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Computer identification of plane regions

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COMPUTER IDENTIFICATION OF PLANE REGIONS

JOZEF ZÁMOŽÍK

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1. INTRODUCTION

Our aim is to give a simple algorithm for the computer identification of the insidedness and the outsidedness of a plane bounded region, i.e. an algorithm by means of which it is possible to determine whether a point belongs to the given region. The region itself can be arbitrarily complicated.

The algorithm can be applied in many ways, for example for determining the contour, the boundary of a region that can be the union, intersection or difference of an arbitrary number of k-tuply connected regions, further, for the graphic solution of an operation on abstract sets or for the determination of the projection outline of a polyhedron, for the automatic hatching of arbitrarily complicated regions etc. These problems belong to the characteristic problems of the computer graphics and image processing. Similar problems are solved e.g. in [1], [2], etc. See also the references in [1] and [4].

The solution of our problem was initiated in [3], where authors describe a special solution of the problem of a polyhedron's projection outline (the polyhedron is given by the wire model encoded by means of the TV camera and the laser beam, respectively).

Section 2 deals with the determination of the so-called ***r**-value* of the boundary of a simply connected region. In Section 3 we determine the so-called ***R**-value* of a region and in Sections 4 and 5 the ***R**-value* for the union, the intersection and the difference of regions. In Section 6 we deal with a k-tuply connected region. Section 7 offers several practical remarks and in conclusion indicates some possibilities of applications.

2. BASIC CONSIDERATIONS

N o t e 1. The constructions and theorems in this paper are elementary consequences of theorems of the set theory, geometry and algebra and they are constructed for practical purposes. Therefore we omit the proofs of theorems. The correctness of theorems and the well-founededness of definitions are evident.

In the plane, the basic element in our consideration will be the so-called simply connected (open) region $\boldsymbol{R}$. It is bounded by a negative by oriented and closed line $\boldsymbol{r}$ (in short, boundary), which does not belong to the region and which does not intersect itself.

Note 2. Section 7 a negative by oriented and closed *broken* line will be considered as such a boundary.

Note 3. The negative orientation of the boundary $\boldsymbol{r}$ of a simply connected region $\boldsymbol{R}$ is the clockwise orientation, so that in the sense of the orientation the region lies to the right from its boundary.

Let us note that every straight line p generally intersects a boundary $\boldsymbol{r}$ in an even number of ordinary points. We will consider a point of fracture on the straight line p or a point of tangency on a tangent p as two identical points of the boundary $\boldsymbol{r}$ and of the line p as well.

Let p be an oriented line and φ_i the oriented angle between the line p and the tangent to the boundary (in this order) at its point of intersection P_i, Let $\{P_i\}$ be the sequence of points of intersection such that each pair of neighbouring points satisfies the following condition: The point P_i is before P_{i+1} (P_{i+1} is behind P_i) if and only if $t_i < t_{i+1}$, where t_i and t_{i+1} are the values of an increasing parameter of the oriented line p at the points P_i and P_{i+1}, respectively.

To an ordinary points of intersetion P_i (for a case $\varphi_i \neq 0$) let us assign the so-called *r-value* 1 or -1, such that (Fig. 1):

$$(1) \qquad \begin{array}{rl} 1 & \text{for} \quad 0 < \varphi_i < \pi \\ -1 & \text{for} \quad \pi < \varphi_i < 2\pi \end{array}$$

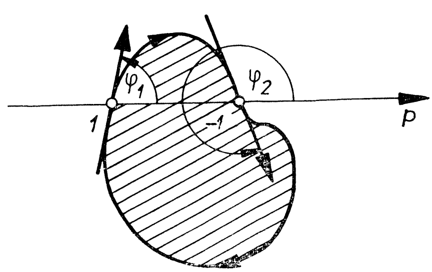

Fig. 1. The *r-value* 1 or -1 of ordinary points.

