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ON COUPLED THERMOELASTIC VIBRATION OF GEOMETRICALLY NONLINEAR THIN PLATES SATISFYING GENERALIZED MECHANICAL AND THERMAL CONDITIONS ON THE BOUNDARY AND ON THE SURFACE

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INTRODUCTION

At the beginning of our paper we consequently derive the vibration problem of a geometrically nonlinear plate in coupled thermoelasticity from the three-dimensional equilibrium condition coupled with the three-dimensional heat condition equation. The equations obtained here represent a novel result in coupled thermoelasticity and plate theory.

Thereby we consider boundary conditions in subdifferential form between the bending moments, the shearing forces and the velocity w' of the vertical displacement and $\partial w'/\partial n$, respectively, as well as between the in-plane stress resultants N and the velocity $\{u'_1, u'_2\}$ of the horizontal displacements[1]). Furthermore, we derive thermal subgradient conditions on the boundary and in the domain (i.e. on the upper and

[1]) Condition upon w, $\partial w/\partial n$, u instead of w', $\partial w'/\partial n$, u' seem to be more "natural", but nonlinear hyperbolic problems with nonclassical boundary conditions of such a type present too hard mathematical difficulties (certain monotopicity properties of the Yosida approximation for subgradients cannot be used), so that we do not know any result of such a type.

lower surface of the plate) including classical (free convexion, isolation) and non-classical conditions (heat control). Some examples illustrate our formulations.

In Section 6 we prove an existence theorem for the problem in a weak variational formulation. Furthermore, we show (Section 7) that the solution is uniquely determined and depends continuously on the given data. Thereby the consequent derivation of the problem appears to be very profitable. Thus some thermoelastic coupling terms (strong nonlinearities and higher order time and spatial derivatives) can be compensated.

Nonlinear isothermal plate problems with non-classical boundary conditions (statistical and dynamical) for von Karman's equations are considered in [8], [9] and [13]. In the formulation without introducing a stress function, existence and uniqueness results for the static problem with certain zero boundary conditions are contained in [3] and [14]. In the dynamical case we do not know any result for generalized boundary conditions between the horizontal stresses N and the horizontal displacements velocity u'. In [4] classical zero boundary conditions between N and u' and either for w' of for $\partial w'/\partial n$ are assumed.

Statements of problem of thermoelasticity are derived in vaiious publications (cf. for example [19], [7], [16]–[18], [1]) and are mathematically treated for the three-dimensional case in [5] and [2]. For thermoelastic plates we only know uncoupled problems with classical boundary conditions.

1. NOTATIONS

Let $\Omega \subset R^2$ be a bounded domain having a Lipschitz-continuous boundary Γ (i.e. $\Omega \in \mathfrak{A}^{0,1}$, cf. [15]). We denote by $L^p(\Omega)$ the space of all measurable functions integrable to the power p with the usual norm. The norm and the scalar product in $L^2(\Omega)$ is denoted by $|.|$ and $(.,.)$, resp. $W^{m,p}(\Omega)$ denotes the usual Sobolev space of L^p-functions which have generalized derivatives up to the order m in $L^p(\Omega)$, equipped with the usual norm denoted here by $\|\cdot\|_{m,p}$. $W^{s,2}(\Gamma)$ ($s \geqq 0$ real) is the well-known Hilbert space of traces (cf. [15]), $W^{-s,2}(\Gamma)$ denotes its dual. Given a Banach space X we denote by $C([0, T]; X)$, $L^p(0, T; X)$ $(1 \leqq p \leqq +\infty)$ the spaces of functions defined on $[0, T]$ with values in X, continuous or strongly measurable and integrable to the power p on $[0, T]$ (or bounded for $p = +\infty$), resp., with the usual norms.

Finally, let $\varphi : X \to (-\infty, +\infty]$ be a proper (i.e. $\varphi \not\equiv +\infty$), convex and lower semi-continuous (shortly written: l.s.c.) functional. We denote by $D(\varphi) = \{u \in X : : \varphi(u) < +\infty\}$ the domain of φ. The subgradient mapping $\partial\varphi : X \to 2^{X^*}$ is defined by

$$\partial\varphi(u) = \{u^* \in X^* : \varphi(v) - \varphi(u) \geqq \langle u^*, v - u\rangle \text{ for all } v \in X\}$$

($\langle u^*, u\rangle$ denotes the value of the functional $u^* \in X^*$ at $u \in X$).

2. THE DERIVATION OF THE TWO-DIMENSIONAL PROBLEM FROM THE SPATIAL SITUATION

1^0 **Basic equations.** Let Ω describe the middle plane of the undeformed plate having the thickness h. Then[2]) $\tilde{\Gamma}_1 = \Omega \times \{+\frac{1}{2}h\}$ and $\tilde{\Gamma}_2 = \Omega \times \{-\frac{1}{2}h\}$ represent the upper and lower surfaces and $\tilde{\Gamma}_3 = \Gamma \times [-\frac{1}{2}h, +\frac{1}{2}h]$ is the vertical boundary strip of the plate. Thus $\tilde{\Gamma} = \tilde{\Gamma}_1 \cup \tilde{\Gamma}_2 \cup \tilde{\Gamma}_3$ is the boundary of the plate $cl\tilde{\Omega} = \bar{\Omega} \times \times [-\frac{1}{2}h, +\frac{1}{2}h]$ considered as a three-dimensional body.

To derive the dynamical problem we assume that all functions occuring in our considerations are continuous and have continuous derivatives of any order we need (for $x \in \bar{\Omega}$, $z \in [-\frac{1}{2}h, +\frac{1}{2}h]$, $t \geqq 0$). When formulating the generalized solutions we will drop this assumption.

The three-dimensional equations of motion in Lagrangian coordinates (cf. [19]) take the form[3]) setting the density $\varrho \equiv 1$)

$$\tilde{u}_i'' + \tilde{s}_{ij,j} = \tilde{f}_i \quad \text{on} \quad \tilde{\Omega} \quad \text{for} \quad t \geqq 0\,, \tag{2.1}$$

where $\tilde{u}(x, z; t) = \{\tilde{u}_1(x, z; t), \tilde{u}_2(x, z; t), \tilde{u}_3(x, z; t)\}$ denotes the displacement vector, $\tilde{f}$ is the density of the body-force vector acting on the undeformed plate and $\tilde{s}_{ij}$ is the Lagrandian stress tensor. The latter is related to Kirchhoff's tensor $\tilde{\sigma}_{ij}$ by

$$\tilde{s}_{ji} = \tilde{\sigma}_{jk}(\delta_{ki} + \tilde{u}_{i,k})\,. \tag{2.2}$$

The strain tensor is defined by

$$\tilde{\varepsilon}_{ij} = \tfrac{1}{2}(\tilde{u}_{i,j} + \tilde{u}_{j,i} + \tilde{u}_{k,i}\tilde{u}_{k,j})\,. \tag{2.3}$$

Denoting the temperature difference (related to an initial temperature T_0) by $\tilde{\theta}(x, z; t)$ we consider the linear thermoelastic constitutive law

$$\tilde{\sigma}_{ij} = a_{ijkl}\,\tilde{\varepsilon}_{kl}(u) - b_{ij}\tilde{\theta}\,, \tag{2.4}$$ [4])

where the coefficients a_{ijkl} satisfy the usual symmetry and ellipticity conditions; b_{ij} is a symmetrical tensor which describes the coupling of the thermal and elastical properties.

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[2]) In all what follows we will equip objects (domains, functions, ...) depending on three variables with a tilde.

[3]) In all what follows, Latin subscripts have the range of integers 1, 2, 3 and Greek indices take the integers 1 and 2. Furthermore, we will use the notation $\tilde{p}_{,\alpha}(x) = \partial\tilde{p}(x)/\partial x_\alpha$, $\tilde{p}_{,3}(x, z) = \partial\tilde{p}(x, z)/\partial z$ and $\tilde{p}'(x, z; t) = \partial\tilde{p}(x, z; t)/\partial t$ for $x = (x_1, x_2) \in \Omega$, $z \in [-\frac{1}{2}h, +\frac{1}{2}h]$, $t \geqq 0$. Summation over repeated subscripts is applied.

[4]) This law has been obtained by considerations of the free energy assuming that the body satisfies Hooke's law under isothermal cooditions (cf. [16]). Here we suppose that also a_{ijkl} and b_{ij} are independent of z, i.e. that the plate is homogeneous along the perpendiculars to the middle surface, which seems to be right in virtue of the linearizations we make in the following.

From the general conservation theorems of thermodynamics we obtain the "three-dimensional" heat conduction equation (cf. for example [19], [16])

$$-\lambda_{ij}\tilde{\theta}_{,ij} + c_\varepsilon\tilde{\theta}' + T_0 b_{ij}\tilde{\varepsilon}'_{ij} = \tilde{\omega} \quad \text{in} \quad \tilde{\Omega} \quad \text{for} \quad t \geqq 0\,, \tag{2.5}$$

provided we restrict our considerations to small temperature differences $\tilde{\theta}$ (i.e. $|\tilde{\theta}| \ll T_0$). Here λ_{ij} denotes the symmetrical heat conduction tensor, c_ε is the specific heat for constant strains and $\tilde{\omega}$ represents the heat sources density.

Let $T < 0$ be arbitrarily chosen and let $\tilde{v} = \{\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ be a virtual displacement with $\tilde{v}(x, z; 0) = \tilde{v}(x, z; T) = 0$. Multiplying (2.1) by $\tilde{v}$ we get (after integration over $\tilde{\Omega} \times (0, T)$) the Hamilton principle (cf. [19])

$$\int_0^T \left\{\iint_{\tilde{\Omega}} [\tilde{\sigma}_{ij}\tilde{\varepsilon}_{ij}(\tilde{u}, \tilde{v}) - \tilde{f}_i\tilde{v}_i - \tilde{u}'_i\tilde{v}'_i]\, \mathrm{d}(x, z) - \int_{\tilde{\Gamma}} \tilde{s}_{ji}\tilde{n}_j\tilde{v}_i\, \mathrm{d}\tilde{\Gamma}\right\} \mathrm{d}t = 0\,, \tag{2.6}$$

which will be the starting point for the derivation of the two-dimensional problem. Here we have used the notation

$$\tilde{\varepsilon}_{ij}(\tilde{u}, \tilde{v}) = \tfrac{1}{2}(\tilde{v}_{i,j} + \tilde{v}_{j,i} + \tilde{u}_{k,j}\tilde{v}_{k,i} + \tilde{u}_{k,i}\tilde{v}_{k,j})\,.$$

Analogously, from (2.5) we get the equation

$$\int_0^T \left\{\iint_{\tilde{\Omega}} [c_\varepsilon\tilde{\theta}'\tilde{\eta} + \lambda_{ij}\tilde{\theta}_{,i}\tilde{\eta}_{,j} + T_0 b_{ij}\tilde{\varepsilon}'_{ij}(\tilde{u})\,\tilde{\eta} - \tilde{\omega}\tilde{\eta}]\, \mathrm{d}(x, z) - \right.$$
$$\left. - \int_{\tilde{\Gamma}} \lambda_{ij}\tilde{\theta}_{,i}\tilde{n}_j\tilde{\eta}\, \mathrm{d}\tilde{\Gamma}\right\} \mathrm{d}t = 0 \tag{2.7}$$

for any $\tilde{\eta}(x, z; t)$ satisfying the continuity properties stated above.

