore, since [13, p. 104]

$$\int_{-1}^{1} w(x) |g(x, \lambda) - L_n(g; x, \lambda)| \, dx \leqq 2 \, G_n(g) \int_{-1}^{1} w(x) \, dx, \tag{2.19}$$

we obtain, because of (2.16) and (2.18), the estimate (2.6) for $R_n^\dagger(f; \lambda)$. The same can also be proved by taking into account that the error term $E_n(g)$ in the quadrature rule (1.2) fulfils

$$E_n(g) = E_n(g - p_n^*(g)) \tag{2.20}$$

and using (2.16) for the estimation of $R_n^\dagger(f; \lambda) = E_n(g)$. Finally, since (1.2) is exact for integrands $g(x)$ which are polynomials of degree $2n - 1$, we can use the polynomial $p_{2n}^*(g; x, \lambda)$ in (2.20), but this will not influence the exponent γ of the rate of convergence of $Q_n^\dagger(f; \lambda)$ to $I(f; \lambda)$.

3. THE INTERPOLATORY QUADRATURE RULE

We consider now the interpolatory quadrature rule (based on the nodes of the corresponding Gaussian rule) $Q_n(f; \lambda)$. Convergence results for this rule were obtained in the case of the weight function

$$w(x) = (1 - x)^{\alpha} (1 + x)^{\beta}, \quad \alpha, \beta > -1 \tag{3.1}$$

by Sanikidze [14], who proved convergence of $Q_n(f; \lambda)$ to $I(f; \lambda)$ if $f \in H^{\mu}[-1, 1]$ in the special case when $\alpha = \beta = -\frac{1}{2}$ (and found also the rate of convergence), by Elliott and Paget [3] in the general case of the weight function (3.1) if $f \in H^1[-1, 1]$, that is, if $f(x)$ possesses a bounded derivative in $[-1, 1]$, and, finally, by Elliott and Paget [4] and, independently, by Sheshko [15] if $f \in H^{\mu}[-1, 1]$. It should be mentioned that the methods used in [4] and [15] for the proof of the convergence are quite different. Moreover, the results of [15] are weaker than those of [4] in the cases when $\max(\alpha, \beta) > -\frac{1}{2}$ since convergence was proved in [15] only if $\mu > \max(\alpha, \beta) + \frac{1}{2}$ in this case although convergence occurs for all values of μ ($\mu \in (0, 1]$) as is clear from the results of [4]. Yet, both methods used in [4] and [15], and particularly that used in [4] and based on the results of [3], are rather complicated.

In the case of a general non-negative integrable weight function $w(x)$, Korneichuk [11] investigated the convergence of $Q_n(f; \lambda)$ to $I(f; \lambda)$ for $f^{(1)} \in C[-1, 1]$ and proved it under appropriate restrictions. Moreover, Elliott and Paget [5] proved by an elegant method the convergence of $Q_n(f; \lambda)$ to $I(f; \lambda)$ under a moderately restrictive assumption on $q_n(\lambda)$, defined by (1.4), for all $f \in H^{\mu}[-1, 1]$ provided that $\mu \in (\frac{1}{2}, 1]$. Here we will generalize this result for all $\mu \in (0, 1]$ and we will give also convergence rates. Clearly, the method used here presents considerable similarities with the methods used in the aforementioned references, but it leads to stronger results, which can be expressed in the form of the following theorem:

Theorem 2. *Under the continuity assumptions of Theorem* 1 *and the further assumptions that*

$$A_4 < w(x) \leqq A_5, \quad x \in [c, d] \subset [-1, 1], \tag{3.2}$$

and

$$|p_n(x)| \leqq A_5, \quad x \in [c, d], \tag{3.3}$$

it follows that

$$R_n(f; \lambda) \leqq A_6 n^{-\delta}, \tag{3.4}$$

where

$$\delta = \min(p_1 + \mu_1, p_2 + \mu_2 - \varepsilon), \tag{3.5}$$

(ε *being an arbitrarily small positive constant*) *for all* $\lambda \in (c, d)$.

Proof. We know [5] that

$$R_n(f; \lambda) = I(r_n(f); \lambda), \tag{3.6}$$

where

$$r_n(f; x) = f(x) - L_n(f; x) \tag{3.7}$$

and $L_n(f; x)$ is the polynomial (of degree $n - 1$) interpolating $f(x)$ at the nodes $x_{i,n}$ (as already mentioned). Hence, we obtain

$$(3.8)\quad R_n(f; \lambda) = \int_{-1}^{c'} w(x) \frac{r_n(f; x)}{x - \lambda} \mathrm{d}x + \int_{c'}^{d'} w(x) \frac{r_n(f; x)}{x - \lambda} \mathrm{d}x + \int_{d'}^{1} w(x) \frac{r_n(f; x)}{x - \lambda} \mathrm{d}x ,$$

$$\lambda \in (c', d') \subset [c', d'] \subset [c, d] .$$

But, because of (2.19) and the fact that $|x - \lambda| \geqq \min(\lambda - c', d' - \lambda)$ when $x \in \in [-1, c'] \cup [d', 1]$, we conclude, since $f \in C^{p_1}[-1, 1]$ and $f^{(p_1)} \in H^{\mu_1}[-1, 1]$, that

$$(3.9)\qquad \int_{-1}^{c'} w(x) \left|\frac{r_n(f; x)}{x - \lambda}\right| \mathrm{d}x + \int_{d'}^{1} w(x) \left|\frac{r_n(f; x)}{x - \lambda}\right| \mathrm{d}x \leqq A_7 n^{-(p_1+\mu_1)}.$$