For the other points P_i (i.e. points of tangency, points of reversal and points of fracture) we define their *r-values* as follows (Fig. 2):

Let the point $P_{i \mp \Delta i}$ with $u_i \mp \Delta u$ for the value of the increasing parameter on the boundary (where Δu (>0) is sufficiently small) not lie on the line p and let its tangent be not parallel to the line p; then we determine $\varphi_{i \mp \Delta i}$ (i.e. the angle between this

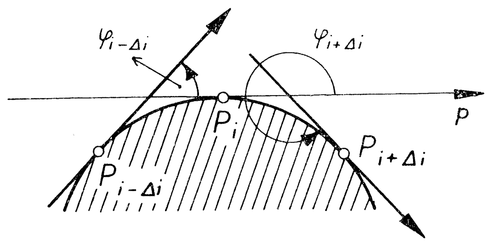

Fig. 2. To the ***r-value*** for the rest of points P_i.

tangent and the line p) and the *r-values* 1 or -1 as in (1) for an ordinary point of intersection. Then the so-called resulting *r-value* at the point P_i will be as given in Table 1.

$P_{i-\Delta i}$ \ $P_{i+\Delta i}$	1	-1
1	1	0
-1	0	-1

Tab. 1.

Several exemples of *r-values* are in Fig. 3.

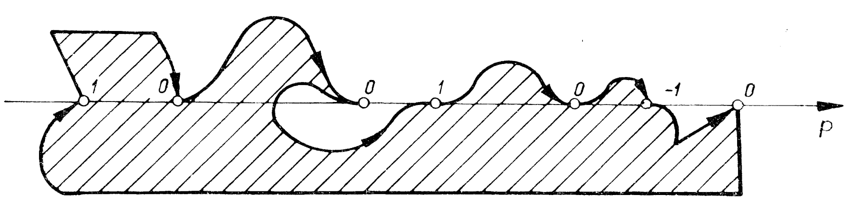

Fig. 3. The ***r-values*** for some points.

Note 4. Let the parameter u_i of a closed line $\boldsymbol{r}$ be from the interval $\langle a, b\rangle$. Moreover if $u_i = a$, then let $u_i - \Delta u = b - \Delta u$; if $u_i = b$, then $u_i + \Delta u = a + \Delta u$.

A special case occurs when part of the boundary is a segment on the line p. We omit this case for the present and we shall deal with it in detail only in the practical part (Section 7).

Note 5. The ***r**-values* at points of reversal and at such points of tangency that the line p is the tangent of the line $\boldsymbol{r}$ are defined in this section for the completeness. With regard to Note 2 we shall not use these values for practical purposes.

3. ***R**-value*

Let M be a point and $\boldsymbol{R}$ a simply connected region in the plane. Let us assign to the point M the so-called ***R**-value* 0 if $M \notin \boldsymbol{R}$ and the ***R**-value* 1 if $M \in \boldsymbol{R}$. The points on a boundary $\boldsymbol{r}$ have the ***R**-value* 0 because they do not belong to the region.

Now we will define the ***R**-value* for the points lying on the line p. Let k_i^p be the ***r**-value* (or the resulting ***r**-value*) at the point P_i and K_i^p the ***R**-value* between P_i and P_{i+1} (or behind the point P_i if P_i is the last common point of the line p and the boundary). Then

$$K_i^p = \sum_{j=1}^{i} k_j^p \tag{2}$$

It is obvious that ***R**-value* of all points on the line p before the point P_1 is 0. Moreover, $K_1^p \neq -1$ and $K_n^p = 0$ where K_n^p is the ***R**-value* with regard to the point P_n that is the last intersection of the line p and of the boundary $\boldsymbol{r}$. These assertions follow directly from the definition of the ***R**-values* and ***r**-values* on the line p (for example the ***r**-value* $k_i^p = -1$ implies that the region with regard to (2) is not negative oriented (Fig. 4)).

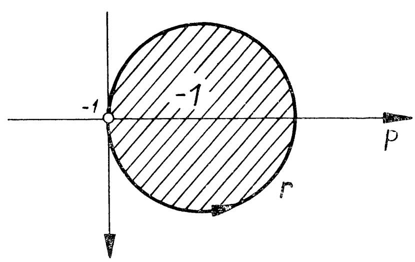

Fig. 4. The ***r**-value* and the ***R**-value* for a positive oriented region.