2^0 Certain linearizations. First we make the usual hypothesis introduced by von Kármán (using Kirchhoff's Hypothesis and neglecting "certain" higher order terms of strains):

$$\tilde{u}_\alpha(x, z; t) = u_\alpha(x, t) - z\, w_{,\alpha}(x, t) \quad \text{for} \quad \alpha = 1, 2\,; \tag{2.8}$$
$$\tilde{u}_3(x, z; t) = w(x, t)\,,$$

$$\tilde{\varepsilon}_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha} + w_{,\alpha}w_{,\beta}) - z\, w_{,\alpha\beta} = \varepsilon^0_{\alpha\beta} - z\, ew_{,\alpha\beta} \quad (\alpha = 1, 2)\,, \tag{2.9}$$
$$\tilde{\varepsilon}_{3i} = 0 \quad \text{for} \quad i = 1, 2, 3\,;$$

(here ε^0_{ab} is the strain tensor in the middle surface of the plate).

For the temperature difference $\tilde{\theta}$ we make the following "plate hypothesis" (cf. [7], [19])

$$\tilde{\theta}(x, z; t) = \theta_1(x, t) + z\,\theta_2(x, t)^{5)}\,. \tag{2.10}$$

[5]) Notice that, since h is small, (2.11) is a good approximation. In fact, set $\theta_1 = \frac{1}{2}(\theta_u + \theta_l)$, $\theta_2 = \frac{1}{2}h(\theta_u - \theta_l)$, where θ_u, θ_l are the temperature differences on the upper and lower surfaces of the plate (i.e. on $\tilde{\Gamma}_1$ or $\tilde{\Gamma}_2$).

Using (2.4) and (2.10) we obtain

$$\tilde{\sigma}_{\alpha\beta} = (\sigma^0_{\alpha\beta} - b_{\alpha\beta}\theta_1) - z(a_{\alpha\beta\gamma\delta}w_{,\gamma\delta} + b_{\alpha\beta}\theta_2)\,, \tag{2.11}$$

where $\sigma^0_{\alpha\beta} = a_{\alpha\beta\gamma\delta}\varepsilon^0_{\gamma\gamma}$ is the "elastic part" of the in-plane stress tensor.

3^0 **The two-dimensional equations.** We suppose[6]) that on the upper surface $\tilde{\Gamma}_1$ a perpendicular load $p = \{0, 0, -p_3\}$ acts while on the lower surface $\tilde{\Gamma}_2$ we have free boundary conditions. Thus we obtain (using (2.8)–(2.11), similar assumptions for the testfunction $\tilde{v}$ in (2.6) and integration over z) the *two-dimensional equilibrium conditions*:

$$w'' - \Delta w'' + \Delta_a^2 w - \frac{12}{h^2}\left[(\sigma^0_{\alpha\beta}w_{,\alpha})_{,\beta} - b_{\alpha\beta}(\theta_1 w_{,\alpha})_{,\beta}\right] + \Delta_b\theta_2 = F_3\,, \tag{2.12}$$

$$u''_\alpha - \sigma^0_{\alpha\beta,\beta} + b_{\alpha\beta}\theta_{1,\beta} = F_\alpha \quad \text{on} \quad \Omega \times (0, T)\,. \tag{2.13}$$

Here we have introduced the following definitions:
Δ is the Laplacian, $\Delta_a^2 w = (a_{\alpha\beta\gamma\delta}w_{,\gamma\delta})_{,\alpha\beta}$, $\Delta_b\theta_2 = b_{\alpha\beta}\theta_{2,\alpha\beta}$, $F_\alpha = h^{-1}f_\alpha$, $F_2 = f_3 - p_3$
In all what follows we assume that

$$w = 0 \quad \text{on} \quad \Gamma_1 \quad \text{and} \quad \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \Gamma_2 \quad \text{for} \quad t \geqq 0\,, \tag{2.14}$$

where Γ_1, $\Gamma_2 \subset \Gamma$ are open subsets with $\operatorname{mes}(\Gamma_1 \cup \Gamma_2) = \operatorname{mes}\Gamma$ (we use these conditions to drop boundary terms generated by integration of $\Delta w''$). Multiplying (2.12), (2.13) by testfunctions, integrating by parts over $\Omega \times (0, T)$ and comparing the equation obtained with (2.6) we finally have the following variational principle, which leads us to the generalized variational formulation of the problem:

$$\begin{aligned}\int_0^T \Bigg\{ \int_\Omega \Bigg[& w''q + w''_{,\alpha}q_{,\alpha} + a_{\alpha\beta\gamma\delta}w_{,\gamma\delta}q_{,\alpha\beta} + b_{\alpha\beta}\theta_2 q_{,\alpha\beta} + \\ & + \frac{12}{h^2}(\sigma^0_{\alpha\beta}w_{,\alpha}q_{,\beta} - b_{\alpha\beta}\theta_1 w_{,\alpha}q_{,\beta})\Bigg] \mathrm{d}x + \\ & + \frac{12}{h^2}\int_\Gamma \left[B(M)\frac{\partial q}{\partial n} + Q(M, N)\, q\right] \mathrm{d}\Gamma \Bigg\} \mathrm{d}t = \int_0^T \int_\Omega F_3 \,.\, q \,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{2.15}$$

and

$$\int_0^T \left\{ \int_\Omega (u''_\alpha v_\alpha + \sigma^0_{\alpha\beta}v_{\alpha,\beta} - b_{\alpha\beta}\theta_1 v_{\alpha,\beta})\,\mathrm{d}x + \int_\Gamma N_\alpha v_\alpha\,\mathrm{d}\Gamma \right\} \mathrm{d}t = \int_0^T \int_\Omega F_\alpha v_\alpha\,\mathrm{d}x\,\mathrm{d}t \tag{2.16}$$

[6]) We assume classical conditions on the upper and lower surfaces of the plate only to simplify our considerations. It is possible to formulate non-classical conditions (for example obstacle conditions, elastic clamping or supporting in the domain of the two-dimensional plate) in the same manner as for the linear static case in [20], Part I.

for all testfunctions q and v_α, where q satisfies (2.14). Here $B(M) = -M_{\alpha\beta}|_\Gamma n_\alpha n_\beta$ represents the bending moments on the boundary related to the moments in the middle plane $M_{\alpha\beta} = a_{\alpha\beta\gamma\delta} w_{,\gamma\delta}$. The term $Q(M, N) = B_2(M) + N_{\alpha\beta}|_\Gamma n_\alpha w_{,\beta}$ denotes the shearing forces on the boundary (cf. [19], [1]), where

$$B_2(M) = \frac{\partial M_{\alpha\beta}}{\partial n}\bigg|_\Gamma n_\alpha n_\beta - \frac{\partial}{\partial s}\Big[n_1 n_2 (M_{11} - M_{22})|_\Gamma + (n_2^2 - n_1^2) M_{12}|_\Gamma - - n_1 n_2 \frac{\partial}{\partial s}(M_{11} - M_{22})|_\Gamma - (n_2^2 - n_1^2) \frac{\partial M_{12}}{\partial s}\bigg|_\Gamma \Big]$$

represents the part of the shearing force which is generated by vertical stress difference on $\tilde{\Gamma}_3$. Whashizu (cf. [19], Chap. 8.5) pointed out that, in general, in geometrically nonlinear plate theory also the in-plane stress resultants

$$N_{\alpha\beta} = \int_{-h/2}^{+h/2} \sigma_{\alpha\beta}\, dz = h(\sigma^0_{\alpha\beta} - b_{\alpha\beta}\theta_1)$$

contribute to the equation of equilibrium (also on the boundary) in the direction of the z-axis due to the inclination $w_{,\alpha}$ of the middle surface; $N_\alpha = N_{\alpha\beta}|_\Gamma n_\beta$ is the in-plane stress resultants vector.

In the following we will consider generalized boundary conditions of the form

$$(2.17)\qquad \left.\begin{array}{l} B(M) \in \partial g_1\left(\dfrac{\partial w'}{\partial n}\right) \\ Q(M, N) \in \partial g_2(w') \\ N \in \partial h(u') \end{array}\right\} \text{on } \Gamma, \quad \text{for} \quad t \geqq 0$$

(g_1, g_2, h are proper, convex, l.s.c. functionals on $\mathbb{R}^1$ or $\mathbb{R}^2$, respectively) which leads us to variational inequalities. We remark (cf. [3], [20]) that (2.17) includes the classical as well as non-classical boundary conditions (e.g. friction).

4^0 Projection of the heat conduction equation. Setting in (2.7) $\tilde{\eta}(x, z; t) = \eta_1(x, t) + z\, \eta_2(x, t)$ and integrating over $z \in [-\frac{1}{2}h, +\frac{1}{2}h]$ we obtain the equation

$$(2.18)\qquad \int_\Omega \Big\{ \frac{12}{h^2} [(c_\varepsilon \theta_1' + T_0 b_{\alpha\beta} \varepsilon^{0\prime}_{\alpha\beta}) \eta_1 + \lambda_{3\alpha} \theta_2 \eta_{1,\alpha} + \lambda_{\alpha\beta} \theta_{1,\alpha} \eta_{1,\beta}] + + \Big[\Big(c_\varepsilon \theta_2' - T_0 b_{\alpha\beta} w_{,\alpha\beta} + \frac{12}{h^2} \lambda_{3\alpha} \theta_{1,\alpha}\Big) \eta_2 + \lambda_{\alpha\beta} \theta_{2,\alpha} \eta_{2,\beta}\Big]\Big\} dx - - h \int_\Gamma \lambda_{ij} \tilde{\theta}_{,i} \tilde{n}_j \tilde{\eta}\, d\tilde{\Gamma} = \int_\Omega (K_1 \eta_1 + K_2 \eta_2)\, dx$$

with

$$K_1(x) = h \int_{-h/2}^{+h/2} \tilde{\omega}(x, z)\, dz\,, \quad K_2(x) = h \int_{-h/2}^{+h/2} \tilde{\omega}(x, z) \,.\, z\, dz\,.$$

If the flux vector is given on the upper and lower surfaces of the plate, i.e. without loss of generality $\tilde{\omega}_b(\tilde{\theta}) = \lambda_{ij}\tilde{\theta}_{,i}\tilde{u}_j = 0$ on $\tilde{\Gamma}_1 \cup \tilde{\Gamma}_2$, we get from (2.18) (taking $\eta_\alpha \in \mathscr{D}(\Omega + (0, T))$) the *two-dimensional heat conduction equations*:

$$c_\varepsilon\theta_1' - \Delta_\lambda\theta_1 - \lambda_{3\alpha}\theta_{2,\alpha} + T_0 b_{\alpha\beta}\varepsilon^{0\prime}_{\alpha\beta} = \frac{h^2}{12}K_1 \tag{2.19}$$

$$c_\varepsilon\theta_2' - \Delta_\lambda\theta_2 + \lambda_{\alpha 3}\theta_{1,\alpha} - T_0\,\Delta_b w' = \frac{1}{h}K_2 \tag{2.20}$$

on $\Omega \times (0, T)$ (under more general conditions on $\tilde{\Gamma}_1 \cup \tilde{\Gamma}_2$ it is not possible to obtain these equations).