Moreover, on the basis of Theorems 6.3 and 6.4 of Freud [6, pp. 114–116], we also conclude that under the assumptions of Theorem 2

$$(3.10)\qquad \max_{x \in [c', d']} |r_n(f; x)| \leqq A_8 n^{\varepsilon/3 - (p_2+\mu_2)} .$$

Then, by taking into account a lemma due to Kalandiya [10, pp. 105–107] together with (3.10), we conclude that

$$(3.11)\qquad \max_{x, \lambda \in [c', d']} \frac{|r_n(f; x) - r_n(f; \lambda)|}{|x - \lambda|^{\varepsilon/3}} \leqq A_9 n^{\varepsilon - (p_2+\mu_2)} .$$

It seems clear from the proof of the aforementioned lemma of Kalandiya, that its results can be slightly generalized to assure the validity of the estimate (3.11).

Now, following a standard device [4], we can write

$$(3.12)\quad \int_{c'}^{d'} w(x) \frac{r_n(f; x)}{x - \lambda} \mathrm{d}x = \int_{c'}^{d'} w(x) \frac{r_n(f; x) - r_n(f; \lambda)}{x - \lambda} \mathrm{d}x + r_n(f; \lambda) \int_{c'}^{d'} \frac{w(x)}{x - \lambda} \mathrm{d}x$$

and on this basis of (3.10) and (3.11) we see (exactly as in [4]) that

$$(3.13)\qquad \left|\int_{c'}^{d'} w(x) \frac{r_n(f; x)}{x - \lambda} \mathrm{d}x\right| \leqq A_{10} n^{\varepsilon - (p_2+\mu_2)} .$$

Moreover, from (3.9) and (3.13), the estimate (3.4) is obtained for the error $R_n(f; \lambda)$ and this completes the proof of Theorem 2. We can also mention that, obviously, for all $\lambda \in (c, d)$ there exist constants c', d' such that $\lambda \in (c', d') \subset [c', d'] \subset [c, d]$ $(c < c' < \lambda < d' < d)$. Of course, it is possible that $[c, d] \equiv [-1, 1]$. Then we obtain the following corollary of Theorem 2:

Corollary. *If* $f \in H^{\mu}[-1, 1]$ *and* (3.2) *and* (3.3) *are satisfied for all* $[c, d] \subset$

$\subset [-1, 1]$, *then* $\lim_{n\to\infty} R_n(f; \lambda) = 0$ *for all* $\lambda \in (-1, 1)$. *Moreover, if* $f \in C^p[-1, 1]$ $(p \geqq 0)$ *and* $f^{(p)} \in H^\mu[-1, 1]$, *then*

$$R_n(f; \lambda) \leqq A_{11} n^{-(p+\mu-\varepsilon)}, \quad \lambda \in (-1, 1). \tag{3.14}$$

Finally, we can mention that the aforementioned theorems reported by Freud [6, pp. 114–116] are generalizations of a previous theorem reported by Szegö [16, pp. 343–344].

Now we will consider the special case of the weight function (3.1), associated with the Jacobi polynomials and the Gauss-Jacobi quadrature rule. In this case (3.2) holds true for every $[c, d] \subset [-1, 1]$. Moreover, for the normalized Jacobi polynomials $p_n^{(\alpha,\beta)}(x)$ (including the Chebyshev and the Legendre polynomials as special cases) the following inequality, reported by Freud [6, pp. 45–46], holds true:

$$(1 - x^2)^\varrho \left|p_n^{(\alpha,\beta)}(x)\right| \leqq A_{12}, \quad x \in (-1, 1), \tag{3.15}$$

ϱ being a positive constant. Hence, (3.3) is also satisfied for all $[c, d] \subset [-1, 1]$ in this case (with an appropriate value assigned to A_5) and the above results are valid in the case of the weight function (3.1). This result is in complete agreement with the results of Elliott and Paget [4] (for the same weight function), but the proof of convergence given here seems somewhat simpler that than in [4] since the latter was based on the estimation of $\sum_{i=1}^{n} |A_{i,n}(\lambda)|$, which is not very simple [3]. Of course, in the special case under consideration, we can use directly Theorem 14.4 of Szegö [16, pp. 333–337] for the proof of (3.10), without reference to the assumptions (3.2) and (3.3).

Clearly, all the above results are not valid for $\lambda = \pm 1$, but this is of no interest for the evaluation of Cauchy type principal value integrals along $[-1, 1]$ since in these cases these integrals do not exist (even in the principal value sense) or, if they do, they do not require the concept of the principal value of an integral for their definition (they are simple singular integrals).

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Souhrn

DALŠÍ VÝSLEDKY O KONVERGENCI DVOU KVADRATURNÍCH FORMULÍ PRO HLAVNÍ HODNOTU INTEGRÁLŮ CAUCHYOVA TYPU

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Jsou dokázány nové výsledky o konvergenci a rychlosti konvergence pro dvě známé kvadraturní formule pro numerický výpočet hlavní hodnoty integrálů Cauchyova typu na konečném intervalu, jmenovitě Gaussovy kvadraturní formule a podobné interpolační formule, používající tytéž uzly jako Gaussova. Hlavní výsledek se týká konvergence interpolační formule pro funkce, které splňují Hölderovu podmínku s exponentem menším nebo rovným $\frac{1}{2}$. Získané výsledky doplňují řadu dřívějších výsledků o konvergenci zmíněných formulí.

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