An example of ***R**-values* on the line p is in Fig. 5.

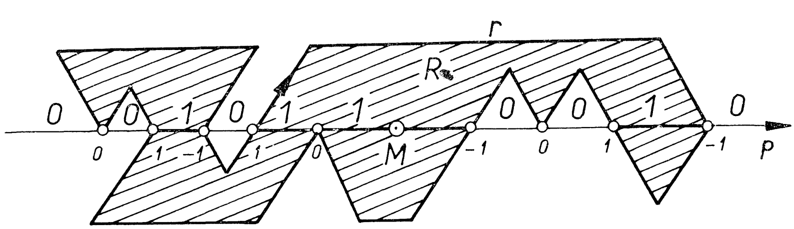

Fig. 5. Example of ***R**-values* on the line p (and at the point M).

Example 1. Using the foregoing considerations we can solve the problem: determine whether a point M belongs to the region $\boldsymbol{R}$.

Algorithm.

1. Let us choose an oriented line p through the piont M and detect its points of intersection with the boundary $\boldsymbol{r}$ (see Fig. 5).

2. If $P_i = M$ for some P_i then $K_i^p = 0$ at M and the point M does not lie in the region (then STOP follows).

3. If $P_i \neq M$ for all P_i then we construct a sequence $\{P_i\}$ by means of the increasing parameter of the line p and we determine all ***r**-values* k_i^p up to the point P_i which is the nearest to M at the same time satisfies $t_P < t_M$ (t_P and t_M are the values of the parameter for the points P_i and M).

4. Let us determine the ***R**-value* K_i^p according to (2). If $K_i^p = 0$ then the point M does not lie in the region; if $K_i^p = 1$ then the point M lies in the region $\boldsymbol{R}$.

4. *R-value* OF A REGION THAT IS THE INTERSECTION OR UNION OF REGIONS

Let ${}^1\boldsymbol{R}, {}^2\boldsymbol{R}, \ldots, {}^s\boldsymbol{R}$ be simply connected regions and p a straight line. If the case $P_h = {}^rP_i = \ldots = {}^vP_j$ occurs, where ${}^rP_i, \ldots, {}^vP_j$ are the intersections of the regions and of the line p then the ***r**-value* (of the boundary) at the point P_h is ${}^rk_i + \ldots + {}^vk_j = k_h$. In Fig. 6 the ***r**-value* for the point $P_5 = {}^1P_2 = {}^2P_4$ is $k_5 = {}^1k_2 + {}^2k_4 = -1 + (-1) = -2$.

Let there exist a point that belongs to each of the regions ${}^1\boldsymbol{R}, {}^2\boldsymbol{R}, \ldots, {}^n\boldsymbol{R}$. Let us assign the ***R**-value* n to this point. Then in general the ***R**-value* of the region $\boldsymbol{R} = {}^1\boldsymbol{R} \cap {}^2\boldsymbol{R} \cap \ldots \cap {}^n\boldsymbol{R}$ obviously is at most n.

Now we determine the ***R**-value* of an intersection of regions on the line p. Let k_h^p be the ***r**-value* (or the resulting ***r**-value*) at the point P_h. If P_h and P_{h+1} are the neighbouring points common to the line p and to one of the boundaries of the given regions then we label by K_h^p the ***R**-value* on the line p between the points P_h and P_{h+1}. If P_h is the last point common to the line p and to one of the boundaries then we

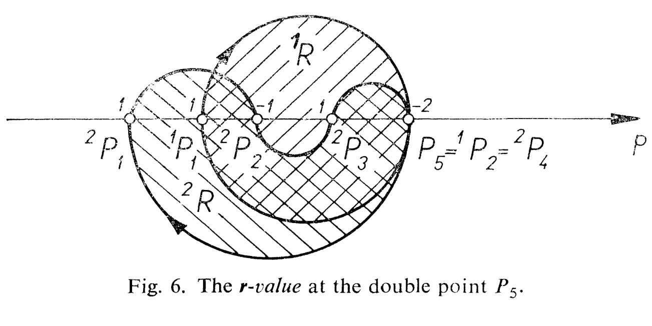

Fig. 6. The ***r**-value* at the double point P_5.