In (2.18) we let the boundary integral in a three-dimensional form. We will consider boundary conditions of the kind

$$-\tilde{\omega}_b(\tilde{\theta}) \in \partial\tilde{\psi}(\tilde{\theta}|_{\tilde{\Gamma}}) \quad \text{on} \quad \tilde{\Gamma}\,, \tag{2.21}$$

where $\tilde{\psi}$ - $\mathbb{R}^3 \to (-\infty, +\infty]$ is a proper, convex, l.s.c. function. In general, it seems to be not possible to transform this condition into two separate conditions upon θ_1 and θ_2. However, with the help of the linear (and in a certain space continuous) mapping $\{\theta_1, \theta_2\} \mapsto \tilde{\theta} = \theta_1 + z\theta_2$, condition (2.21) represents indeed two-dimensional boundary conditions. We will give an exact explanation in Section 4.

3. VARIATIONAL FORMULATION OF THE GENERALIZED SOLUTION

We define $V = [W^{1,2}(\Omega)]^2$ and $H = [L^2(\Omega)]^2$ with the scalar products $(u, v)_V = (u_1, v_1)_{1,2} + (u_2, v_2)_{1,2}$ and $(u, v)_H = (u_1, v_1) + (u_2, v_2)$ for $u, v \in V$ or H, respectively. Let $\Gamma_1, \Gamma_2 \subset \Gamma$ be open subsets with mes $(\Gamma_1 \cup \Gamma_2) =$ mes Γ. In connection with (2.14) we define the subspaces

$$X = \left\{ w \in W^{2,2}(\Omega) : w|_\Gamma = 0 \text{ a.e. on } \Gamma_1\,, \ \frac{\partial w}{\partial n}\bigg|_\Gamma = 0 \text{ a.e. on } \Gamma_2 \right\}$$

and

$$Y = \{ w \in W^{1,2}(\Omega) : w|_\Gamma = 0 \text{ almost everywhere on } \Gamma_2 \}$$

equipped with the scalar products of $W^{2,2}(\Omega)$ and $W^{1,2}(\Omega)$, respectively.

Furthermore, let proper, convex and l.s.c. functionals $\Phi_1 : L^2(0, T; X) \to (-\infty, +\infty]$ and $\Phi_2, \Phi_3 : L^2(0, T; V) \to (-\infty, +\infty]$ be given, shaped by

$$\Phi_j(v) = \begin{cases} \int_0^T \varphi_j(v)\,\mathrm{d}t & \text{if } \varphi_j(v) \in L^1(0, T) \\ +\infty & \text{otherwise} \end{cases}$$

for $j = 1, 2, 3$, where $\varphi_1 : X \to (-\infty, +\infty]$ and $\varphi_i : V \to (-\infty, +\infty]$ $(i = 2, 3)$ are proper, convex and lower semi-continuous functionals.

In accordance with (2.15), (2.16), (2.18) we introduce the following notation of bilinear and trilinear forms:

$$a(w, q) = \int_\Omega a_{\alpha\beta\gamma\delta} w_{,\gamma\delta} q_{,\alpha\beta} \, dx ,$$

$$a_1(\theta, \eta) = \frac{12c_\varepsilon}{h^2} (\theta_1, \eta_1) + c_\varepsilon(\theta_2, \eta_2) ,$$

$$b_1(\eta_1, w, q) = \int_\Omega b_{\alpha\beta} \eta_1 w_{,\alpha} q_{,\beta} \, dx ,$$

$$b_2(\eta_2, q) = \int_\Omega b_{\alpha\beta} \eta_2 q_{,\alpha\beta} \, dx ,$$

$$d_1(\eta_1, v) = \int_\Omega b_{\alpha\beta} \eta_1 v_{\alpha,\beta} \, dx ,$$

$$d_2(\eta, \theta) = \frac{12}{h^2} \int_\Omega \lambda_{\alpha\beta} \eta_{1,\alpha} \theta_{1,\beta} \, dx + \int_\Omega \lambda_{\alpha\beta} \eta_{2,\alpha} \theta_{2,\beta} \, dx ,$$

$$d_3(\theta_2, \eta_1) = \int_\Omega \lambda_{3\alpha} \theta_2 \eta_{1,\alpha} \, dx$$

for $\eta, \theta, v \in V$, $w, q, \in X$. For the sake of simplicity, we suppose that all coefficients are constants. Taking into account the boundary conditions (2.17), (2.14), (2.21) we give

Definition. *The triple* $\{w, u, \theta\} \in L^2(0, T; X \times V \times V)$ *with* $\{w', u', \theta'\} \in L^2 .$ $. (0, T; X \times V \times H)$ *and* $\{w'', u''\} \in L^2(0, T; Y \times H)$ *is called a generalized solution of the thermoelastic dynamical problem of the nonlinear plate theory if*

(i) *the inequality*

$$(3.1) \quad \int_0^T \Big\{ (w'', q - w')_{1,2} + a(w, q - w') + \frac{12}{h^2} \int_\Omega \sigma^0_{\alpha\beta} w_{,\alpha} (q - w')_{,\beta} \, dx - \\ - \frac{12}{h^2} b_1(\theta_1, w, q - w') + b_2(\theta_2, q - w') \Big\} \, dt + \\ + \Phi_1(q) - \Phi_1(w') \geqq \int_0^T (F_3, q - w') \, dt$$

is valid for all $q \in L^2(0, T; X)$,

(ii) *the inequality*

$$(3.2) \qquad \int_0^T \left\{ (u'', v - u')_H + \int_\Omega \sigma^0_{\alpha\beta}(v_\alpha - u'_\alpha)_{,\beta} \, dx + d_1(\theta_1, v - u') \right\} dt + $$
$$+ \Phi_2(v) - \Phi_2(u') \geqq \int_0^T (F_\alpha, v_\alpha - u'_\alpha) \, dt$$

is satisfied for all $v \in L^2(0, T; V)$,

(iii) *the inequality*

$$(3.3) \qquad \int_0^T \left\{ a_1(\theta', \eta - \theta) + d_2(\theta, \eta - \theta) - T_0 \, b_2(\eta_2 - \theta_2, w') + \right.$$
$$+ \frac{12}{h^2} [d_3(\theta_2, \eta_1 - \theta_1) + d_3(\eta_2 - \theta_2, \theta_1)] +$$
$$\left. + \frac{12}{h^2} T_0 \int_\Omega b_{\alpha\beta} \varepsilon^{0\prime}_{\alpha\beta}(u, w)(\eta_1 - \theta_1) \, dx \right\} dt +$$
$$+ \Phi_3(\eta) - \Phi_3(\theta) \geqq \int_0^T (K, \eta - \theta)_H \, dt$$

is true for all $\eta \in L^2(0, T; V)$ *and*

(iv) *the initial conditions* $w(0) = w_0$, $w'(0) = w_1$, $u(0) = u_0$, $u'(0) = v_0$, $\theta(0) = \{T_0, 0\}$ *are fulfilled, where*

$$\varepsilon^0_{\alpha\beta}(u, w) = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha} + w_{,\alpha} w_{,\beta}) \quad \textit{and} \quad \sigma^0_{\alpha\beta} = a_{\alpha\beta\gamma\delta} \varepsilon^0_{\gamma\delta}(u, w) \, .$$

Remark. Let $\{u, w, \theta\}$ be a classical solution of our problem, i.e. (2.12), (2.13) and (2.5) hold for $\tilde{\theta} = \theta_1 + z\theta_2$ and the boundary conditions (2.14), (2.17) and (2.21) are satisfied, where u, w, θ are sufficiently smooth. Setting

$$\varphi_{11}\left(\frac{\partial w'}{\partial n}\bigg|_\Gamma\right) = \begin{cases} \int_{\Gamma \setminus \Gamma_1} g_1\left(\dfrac{\partial w'}{\partial n}\right) d\Gamma & \text{if } \; g_1\left(\dfrac{\partial w'}{\partial n}\right) \in L^1(\Gamma \setminus \Gamma_1) \\ +\infty & \text{otherwise}, \end{cases}$$

$$\varphi_{12}(w'|_\Gamma) = \begin{cases} \int_{\Gamma \setminus \Gamma_2} g_2(w') \, d\Gamma & \text{if } \; g_2(w') \in L^1(\Gamma \setminus \Gamma_2) \\ +\infty & \text{otherwise} \end{cases}$$

for $t \geqq 0$ and

$$(3.4) \qquad \varphi_1(v) = \varphi_{11}(v|_\Gamma) + \varphi_{12}\left(\frac{\partial v}{\partial n}\bigg|_\Gamma\right) \quad \text{for} \quad v \in X$$

and

$$(3.5)\qquad \varphi_2(u) = \begin{cases} \int_\Gamma h_1(u|_\Gamma)\,\mathrm{d}\Gamma & \text{if } h_1(u|_\Gamma) \in L^1(\Gamma) \\ +\infty & \text{otherwise,} \end{cases}$$

we obtain from (2.14)–(2.17) the inequalities (3.1) and (3.2) (for the derivation of (3.3) we refer to Section 4).

Conversely, if $\{u, w, \theta\}$ is a generalized solution we obtain the equilibrium equations (2.12), (2.13) in the sense of distributions and, under some slight smoothness assumptions on the in-plane stress resultants, the boundary conditions (2.17) in the sense of traces. For the interpretation of (3.3) we refer to Section 4. It is well known that $w \in C([0, T]; X)$, $u \in C([0, T]; V)$, $w' \in C([0, T]; Y)$ and $u', \theta \in C([0, T]; H)$ so that the condition (iv) of our definition makes sense.

4. EXAMPLES OF THERMAL CONDITIONS ON THE BOUNDARY AND ON THE SURFACE OF THE PLATE

Let $\{u, w, \theta\}$ be a classical solution of our problem (cf. Remark, Sect. 3). The mapping $P : V \to W^{1,2}(\tilde{\Omega})$ defined by

$$(4.1)\qquad [P(\theta_1, \theta_2)]\,(x, z) = \theta_1(x) + z\,\theta_2(x) = \tilde{\theta}(x, z)$$

for a.a. $x \in \Omega$, $z \in [-\frac{1}{2}h, +\frac{1}{2}h]$ is obviously continuous and linear. By the continuity of the trace operator defined on $W^{1,2}(\tilde{\Omega})$ it follows that the functional defined by

$$(4.2)\qquad \varphi_3(\theta) = \tilde{\psi}(P(\theta)|_{\tilde{\Gamma}}) \quad \text{for all} \quad \theta \in V$$

is convex and lower semi-continuous if $\tilde{\psi} : W^{1/2,2}(\tilde{\Gamma}) \to (-\infty, +\infty]$ in (2.21) is. We assume that φ_3 is proper if $\tilde{\psi}$ is (i.e. $\tilde{\psi}$ is compatible with the linearization P; there exists $\tilde{\theta}_0 \in \operatorname{Im} P$ with $\tilde{\psi}(\tilde{\theta}_0|_{\tilde{\Gamma}}) \neq +\infty$). Thus we obtain (3.3) with φ_3 defined by (4.2).