label by K_h^p the ***R**-value* on the line p behind the point P_h. Then the ***R**-value* of the points of the line p between the points P_h and P_{h+1} (or behind the point P_h) is:

$$K_h^p = \sum_{g=1}^{h} k_g^p \,. \tag{3}$$

The ***R**-value* at the point P_h is $K_h^p - k_h^p$.

The competence of our definition follows from the set operation of intersection because $0 \leqq K_h^p \leqq n$.

Note that the construction of the ***R**-value* immediately implies

$$K_h^p = K_{h-1}^p + k_h^p \,. \tag{4}$$

Example 2. Let regions ${}^1\boldsymbol{R}, {}^2\boldsymbol{R}, \ldots, {}^n\boldsymbol{R}$ and a point M be given in the plane. Determine the ***R**-value* of the intersection of those regions which contain the point M.

Algorithm.

1. We choose an oriented line p through the point M and detect its intersections with the boundaries of the regions.

2. Let us construct a sequence $\{P_h\}$ by means of the increasing parameter of the line p and let us determine all the resulting ***r**-values* k_h^p up to the point P_h that is the nearest point to the point M for which $t_P \leqq t_M$ (t_P and t_M are the values of the parameter of the line p at the point P_h and M).

3. We determine the ***R**-value* K_h^p using (3). If $P_h \neq M$ then the ***R**-value* of the intersection at the point M is K_h^p; if $P_h = M$ then this ***R**-value* is $K_h^p - k_h^p$.

In Fig. 7 the point M lies in the intersection of three out of four regions because $K_6^p = 1 + 1 - 1 + 1 + 0 + 1 = 3$. The point P_6 has the ***R**-value* $K_6^p - 1 = 3 - 1 = 2$.

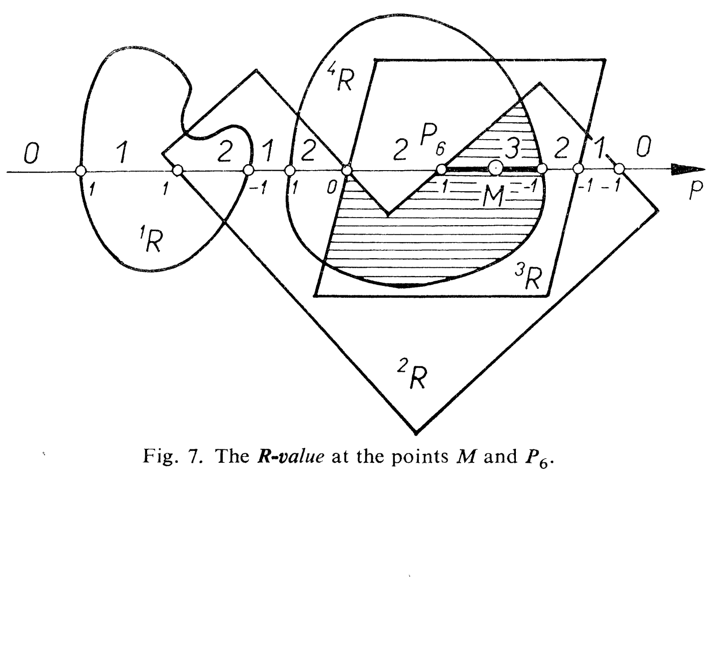

Fig. 7. The ***R**-value* at the points M and P_6.

The above algorithm can be used e.g. for a solution of the following problems:

1. To mark (e.g. by hatching) the intersection for a given ***R-value*** (if it exists).
2. To mark the boundary of this intersection.
3. To mark the union of an arbitrary number of region.
4. To mark the boundary of this union.