Let $\tilde{\psi}$ be decomposable into

$$(4.3)\qquad \tilde{\psi} = \tilde{\psi}_1 + \tilde{\psi}_2 + \tilde{\psi}_3\,,$$

where $\tilde{\psi}_i$ is concentrated on $\tilde{\Gamma}_i$ for $i = 1, 2, 3$, (i.e. if $\tilde{v}, \tilde{u} \in W^{1/2,2}(\tilde{\Gamma})$ with $\tilde{v} = \tilde{u}$ a.e. on $\tilde{\Gamma}_i$ then $\tilde{\psi}_i(\tilde{v}) = \tilde{\psi}_i(\tilde{u})$ (for example $\int_{\tilde{\Gamma}_i} \tilde{v}\,\mathrm{d}\tilde{\Gamma}$). Then for the functional φ_3 defined by (4.2) we have the decomposition $\varphi_3 = \varphi_{31} + \varphi_{32} + \varphi_{33}$ with $\varphi_{3i}(\theta) = \tilde{\psi}_i(P(\theta)|_{\tilde{\Gamma}})$ for $\theta \in V$. But $P(\theta)\,(x, z)|_{\tilde{\Gamma}_{1,2}} = \theta_1(x) \pm \frac{1}{2}h\,\theta(x)$ for a.a. $x \in \Omega$, $z \in [-\frac{1}{2}h, +\frac{1}{2}h]$. Thus only φ_{33} represents a (two-dimensional) boundary functional (depends only on $\theta|_\Gamma$), while φ_{31}, φ_{32} characterize the thermal state in the domain Ω (exactly on the upper and lower surfaces of the plate).

If $\{u, w, \theta\}$ is a generalized solution defined in Selection 3 we obtain the two-dimensional heat conduction equation (2.18)–(2.19) in the sense of distributions under the assumption that $\tilde{\psi}_1, \tilde{\psi}_2$ are linear continuous on $L^2(\tilde{\Gamma})$ (with additional parts in K_1, K_2). Furthermore, we then have $\omega_b(\theta|_\Gamma) = \tilde{\omega}_b(P(\theta))|_{\Gamma_3} \in L^2(\Gamma)$ and $\omega_b(\theta|_\Gamma) \in \partial\psi(\theta|_\Gamma)$, where ψ is the boundary functional related φ_{33}. In the general case of $\tilde{\psi}_1, \tilde{\psi}_2$ we have multivalued heat conduction "equations" in the subgradient form.

We now give a typical example of thermal boundary conditions:

Example 1 (heat control with limited heat and cool capacity): The temperature difference $\tilde{\theta}$ on the boundary of the three-dimensional plate should be held between two limit temperatures $\tau_1, \tau_2 \in L^2(\tilde{\Gamma})$ with $\tau_1 \leqq 0 \leqq \tau_2$ a.e. on $\tilde{\Gamma}$. If $\tau_1 < \tilde{\theta} < \tau_2$, a free heat exchange with the surroundings (with a proportionality coefficient $k_0 \in L^\infty(\tilde{\Gamma})$) occurs. If $\tilde{\theta}$ exceeds the limit $\tau_1(\tau_2)$ the plate will be heated (cooled), the additional outward heat flux $-\omega_b - k_0\tilde{\theta}|_{\tilde{\Gamma}}$ being proportional (with positive coefficients $k_1, k_2 \in L^\infty(\tilde{\Gamma})$, resp.) to the value of excess. The heat and control capacity let be limited, i.e. the additional heat flux can only reach certain quantities $g_1, g_2 \in L^2(\tilde{\Gamma})$ with $g_1 \leqq 0$, $g_2 \geqq 0$ a.e. on $\tilde{\Gamma}$. This situation corresponds to the functional

$$\tilde{\psi}(\tilde{v}) = \int_{\tilde{\Gamma}} g_0(x, z; \tilde{v}(x, z)) \, d\tilde{\Gamma}$$

which is defined for $\tilde{v} \in L^2(\tilde{\Gamma})$, where

$$g_0(x, z; s) = \begin{cases} k_0 s^2 + g_1 \cdot s & \text{if} \quad s \leqq \tau_1 + g_1/k_1 \\ (k_0 + k_1)\, s^2 - k_1\tau_1 s & \text{if} \quad \tau_1 + g_1/k_1 \leqq s \leqq \tau_1 \\ k_0 s^2 & \text{if} \quad \tau_1 \leqq s \leqq \tau_2 \\ (k_0 + k_2)\, s^2 - k_2\tau_2 s & \text{if} \quad \tau_2 \leqq s \leqq \tau_2 + g_2/k_2 \\ k_0 s^2 + g_2 s & \text{if} \quad s \geqq \tau_2 + g_2/k_2 \end{cases}$$

for $s \in \mathbb{R}$.

Remark 4.1. The case $k_0 \equiv 0$ (no heat exchange with the surroundings occurs if $\tilde{\theta}$ does not exceed the two temperature limits) is considered in [5] for a three-dimensional body. In some cases we can drop the summability properties of k_1, k_2, τ_1, τ_2. For example, it makes sense to set either $\tau_1 \equiv -\infty$ or $\tau_2 \equiv +\infty$, i.e. the temperature must stay either below or above a fixed limit. Also the classical boundary conditions can be considered as a specialization of g_0, for example we obtain

$$g_0(x, z; s) = k_0(x, z)\, s^2 \quad \text{for} \quad s \in \mathbb{R}\,, \quad \text{a.a.} \quad (x, z) \in \tilde{\Gamma}\,,$$

(free heat flux, and for $k_0 \equiv 0$ Neumann's problem) and

$$g_0(x, z; s) = \{0 \text{ if } s = 0 \quad \text{or} \quad +\infty \text{ otherwise}\}$$

(Dirichlet's conditions).

Remark 4.2. For extensive discussions of mechanical boundary conditions we refer to [20]. In a similar way we are able to deal with mixed thermal conditions, i.e. various conditions on certain parts of the three-dimensional boundary of the plate.

5. RESULTS

Theorem 5.1. *Suppose that there exist minima of the functionals* φ_i $(i = 1, 2, 3)$ *realized by the initial values, i.e.*

$$(5.1)\qquad \begin{aligned} \varphi_1(w_1) &\leqq \varphi_1(q) \quad \forall q \in X \\ \varphi_2(v_0) &\leqq \varphi_2(v) \quad \forall v \in V \\ \varphi_3(\{T_0, 0\}) &\leqq \varphi_3(\eta) \quad \forall \eta \in V, \quad T_0 = \text{const.} > 0 . \end{aligned}$$

Furthermore, let

$$(5.2)\qquad F_i, F'_i, K_\alpha, K'_\alpha \in L^2(0, T; L^2(\Omega)) \quad \textit{for} \quad i = 1, 2, 3 \quad \textit{and} \quad \alpha = 1, 2 ;$$
$$w_0 \in W_0^{3,2}(\Omega), \quad w_1 \in X, \quad u_0 \in [W_0^{2,2}(\Omega)]^2, \quad v_0 \in V .$$

Then there exists a generalized solution $\{w, u, \theta\}$ *of the dynamical thermoelastic plate problem in the sense of the definition given in Section* 3 *with*

$$(5.3)\qquad \begin{aligned} &w, w' \in L^\infty(0, T; X), \quad w'' \in L^\infty(0, T; Y) \\ &u, u' \in L^\infty(0, T; V), \quad u'' \in L^\infty(0, T; H) \\ &\theta, \theta' \in L^2(0, T; V) \cap L^\infty(0, T; H) . \end{aligned}$$

Theorem 5.2. *Under the assumptions of Theorem* 5.1 *the solution depends continuously on the given data* $\{w_0, w_1, u_0, v_0, F, K\}$ *and is uniquely determined by a fixed set of data. More precisely: If* $\{\hat{w}_0, \hat{w}_1, \hat{u}_0, \hat{v}_0, \hat{F}, \hat{K}\}$ *and* $\{w_0^*, w_1^*, u_0^*, v_0^*, F^*, K^*\}$ *are two sets of data possessing the properties given in Theorem* 5.1 *and* $\{\hat{w}, \hat{u}, \hat{\theta}\}$, $\{w^*, u^*, \theta^*\}$ *are the corresponding solutions, then we have*

$$(5.4)\qquad \begin{aligned} &\|\hat{w}(t) - w^*(t)\|_{2,2}^2 + \|\hat{w}'(t) - w^{*\prime}(t)\|_{1,2}^2 + \|\hat{u}(t) - u^*(t)\|_V^2 + \\ &+ |\hat{u}'(t) - u^{*\prime}(t)|_H^2 + |\hat{\theta}(t) - \theta^*(t)|_H^2 + \|\hat{\theta} - \theta^*\|_{L^2(0,T;V)}^2 \leqq \\ &\leqq c_0\{\|\hat{w}_0 - w_0^*\|_{2,2}^2 + \|\hat{w}_0 - w_0^*\|_{1,4}^4 + \|\hat{w}_1 - w_1^*\|_{1,2}^2 + \\ &+ \|\hat{u}_0 - u_0^*\|_V^2 + |\hat{v}_0 - v_0^*|_H^2 + \|\hat{F} - F^*\|_{(L^2(0,T;L^2(\Omega))^3}^2 + \|\hat{K} - K^*\|_{L^2(0,T;H)}^2 . \end{aligned}$$

Remark 5.1. Duvaut and Lions ([4]) have shown an existence and uniqueness theorem for the isothermal case (set formally $\theta \equiv 0$, $\Phi_3 \equiv 0$, $K \equiv 0$ in our considerations) with $\Phi_2 \equiv 0$ and $w = 0$ on Γ. Even for isothermal problems we do not know any results with boundary conditions for horizontal displacements (or in-plane stresses) in the middle surface or between the perpendicular displacements velocity w' and the shearing forces $Q(M, N)$ in such a general form as we have just introduced.

Remark 5.2. The existence of minima of the functionals φ_i can be shown for all examples of functionals known to the author (in the field of thermoelasticity). If this is not directly satisfied then (in the known cases) the functionals (or the parts of it which do not possess a minimum) are linear and continuous. In this case the functionals (or their parts mentioned) can be handled without difficulties and without any regularization by a smoother functional.

Remark 5.3. The conditions (5.1), (5.2) can be slightly generalized (as in [6]).

6. PROOF OF THEOREM 5.1

1⁰ We prove the result stated in Section 5 with the help of two approximations. First we "smooth" the functionals φ_i by the Yosida approximation of $\partial\varphi_i$ dependent on a parameter $\varepsilon > 0$. Using Galerkin's method for this regularized problem we obtain by certain a-priori-estimates and by passing to the limit for $\varepsilon \to 0$ the result desired.

Thus we define for $\varepsilon > 0$ the approximating functional φ_ε of a proper, convex and l.s.c. functional $\varphi : U \to (-\infty, +\infty]$ (U is a Hilbert space) by

(6.1) $$\varphi_\varepsilon(p) = \frac{1}{2\varepsilon} \|p - J_\varepsilon p\|_U^2 + \varphi(J_\varepsilon p) \quad \text{for} \quad p \in U\,,$$

where

$$J_\varepsilon = (I + \varepsilon\,\partial\varphi)^{-1} \quad (I \text{ is the identical mapping})\,.$$

It is well-know that

(6.2) $$\lim_{\varepsilon\to 0} \varphi_\varepsilon(p) = \varphi(p) \quad \text{for all} \quad p \in U\,,$$

(6.3) φ_ε is Fréchet-differentiable and the derivative is monotone and Lipschitz-continuous (with the Lipschitz constant $1/\varepsilon$) in U.

(6.4) If φ possesses a minimum in $p_0 \in U$ then $J_\varepsilon p_0 = p_0$ and $\varphi(p_0) = \varphi_\varepsilon(p_0) \leqq \varphi_\varepsilon(q) \leqq \varphi(q) \quad \forall q \in U\,, \quad \varepsilon > 0\,.$

2⁰ Approximate solutions

Let $\{q_j\} \in X$ be a basis in X and let $\{v_j\} \in V$ be a basis in V in the following sense: (i) $\{q_j\}$ ($\{v_j\}$) is a linearly independent system and (ii) the set of all linear combinations of $\{q_j\}$ ($\{v_j\}$) is dense in X (in V, respectively).

We now define the approximating solution $\{w_m, u_m, \theta_m\}$ by

$$w_m(t) = \sum_{i=1}^{m} g_{im}(t)\, q_i, \quad u_m(t) = \sum_{i=1}^{m} h_{im}(t)\, v_i, \quad \theta_m(t) = \sum_{i=1}^{m} k_{im}(t)\, v_i ,$$

so that

$$(6.5) \qquad (w''_m(t), q_r)_{1,2} + a(w_m(t), q_r) + \frac{12}{h^2}\int_\Omega \sigma^{0(m)}_{\alpha\beta}\, w_{m,\alpha} q_{r,\beta}\, dx - \\ - \frac{12}{h^2} b_1(\theta_{m1}, w_m, q_r) + b_2(\theta_{m2}, q_r) + (\varphi'_{1\varepsilon}(w'_m), q_r)_{2,2} = (F_3, q_r)\,,$$

$$(6.6) \qquad (u''_m, v_r)_H + \int_\Omega \sigma^{0(m)}_{\alpha\beta} v_{r\alpha,\beta}\, dx - d_1(\theta_{m1}, v_r) + (\varphi'_{2\varepsilon}(u'_m), v_r)_V = (F_\alpha, v_{r\alpha})$$

and

$$(6.7) \qquad a_1(\theta'_m, v_r) + d_2(\theta_m, v_r) - T_0\, b_2(v_{r2}, w'_m) + \\ + \frac{12}{h^2}\left[d_3(\theta_{m2}, v_{r1}) + d_3(v_{r2}, \theta_{m1}) + T_0 \int_\Omega b_{\alpha\beta}\varepsilon^{0\prime}_{\alpha\beta}(u_m, w_m)\, v_{r1}\, dx\right] + \\ + (\varphi'_{3\varepsilon}(\theta_m), v_r)_V = (K, v_r)_H$$

are valid for $r = 1, 2, \ldots, m$ and for all $t \in [0, T]$ and the following initial conditions are satisfied: $w_m(0) = w_0$, $w'_m(0) = w_1$, $u_m(0) = u_0$, $u'_m(0) = v_0$, $\theta_{m1}(0) = T_0$, $\theta_{m2}(0) = 0$.

By the theory of ordinary differential equations a solution of this problem with absolutely continuous time derivative of the highest order is obtained (at least for $[0, t_m] \subset [0, T]$, 3^0 and 4° yield $t_m = T$);

3^0 A-priori-estimates, A

First we multiply (6.5)–(6.7) by $g'_{rm}(t) - g'_{rm}(0)$, $h'_{rm}(t) - h'_{rm}(0)$, $k_{rm}(t) - k_{rm}(0)$, respectively and sum over $r = 1, 2, \ldots, m$. Adding now the three equations obtained we have (here we drop the fixed index m)

$$(6.8) \qquad \frac{1}{2}\frac{d}{dt}\left\{\frac{h^2}{12}\|w'(t)\|^2_{1,2} + a(w(t), w(t)) + |u'(t)|^2_H \right. \\ \left. a_1(\theta(t), \theta(t)) + \int_\Omega a_{\alpha\beta\gamma\delta}\varepsilon^0_{\alpha\beta}\varepsilon^0_{\alpha\beta}\, dx\right\} + \\ + \frac{h^2}{12T_0} d_2(\theta(t), \theta(t)) + U_1 = U_2 + \frac{h^2}{12}(F_3(t), w'(t) - w_1) + \\ + (F_\alpha(t), u'_\alpha(t) - v_{0\alpha}) + \frac{h^2}{12T_0}(K(t), \theta(t) - \{T_0, 0\})_H\,,$$

where

$$U_1 = (\varphi'_{1\varepsilon}(w'), w' - w_1)_{2,2} + (\varphi'_{2\varepsilon}(u'), u' - v_0)_V + \\ + (\varphi'_{3\varepsilon}(\theta), \theta - \{T_0, 0\})_V - b_1(\theta_1, w, w') + \\ + \frac{h^2}{12} b_2(\theta_2, w') - d_1(\theta_1, u') - \frac{h^2}{12} b_2(\theta_2, w') + \\ + \frac{2}{T_0} d_3(\theta_2, \theta_1) + \int_\Omega b_{\alpha\beta}\varepsilon^{0\prime}_{\alpha\beta}(u, w)\, \theta_1\, dx$$

(here $w, w', u, u', \theta_1, \theta_2$ depend on t while the initial values w_1, v_0, T_0 of course do not) and

$$U_2 = \frac{1}{2}\frac{d}{dt}\left\{\frac{h^2}{12}(w'(t), w_1) + (u'(t), v_0)_H + \frac{12c_\varepsilon}{h^2}(\theta_1(t), T_0)\right\} + a(w(t), w_1) + \\ + \int_\Omega \sigma^0_{\alpha\beta}(t)\,(w_{,\alpha}(t)\, w_{1,\beta} + v_{0\alpha,\beta})\, dx - \\ - b_1(\theta_1(t), w(t), w_1) + \frac{h^2}{12} b_2(\theta_2(t), w_1) - d_1(\theta_1(t), v_0) - \\ - \frac{h^2}{12} b_2(\theta_2(t), w_1) + T_0 \frac{d}{dt}\int_\Omega b_{\alpha\beta}\varepsilon^0_{\alpha\beta}(u, w)\, dx$$

(here we used the identities $T_0 = \text{const.}$ and $d_3(\theta, \text{const.}) = d_2(\theta_2, \text{const.}) = 0$). Furthermore, we have

$$a_1(\theta, \theta) \geqq c_1|\theta|^2_H \quad \text{for all} \quad \theta \in H \tag{6.9}$$

and

$$d_2(\theta, \theta) + |\theta|^2_H \geqq c_2\|\theta\|^2_V \quad \text{for all} \quad \theta \in V \tag{6.10}$$

($c_1, c_2 = \text{const.} > 0$). Moreover, $\varphi'_{i\varepsilon}$ $(i = 1, 2, 3)$ are monotone mappings and by (5.1) these parts of U are greater than zero; the b_2-terms in U cancel each other and the d_3-term can be estimated as follows:

$$|d_3(\theta_2, \theta_1)| = \left|\int_\Omega \lambda_{3\alpha}\theta_2\theta_{1,\alpha}\, dx\right| \leqq \delta\, .\, \|\theta\|^2_V + c_3\, .\, |\theta|^2_H \tag{6.11}$$

with $\delta > 0$ arbitrary. The sum of the remaining parts of U vanishs (since $b_{\alpha\beta}$ is symmetrical). Thus we have, integrating (6.8) over $t \in [0, T]$ and using (6.9)–(6.11)

$$\|w'(t)\|^2_{1,2} + \|w(t)\|^2_{2,2} + |u'(t)|^2_H + \int_\Omega \varepsilon^0_{\alpha\beta}\varepsilon^0_{\alpha\beta}\, dx + |\theta(t)|^2_H + \int_0^t \|\theta(\tau)\|^2_V\, d\tau \leqq \tag{6.12}$$

$$\leqq c_1 \left\{\|w_1\|^2_{1,2} + \|w_0\|^2_{2,2} + |v_0|^2_H + \int_\Omega \varepsilon^0_{\alpha\beta}(0)\, \varepsilon^0_{\alpha\beta}(0)\, dx + \right.$$

$$+ \int_0^t [U_2(\tau) + (F_3(\tau), w'(\tau) - w_1) + (F_\alpha(\tau), u'_\alpha(\tau) - v_{0\alpha}) +$$

$$+ (K(\tau), \theta(\tau) - \{T_0, 0\})_H + \|w'(\tau)\|^2_{1,2} + |\theta(\tau)|^2_H + \delta\|\theta(\tau)\|^2_V] \, d\tau \Big\} .$$

Since $W^{2,2}(\Omega)$ is continuously imbedded into $W^{1,4}(\Omega)$ we have

$$(6.13) \qquad \left| \int_\Omega a_{\alpha\beta\gamma\delta} \varepsilon^0_{\gamma\delta} \varepsilon^0_{\alpha\beta} \, dx \right| \leqq c(\|u\|_V + \|w\|^2_{2,2})^2$$

and analogously

$$(6.14) \qquad \int_0^t |U_2(\tau)| \, d\tau \leqq \delta \left(\|w'(t)\|^2_{1,2} + |u'(t)|^2_H + |\theta(t)|^2_H + \int_\Omega \varepsilon^0_{\alpha\beta} \varepsilon^0_{\alpha\beta} \, dx \right) +$$

$$+ c \left\{ \|w_0\|^2_{2,2} + \|w_1\|^2_{2,2} + \|v_0\|^2_V + \|u_0\|^2_V + 1 + \int_0^t [\|w(\tau)\|^2_{2,2} + |\theta(\tau)|^2_H] \, d\tau \right\}$$

for $\delta > 0$ arbitrary. Using (6.13), (6.14) and choosing $\delta = 1/2c_1$ we compensate these terms with the left hand side of (6.12) and obtain by Gronwall's lemma:

$$(6.15) \qquad \|w'_m(t)\|^2_{1,2} + \|w_m(t)\|^2_{2,2} + |u'_m(t)|^2_H + \int_\Omega \varepsilon^{0(m)}_{\alpha\beta}(t) \, \varepsilon^{0(m)}_{\alpha\beta}(t) \, dx +$$

$$+ |\theta_m(t)|^2_H + \int_0^t \|\theta_m(\tau)\|^2_V \, d\tau \leqq \text{const} .$$

for all $m = 1, 2, \ldots; t \in [0, T]$ and all $\varepsilon > 0$. Adding $|u_m(t)|^2_H$ on both sides of (6.15) we conclude by Korn's inequality (cf. [10]) that

$$(6.16) \qquad \|u'_m(t)\|_V \leqq \text{const} .$$

4⁰ A-priori-estimates, B

Setting $t = 0$ in (6.5)–(6.7) and multiplying $g''_{rm}(0)$, $h''_{rm}(0)$, $k'_{rm}(0)$ we obtain using (5.1), (5.2), (6.4)

$$(6.16) \qquad \begin{aligned} \|w''_m(0)\|_{1,2} &\leqq \text{const} ., \\ |u''_m(0)|_H &\leqq \text{const} ., \\ |\theta'_m(0)|_H &\leqq \text{const} . \end{aligned}$$

for all $\varepsilon > 0$, $m = 1, 2, \ldots$

We now differentiate (6.5)–(6.7) (with respect to t), multiply it by $g''_{mr}(t)$, $h''_{rm}(t)$, $k'_{rm}(t)$, respectively and sum over $r = 1, 2, \ldots, m$. Adding the three equations obtained we have (here we drop again the index m)

$$(6.17) \qquad \frac{1}{2} \frac{d}{dt} \left\{ \|w''\|^2_{1,2} + a(w', w') + \frac{12}{h^2} |u''|^2_H + \frac{1}{T_0} a_1(\theta', \theta') + \right.$$

$$+ \int_\Omega (\sigma^0_{\alpha\beta} w_{,\alpha})' \, w''_{,\beta} \, \mathrm{d}x + \int_\Omega \sigma^{0\prime}_{\alpha\beta} u''_{\alpha,\beta} \, \mathrm{d}x \Big\} + \frac{1}{T_0} \, d_2(\theta', \theta') + U_3 =$$

$$= (F'_3, w'') + (F'_\alpha, u''_\alpha) + (K', \theta')_H \,,$$

where

$$U_3 = \left(\frac{\mathrm{d}}{\mathrm{d}t} \, \varphi'_{1\varepsilon}(w'), w'' \right)_{2,2} + \left(\frac{\mathrm{d}}{\mathrm{d}t} \, \varphi'_{2\varepsilon}(u'), u'' \right)_V + \left(\frac{\mathrm{d}}{\mathrm{d}t} \, \varphi'_{3\varepsilon}(\theta), \theta' \right)_V -$$

$$- \frac{12}{h^2} \left[b_1(\theta'_1, w, w'') + b_1(\theta_1, w', w'') \right] + b_2(\theta'_2, w'') - \frac{12}{h^2} \, d_1(\theta'_1, u'') -$$

$$- b_2(\theta'_2, w'') + \frac{24}{h^2 T_0} \, d_3(\theta'_2, \theta'_1) + \frac{12}{h^2} \int_\Omega b_{\alpha\beta} \varepsilon^{0\prime\prime}_{\alpha\beta}(u, w) \, \theta'_1 \, \mathrm{d}x \,.$$

An easy calculation shows that

$$\int_\Omega (\sigma^0_{\alpha\beta} w_{,\alpha})' \, w''_{,\beta} \, \mathrm{d}x + \int_\Omega \sigma^{0\prime}_{\alpha\beta} u''_{\alpha,\beta} \, \mathrm{d}x = \tag{6.18}$$

$$= \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \int_\Omega a_{\alpha\beta\gamma\delta} \varepsilon^{0\prime}_{\alpha\beta} \varepsilon^{0\prime}_{\gamma\delta} \, \mathrm{d}x + \int_\Omega \sigma^0_{\alpha\beta} w'_{,\alpha} w'_{,\beta} \, \mathrm{d}x \right\} - \frac{3}{2} \int_\Omega \sigma^{0\prime}_{\alpha\beta} w'_{,\alpha} w'_{,\beta} \, \mathrm{d}x \,.$$

The first term on the right hand side can be handled as in (6.15)–(6.16), while the remaining are estimated with the help of the following interpolation inequality:

$$\left| \int_\Omega (\eta_{,\alpha})^2 \, (\eta_{,\beta})^2 \, \mathrm{d}x \right|^{1/2} \leqq c \|\eta\|_{2,2} \, \|\eta\|_{1,2} \quad \text{for} \quad \eta \in X \,. \tag{6.19}$$

By the monotonicity of Yosida's approximation, the $\varphi'_{i\varepsilon}$-terms in U_3 are greater than zero and the b_2-terms cancel each other. Furthermore, we have

$$|d_3(\theta'_2, \theta'_1)| = \left| \int_\Omega \lambda_{3\alpha} \theta'_2 \theta'_{1,\alpha} \, \mathrm{d}x \right| \leqq \lambda \|\theta'\|_V^2 + c_\lambda |\theta|_H^2 \tag{6.20}$$

for all $\lambda > 0$. It suffices to consider the remaining terms of U_3:

$$- b_1(\theta'_1, w, w'') - b_1(\theta_1, w', w'') - d_1(\theta'_1, u'') + \int_\Omega b_{\alpha\beta} \varepsilon^{0\prime\prime}_{\alpha\beta} \theta'_1 \, \mathrm{d}x =$$

$$= \int_\Omega b_{\alpha\beta} (w'_{,\alpha} w'_{,\beta} \theta'_1) \, \mathrm{d}x - \tfrac{1}{2} \int_\Omega b_{\alpha\beta} \theta_1 (w'_{,\alpha} w'_{,\beta})' \, \mathrm{d}x =$$

$$= \tfrac{3}{2} \int_\Omega b_{\alpha\beta} (w'_{,\alpha} w'_{,\beta} \theta'_1) \, \mathrm{d}x - \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega b_{\alpha\beta} \theta_1 w'_{,\alpha} w'_{,\beta} \, \mathrm{d}x \,.$$

Thus we obtain from (6.17) with the help of (6.9), (6.10), (6.15), (6.16), (6.18)–(6.20) after integration over $(0, t)$ the inequality

$$(6.21)\qquad \|w''(t)\|_{1,2}^2 + \|w'(t)\|_{2,2}^2 + |u''(t)|_H^2 + |\theta'(t)|_H^2 + \\ + \int_\Omega \varepsilon^{0\prime}_{\alpha\beta}(t)\, \varepsilon^{0\prime}_{\alpha\beta}(t)\, dx + \int_0^t \|\theta'(\tau)\|_V^2\, d\tau \leqq \\ \leqq \tfrac{1}{2}\int_0^t \|\theta'(\tau)\|_V^2\, d\tau + c_1 + c_2 \left|\int_\Omega b_{\alpha\beta} w'_{,\alpha}(t)\, w'_{,\beta}(t)\, \theta_1(t)\, dx\right| + \\ + c_3 \int_0^t \left\{|\theta'(\tau)|_H^2 + \|w''(\tau)\|_{1,2}^2 + |u''(\tau)|_H^2 + \left|\int_\Omega b_{\alpha\beta}\, w'_{,\alpha}(\tau)\, w'_{,\beta}(\tau)\, \theta'_1(\tau)\, dx\right|\right\} d\tau\,.$$

Estimating the $b_{\alpha\beta}$-terms with the help of (6.19), (6.15) and using the same argument for the $\varepsilon^{0\prime}$-terms as in Section 2^0 we obtain from (6.21) by Gronwall's lemma

$$(6.22)\qquad \|w''_m(t)\|_{1,2} + \|w'_m(t)\|_{2,2} + |u''_m(t)|_H + \\ + \|u'_m(t)\|_V + |\theta_m(t)|_H + \|\theta_m\|_{L^2(0,T;V)} \leqq \text{const.}$$

for all $m = 1, 2, \ldots;\ \varepsilon > 0$ and a.a. $t \in (0, T)$. Since the Yosida approximation is Lipschitzian, it follows from (6.22) that

$$(6.23)\qquad \|\varphi'_{1\varepsilon}(w'_m(t))\|_{2,2} + \|\varphi_{2\varepsilon}(u'_m(t))\|_V + \|\varphi_{3\varepsilon}(\theta_m)\|_{L^2(0,T;V)} \leqq c/\varepsilon$$

$(c = \text{const.} > 0)$ for all $m = 1, 2, \ldots;\ t \in (0, T)$.

5^0 Passing to the limit for $m \to \infty$

From the estimates (6.15), (6.16), (6.22) it follows that a subsequence $\{w_n, u_n, \theta_n\} \subset \subset \{w_m, u_m, \theta_m\}$ (as well as $\{w'_n, u'_n, \theta'_n\}$) converges weakly* in $L^\infty(0, T; X) \times L^\infty\,.\,(0, T; V) \times [L^\infty(0, T; H) \cap L^2(0, T; V)]$ to $\{w_\varepsilon, u_\varepsilon, \theta_\varepsilon\}$ (resp. $\{w'_\varepsilon, u'_\varepsilon, \theta'_\varepsilon\}$) and $\{w''_n, u''_n\}$ in $L^\infty(0, T; Y) \times L^\infty(0, T; H)$. Then (6.23) yields that there exist $\chi_{1\varepsilon} \in L^\infty\,.\,(0, T; X)$, $\chi_{2\varepsilon} \in L^\infty(0, T; V)$, $\chi_{3\varepsilon} \in L^2(0, T; V)$ which are weak* limits of $\{\varphi'_{1\varepsilon}(w'_n)\}$, $\{\varphi'_{2\varepsilon}(u'_n)\}$, $\{\varphi'_{3\varepsilon}(\theta_n)\}$.

Let $q \in L^2(0, T; X)$ be arbitrary. Since the space $Z = \{w \in L^2(0, T; X)$ with $w' \in \in L^2(0, T; X)\}$ is compactly imbedded into $L^2(0, T; W^{1,4}(\Omega))$ it follows that

$$(6.24)\qquad \int_0^T \int_\Omega \sigma^{0(n)}_{\alpha\beta} w_{n,\alpha} q_{,\beta}\, dx\, dt \xrightarrow[n\to\infty]{} \int_0^T \int_\Omega \sigma^{0(\varepsilon)}_{\alpha\beta} w_{\varepsilon,\alpha} q_{,\beta}\, dx\, dt$$

and in a simiand in a similar way

$$(6.25)\qquad \int_0^T \int_\Omega \sigma^{0(n)}_{\alpha\beta} v_{\alpha,\beta}\, dx\, dt \xrightarrow[n\to\infty]{} \int_0^T \int_\Omega \sigma^{0(\varepsilon)}_{\alpha\beta} v_{\alpha,\beta}\, dx\, dt$$

for every $v \in L^2(0, T; V)$. Since for fixed $q \in X$, $v \in V$, $b_1(\cdot, \cdot, q)$ is a continuous bilinear form on $L^2(\Omega) \times X$, $b_2(\cdot, q)$ and $d_1(\cdot, v)$ are linear, continuous functionals on $L^2(\Omega)$ and H, respectively, we obtain from (6.5), (6.6) (noting that $w_n \xrightarrow[n\to\infty]{} w_\varepsilon$ strongly $L^2(0, T; X)$ $\theta_n \xrightarrow[n\to\infty]{} \theta_\varepsilon$ strongly in L^2 in $(0, T; H)$) by (6.24), (6.25) the identity

$$(6.26)\qquad \int_0^T \Big\{(w_\varepsilon'', q)_{1,2} + a(w_\varepsilon, q) + \frac{12}{h^2}\int_\Omega \sigma_{\alpha\beta}^{0(\varepsilon)} w_{\varepsilon,\alpha} q_{,\beta}\, \mathrm{d}x - - \frac{12}{h^2} b_1(\theta_{\varepsilon 1}, w_\varepsilon, q) + b_2(\theta_{\varepsilon 2}, q) + (\chi_{1\varepsilon}, q)_{2,2} - (F_3, q)\Big\}\, \mathrm{d}t = 0$$

for all $q \in L^2(0, T; X)$ and

$$(6.27)\qquad \int_0^T \Big\{(u_\varepsilon'', v)_H + \int_\Omega \sigma_{\alpha\beta}^{0(\varepsilon)} v_{\alpha,\beta}\, \mathrm{d}x - d_1(\theta_{\varepsilon 1}, v) + (\chi_{2\varepsilon}, v)_V - (F_\alpha, v_\alpha)\Big\}\, \mathrm{d}t = 0$$

for $v \in L^2(0, T; V)$.

In (6.7) all terms, except the $\varphi_{3\varepsilon}'$-term, are linear and continuous in V with respect to θ_n and $a_1(\cdot, \eta)$ is linear and continuous in H for fixed $\eta \in V$. Thus we can easily pass to the limit in these terms. For fixed $\eta \in V$, $b_2(\eta, \cdot)$ is linear and continuous on X, and the expression

$$\int_\Omega b_{\alpha\beta}\varepsilon^{01}(u_n, w_n)\, \eta\, \mathrm{d}x$$

is linear and continuous in V with respect to u_n' and bilinear and continuous in $X \times Y$ with respect to $\{w_n, w_n'\}$. Thus we obtain from (6.7):

$$(6.28)\qquad \int_0^T \Big\{a_1(\theta_\varepsilon', \eta) + d_2(\theta_\varepsilon, \eta) - T_0\, b_2(\eta_2, w') + + \frac{12}{h^2}\Big[d_3(\theta_{\varepsilon 2}, \eta_1) + d_3(\eta_2, \theta_{\varepsilon 1}) + T_0 \int_\Omega b_{\alpha\beta}\varepsilon_{\alpha\beta}^{0\prime}(u_\varepsilon, w_\varepsilon)\, \eta_1\, \mathrm{d}x\Big] + + (\chi_{3\varepsilon}, \eta)_V - (K, \eta)_H\Big\}\, \mathrm{d}t = 0$$

for all $\eta \in L^2(0, T; V)$. By the usual monotonicity argument (cf. [6], Chap. 1, 5.6.1) we finally have

$$(6.29)\qquad \chi_{1\varepsilon} = \varphi_{1\varepsilon}'(w_\varepsilon'), \quad \chi_{2\varepsilon} = \varphi_{2\varepsilon}'(u_\varepsilon'), \quad \chi_{3\varepsilon} = \varphi_{3\varepsilon}'(\theta_\varepsilon).$$

6^0 Passing to the limit for $\varepsilon \to 0$

As in 5^0 we obtain (turning, if necessary, to subsequences denoted also by $\{w_\varepsilon\}$, $\{u_\varepsilon\}$, $\{\theta_\varepsilon\}$) for $\varepsilon \to 0$ the weak* limits w, u, θ in the same spaces as w_ε, u_ε, θ_ε (with the same convergence properties of their time derivatives).

Replacing now in (6.26) q by $q - w'_\varepsilon$ ($q \in L^2(0, T; X)$), in (6.27) v by $v - u'_\varepsilon$ ($v \in L^2(0, T; V)$) and η by $\eta - \theta_\varepsilon$ in (6.28) ($\eta \in L^2(0, T; V)$) we have

$$\tag{6.30} \int_0^T \Big\{ (w''_\varepsilon, q)_{1,2} + a(w_\varepsilon, q) + \frac{12}{h^2} \int_\Omega \sigma^{0(\varepsilon)}_{\alpha\beta} w_{\varepsilon,\alpha} q_{,\beta} \, dx - - \frac{12}{h^2} b_1(\theta_{\varepsilon 1}, w_\varepsilon, q) + b_2(\theta_{\varepsilon 2}, q) + \varphi_{1\varepsilon}(q) - (F_3, q - w') \Big\} \, dt \geqq \geqq \int_0^T \Big\{ (w''_\varepsilon, w'_\varepsilon)_{1,2} + a(w_\varepsilon, w'_\varepsilon) + \frac{12}{h^2} \int_\Omega \sigma^{0(\varepsilon)}_{\alpha\beta} w_{\varepsilon,\alpha} w'_\varepsilon \, dx - - \frac{12}{h^2} b_1(\theta_{\varepsilon 1}, w_\varepsilon, w'_\varepsilon) + b_2(\theta_{\varepsilon 2}, w'_\varepsilon) + \varphi_{1\varepsilon}(w'_\varepsilon) \Big\} \, dt \,,$$

$$\tag{6.31} \int_0^T \Big\{ (u''_\varepsilon, v)_H + \int_\Omega \sigma^{0(\varepsilon)}_{\alpha\beta} v_{\alpha,\beta} \, dx - d_1(\theta_{\varepsilon 1}, v) + \varphi_{2\varepsilon}(v) - (F_\alpha, v_\alpha - u'_{\varepsilon\alpha}) \Big\} \, dt \geqq \geqq \int_0^T \Big\{ (u''_\varepsilon, u'_\varepsilon)_H + \int_\Omega \sigma^{0(\varepsilon)}_{\alpha\beta} (u'_\varepsilon)_{\alpha,\beta} \, dx - d_1(\theta_{\varepsilon 1}, u'_\varepsilon) + \varphi_{2\varepsilon}(u'_\varepsilon) \Big\} \, dt$$

and

$$\tag{6.32} \int_0^T \Big\{ a_1(\theta'_\varepsilon, \eta) + d_2(\theta_\varepsilon, \eta) - T_0 \, b_2(\eta_2, w'_\varepsilon) + + \frac{12}{h^2} \Big[d_3(\theta_{\varepsilon 2}, \eta_1) + d_3(\eta_2, \theta_{\varepsilon 1}) + T_0 \int_\Omega b_{\alpha\beta} \varepsilon^{0\prime}_{\alpha\beta}(u_\varepsilon, w_\varepsilon) \, \eta_1 \, dx \Big] + + \varphi_{3\varepsilon}(\eta) - (K, \eta - \theta_\varepsilon)_H \Big\} \, dt \geqq \int_0^T \Big\{ a_1(\theta'_\varepsilon, \theta_\varepsilon) + d_2(\theta_\varepsilon, \theta_\varepsilon) - T_0 \, b_2(\theta_{\varepsilon 2}, w'_\varepsilon) + + \frac{12}{h^2} \Big[2 d_3(\theta_{\varepsilon 2}, \theta_{\varepsilon 1}) + T_0 \int_\Omega b_{\alpha\beta} \varepsilon^{0\prime}_{\alpha\beta}(u_\varepsilon, w_\varepsilon) \, \theta_{\varepsilon 1} \, dx \Big] + \varphi_{3\varepsilon}(\theta_\varepsilon) \Big\} \, dt \,.$$

On the left hand sides of (6.30)–(6.32) we to the limit for $\varepsilon \to 0$ as in Section 4⁰ (using (6.2)). In a similar way as in Section 5⁰ we get the limit for $\varepsilon \to 0$ for all terms on the right hand side, except the $\varphi_{i\varepsilon}$-terms (noting that $\{w_\varepsilon, u_\varepsilon, \theta_\varepsilon\}$ converges strongly in $L^2(0, T; X) \times L^2(0, T; V) \times L^2(0, T; H)$ and w'_ε in $L^2(0, T; W^{1,4}(\Omega))$). If the following inequalities are satisfied we obtain from (6.30)–(6.32) the inequalities (3.1)–(3.3) of the definition of the generalized solution of the thermoelastic dynamical problem (considering the $\liminf_{\varepsilon \to 0}$ on both sides of (6.30)–(6.32)):

$$\tag{6.33} \liminf_{\varepsilon \to 0} \int_0^T \varphi_{i\varepsilon}(z_{i\varepsilon}(t)) \, dt \geqq \Phi_i(z_i)$$

for $i = 1, 2, 3$; where $z_{1\varepsilon} = w'_\varepsilon$, $z_{2\varepsilon} = u'_\varepsilon$, $z_{3\varepsilon} = \theta_\varepsilon$, $z_1 = w'$, $z_2 = u'$, $z_3 = \theta$.

It remains to prove (6.33). We only show the validity of the first inequality of (6.33). From (6.30) it follows that

$$\int_0^T \varphi_{1\varepsilon}(w_\varepsilon'(t))\,\mathrm{d}t \leqq \int_0^T \Big\{\varphi_{1\varepsilon}(q) - (F_3, q - w_\varepsilon') + \int_\Omega \sigma_{\alpha\beta}^{0(\varepsilon)} w_{\varepsilon,\alpha}(q - w_\varepsilon')\,\mathrm{d}x + $$

$$+ a(w_\varepsilon, q - w_\varepsilon') + (w_\varepsilon'', q - w_\varepsilon')_{1,2} + b_2(\theta_{\varepsilon 2}, q - w_\varepsilon') - \frac{12}{h^2} b_1(\theta_{\varepsilon 1}, w_\varepsilon, q - w_\varepsilon')\Big\}\,\mathrm{d}t$$

for all $q \in L^2(0, T; X)$. We have

$$|b_1(\theta_{\varepsilon 1}, w_\varepsilon, q - w_\varepsilon')| + |b_2(\theta_{\varepsilon 2}, q - w_\varepsilon')| \leqq$$

$$\leqq c(|\theta_\varepsilon|_H \, \|w_\varepsilon\|_{2,2}\,(1 + \|w_\varepsilon'\|_{2,2}) + |\theta_\varepsilon|_H^2 + \|w_\varepsilon'\|_{2,2}^2)\,.$$

Thus we find from (6.15), (6.16), (6.22) using again (6.19) that

$$\int_0^T \varphi_{1\varepsilon}(w_\varepsilon'(t))\,\mathrm{d}t \leqq \text{const.} \quad \text{for all} \quad \varepsilon > 0\,.$$

With the help of (5.1) we obtain

$$\frac{1}{2\varepsilon}\int_0^T \|w_\varepsilon'(t) - J_{1\varepsilon}(w_\varepsilon'(t))\|_{2,2}^2\,\mathrm{d}t \leqq \int_0^T \varphi_{1\varepsilon}(w_\varepsilon'(t))\,\mathrm{d}t - T\,.\,\varphi_1(w_1) \leqq \text{const. for all } \varepsilon > 0$$

and, finally, the inequality (6.22) yields

$$J_{1\varepsilon}(w_\varepsilon') \to w \quad \text{weakly in} \quad L^2(0, T; X)$$

and by the definition of $\varphi_{1\varepsilon}$,

$$\liminf_{\varepsilon\to 0} \int_0^T \varphi_{1\varepsilon}(w'(t))\,\mathrm{d}t \geqq \liminf_{\varepsilon\to 0} \Phi_1(J_{1\varepsilon}(w_\varepsilon')) \geqq \Phi_1(w')\,.$$

The other inequalities of (6.33) follow similarly, q.e.d.

7. PROOF OF THEOREM 5.2

We define $w = \hat{w} - w^*$, $u = \hat{u} - u^*$, $\theta = \hat{\theta} - \theta^*$ and analogously $w_0 - \hat{w}_0 - w_0^*$, $F = \hat{F} - F^*, \ldots$. Substituting now, $\{\hat{w}, \hat{u}, \hat{\theta}\}$ or $\{w^*, u^*, \hat{\theta}^*\}$ for $\{w, u, \theta\}$ or $\{q, v, \eta\}$, respectively, in the inequalities (3.1)–(3.3) of the definition of the solution, we obtain by adding then the three inequalities (using the form not integrated over t, which is equivalent to (3.1)–(3.3)

$$\text{(7.1)} \qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{h^2}{12}(\|w'\|_{1,2}^2) + a(w, w) + |u'|_H^2 + \frac{1}{T_0} a_1(\theta, \theta)\Big\} +$$

$$+ \frac{1}{T_0} d_2(\theta, \theta) + R + R_1 \leqq 0\,,$$

where R is defined by

$$R = \int_\Omega [(\hat{\sigma}^0_{\alpha\beta}\hat{w}_{,\alpha} - \sigma^{0*}_{\alpha\beta}w^*_{,\alpha})\, w'_{,\beta} + (\hat{\sigma}^0_{\alpha\beta} - \sigma^{0*}_{\alpha\beta})\, u'_{\alpha,\beta}]\, \mathrm{d}x$$

and R_1 by

$$R_1 = b_1(\theta_1^*, w^*, w') - b_1(\hat{\theta}_1, \hat{w}, w') + d_1(\theta_1, u') + \frac{2}{T_0} d_3(\theta_2, \theta_1) -$$

$$- \int_\Omega b_{\alpha\beta}(\hat{\varepsilon}^0_{\alpha\beta} - \varepsilon^{0*}_{\alpha\beta})'\, \theta_1\, \mathrm{d}x + (F_3, w') + (F_\alpha, u'_\alpha) + (K, \hat{\theta})_H\,.$$

Denoting $\bar{a}(\varepsilon) = \int_\Omega a_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}\varepsilon_{\alpha\beta}\, \mathrm{d}x$ for $\varepsilon \in [L^2(\Omega)]^4$ we have

$$R = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \hat{\sigma}^0_{\alpha\beta} w_{,\alpha} w_{,\beta}\, \mathrm{d}x - \frac{1}{2}\int_\Omega \hat{\sigma}^{0\prime}_{\alpha\beta} w_{,\alpha} w_{,\beta}\, \mathrm{d}x +$$

$$+ \int_\Omega (\hat{\sigma}^0_{\alpha\beta} - \sigma^{0*}_{\alpha\beta})\, w_{,\alpha}\hat{w}'_{,\beta}\, \mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bar{a}(\hat{\varepsilon}^0 - \varepsilon^{0*})$$

and by (6.19), (5.3)

$$\left|\int_\Omega \hat{\sigma}^0_{\alpha\beta} w_{,\alpha} w_{,\beta}\, \mathrm{d}x\right| \leqq c_1 \int_0^t \|w'(\tau)\|^2_{1,2}\, \mathrm{d}\tau + \frac{1}{2}\|w(t)\|^2_{2,2}\,, \tag{7.2}$$

$$\left|\int_\Omega \hat{\sigma}^{0\prime}_{\alpha\beta} w_{,\alpha} w_{,\beta}\, \mathrm{d}x\right| \leqq c_2(\|\hat{u}'\|_V + \|\hat{w}\|_{1,4}\, \|\hat{w}'\|_{1,4})\, \|w\|^2_{1,4} \leqq c_3\|w\|^2_{2,2}\,, \tag{7.3}$$

$$\left|\int_\Omega (\hat{\sigma}^0_{\alpha\beta} - \sigma^{0*}_{\alpha\beta})\, w_{,\alpha}\hat{w}'_{,\beta}\, \mathrm{d}x\right| \leqq \bar{a}(\hat{\varepsilon}^0 - \varepsilon^{0*}) + \int_\Omega (w_{,\alpha})^2\, (\hat{w}'_{,\beta})^2\, \mathrm{d}x \leqq \tag{7.4}$$

$$\leqq \bar{a}(\hat{\varepsilon}^0 - \varepsilon^{0*}) + c_4\|w\|^2_{2,2}\,.$$

Now we transform the strongly nonlinear terms of R_1. We have

$$b_1(\theta_1^*, w^*, w') - b_1(\theta_1, \hat{w}, w') + d_1(\theta_1, u') - \int_\Omega b_{\alpha\beta}(\hat{\varepsilon}^0_{\alpha\beta} - \varepsilon^{0*}_{\alpha\beta})'\, \theta_1\, \mathrm{d}x =$$

$$= \int_\Omega b_{\alpha\beta}(\theta_1 w_{,\alpha}\hat{w}'_{,\beta} - \hat{\theta}_1 w_{,\alpha} w'_{,\beta})\, \mathrm{d}x \leqq$$

$$\leqq c_5 \cdot \{|\theta(\tau)|^2_H + \|w(\tau)\|^2_{2,2}\} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega b_{\alpha\beta}\hat{\theta}_1 w_{,\alpha} w_{,\beta}\, \mathrm{d}x\,.$$

Using this and (7.2)–(7.4) we obtain by integrating (7.1) over $(0, t)$

$$\|w'(t)\|^2_{1,2} + a(w(t), w(t)) + |u'(t)|^2_H + a_1(\theta(t), \theta(t)) + \tag{7.5}$$

$$+ \int_0^t d_2(\theta(\tau), \theta(\tau))\,\mathrm{d}\tau + \bar{a}(\hat{\varepsilon}^0(t) - \varepsilon^{0*}(t)) \leqq$$

$$\leqq c_6 \Big\{ \|w_1\|_{1,2}^2 + \|w_0\|_{2,2}^2 + |v_0|_H^2 + \|u_0\|_V^2 + \|w_0\|_{1,4}^4 +$$

$$+ \int_\Omega b_{\alpha\beta}\, \hat{\theta}_1(t)\, w_{,\alpha}(t)\, w_{,\beta}(t)\,\mathrm{d}x + \int_0^t [\|w'(\tau)\|_{1,2}^{2} + \|(\tau)\|_{2,2}^{2} + \bar{a}(\hat{\varepsilon}^0(\tau) - \varepsilon^{0*}(\tau)) +$$

$$+ |u'(\tau)|_H^2 + |\theta(\tau)|_H^2]\,\mathrm{d}\tau + \|F\|^2_{(L^2(0,T;L^2(\Omega)))^3} + \|K\|^2_{L^2(0,T;H)} + \tfrac{1}{4}\|w(t)\|_{2,2}^2 \Big\} .$$

The remaining $b_{\alpha\beta}$-term are estimated in the usual way with the help of (6.19). Thus it follows from (7.5) by Gronwall's lemma that

$$\|w'(t)\|_{1,2}^2 + \|w(t)\|_{2,2}^2 + |u'(t)|_H^2 + |\theta(t)|_H^2 +$$

$$+ \int_0^t \|\theta(\tau)\|_V^2\,\mathrm{d}\tau + \bar{a}(\hat{\varepsilon}^0(t) - \varepsilon^{0*}(t)) \leqq$$

$$\leqq c_7\{\|w_0\|_{2,2}^2 + \|w_0\|_{1,4}^4 + \|w_1\|_{1,2}^2 + \|u_0\|_V^2 + |v_0|_H^2 +$$

$$+ \|F\|^2_{(L^2(0,T;L^2(\Omega)))^3} + \|K\|^2_{L^2(0,T;H)}\} .$$

Using again Korn's inequality (cf. (6.15), (6.16)) we obtain the inequality (5.4) q.e.d.

References

[1] *B. A. Boley, J. H. Weiner:* Theory of Thernal Stresses. Wiley & Sons, Inc., New York 1960.

[2] *C. Dafermos:* On the Existence and the Asymptotic Stability of Solutions to the Equations of Linear Thermoelasticity. Arch. Rat. Mech. Anal., 29 (1968), 241—271.

[3] *G. Duvaut, J. L. Lions:* Problèmes unilateraux dans la théorie de la flexion forte des plaques, I: Le cas stationaire. J. Méch., 13 (1974), 51—74.

[4] *G. Duvaut, J. L. Lions:* Problèmes unilateraux dans la théorie de la flexion forte des plaques, II: Le cas d'évolution. J. Méch., 13 (1974), 245—266.

[5] *G. Duvaut:* Inéquations en thermoélasticité et magnétohydrodynamique. Arch. Rat. Mech. Anal., 46 (1972), 241—279.

[6] *G. Duvaut:* Les inéquations en méchanique et en physique. Dunod, Paris 1972.

[7] *S. A. Gribanov, V. N. Ogibalov*: Thermostability of Plates and Shells (in Russian). Izd. Mosc. Univ. 1968.

[8] *I. Hlaváček, J. Naumann:* Inhomogeneous Boundary Value Problems for the von Kármán's Equations, I. Apl. Matem., 19 (1974), 253—269.

[9] *I. Hlaváček, J. Naumann:* Inhomogeneous Boundary Value Problems for the von Kármán's Equations, II. Apl. Matem., 20 (1975), 280—297.

[10] *I. Hlaváček, J. Nečas:* On Inequalities of Korn's Type, I: Boundary value problems for elliptic systems of partial differential equations, II: Applications to linear elasticity, Arch. Rat. Mech. Anal., 36 (1970), 305—334.

[11] *R. Hünlich, J. Naumann:* On General Boundary Value Problems and Duality in Linear Elasticity, I. Apl. Matem., 23 (1978).

[12] *J. L. Lions, E. Magénès:* Problèmes aux limites non homogènes et applications, vol. 1. Dunod, Paris 1968.
[13] *J. Naumann:* On Some Unilateral Boundary Value Problems for the von Kármán's Equations, part I: The coercive case. Apl. Matem., 20 (1975), 96—125.
[14] *J. Naumann:* On Some Unilateral Boundary Value Problems in Non-linear Plate Theory. Beiträge zur Analysis, 10 (1977), 119—134.
[15] *J. Nečas:* Les méthodes dirèctes en théorie des équations elliptiques. Academia, Prague 1967.
[16] *W. Nowacki*: Theory of Elasticity (in Russian). Mir, Moscow 1975.
[17] *W. Nowacki:* Thermoelasticity. Intern. Series of Monographs on Aeronautics and Astronautics, Div. I: Solid and Structural Mechanics, Pergamon Press 1962.
[18] *W. Nokacki*: Dynamical Problems of Thermoelasticity (in Russian). Mir, Moscow 1970.
[19] *K. Washizu:* Variational Methoús in Elasticity and Plasticity. Pergamon Press 1974.
[20] *H. - U. Wenk:* Functional-analytical Investigations on Unilateral Problems in Plate Theory (in German). Thesis A, Humboldt-Univ., Berlin 1978.

Souhrn

O SDRUŽENÝCH TERMOELASTICKÝCH VIBRACÍCH GEOMETRICKY NELINEÁRNÍCH TENKÝCH DESEK, SPLŇUJÍCÍCH ZOBECNĚNÉ MECHANICKÉ A TEPELNÉ PODMÍNKY NA HRANICI A NA POVRCHU

HANS-ULLRICH WENK

Práce se zabývá termoelastickou úlohou pro vibrace tenké desky a tím navazuje na sérii článků o von Kármánových rovnicích. Práce obsahuje odvození variační formulace úlohy, důkaz existence a závislosti řešení na počátečních a okrajových podmínkách a několik příkladů různých „netradičních" okrajových podmínek.

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