We solve Problems 1 and 2 so that we detect (on a suitable number of parallel straight lines) such segment $P_h P_{+1}$ whose inside points have the given ***R-value***. Then the points P_h and P_{h+1} lie on the boundary of the intersection.

Problems 3 and 4 (concerning the union) is solved by detecting (on the suitable number of a parallel straight lines) the line segments $P_h\ P_{h+1}$ whose in their inside points have the ***R-value*** at least 1. The points P_h, P_{h+1} (on parallel lines) with the ***R-values*** 0 are the points of the boundary of the union.

Note 6. The ***R-value*** of the inside points of the segment $P_h P_{h+1}$ can be determined advantageously by means of (4).

5. *R-value* OF THE DIFFERENCE OF REGIONS

Let us label by $\bar{\boldsymbol{R}}$ a simply connected region in the plane with a positive by oriented boundary (i.e. counterclockwise, so that in the sense of this orientation the region lies to the left from its boundary). We retain the method for determining the ***r-values*** of the boundary and of the ***R-values*** of the region as in the previous section. Then the ***R-value*** of the inside of the region is -1 (Fig. 8).

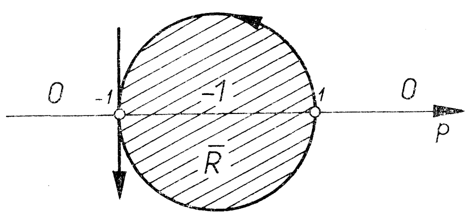

Fig. 8. The ***r-values*** and the ***R-value*** for the $\bar{\boldsymbol{R}}$ region.

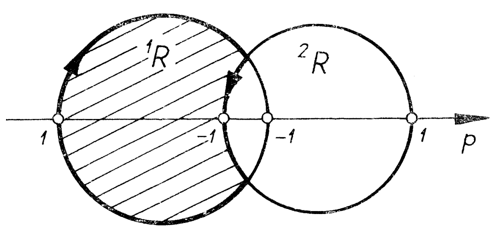

Fig. 9. The difference of regions $^1\boldsymbol{R}$, $^2\boldsymbol{R}$ by means of the union $^1\boldsymbol{R} \cup {}^2\boldsymbol{R}$.

Let us have two regions ${}^1\boldsymbol{R}$ and ${}^2\boldsymbol{R}$. We construct their difference ${}^1\boldsymbol{R} \setminus {}^2\boldsymbol{R}$ by means of the union ${}^1\boldsymbol{R} \cup {}^2\bar{\boldsymbol{R}}$ (Fig. 9).

6. k-TUPLY CONNECTED REGIONS

Using the region $\bar{\boldsymbol{R}}$ we can simply define a k-tuply connected region (i.e. a region with the boundary that is formed by k disjoint closed lines).

Definition. *If* ${}^1\boldsymbol{R}, {}^2\boldsymbol{R}, \ldots, {}^n\boldsymbol{R}$ *are simply connected regions, if* ${}^2\boldsymbol{R} \cap {}^1\boldsymbol{R} = {}^2\boldsymbol{R}$, ${}^3\boldsymbol{R} \cap {}^1\boldsymbol{R} = {}^3\boldsymbol{R}, \ldots, {}^n\boldsymbol{R} \cap {}^1\boldsymbol{R} = {}^n\boldsymbol{R}$ *and the intersections of the other pairs of regions*

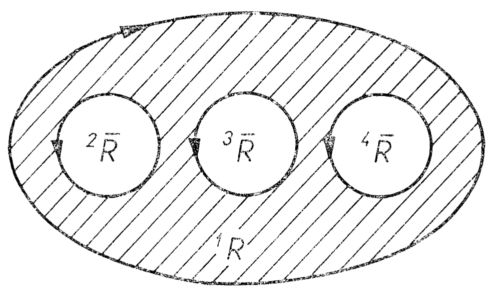

Fig. 10. To the k-tuply connected region.

are empty then the region ${}^1\boldsymbol{R} \cup {}^2\bar{\boldsymbol{R}} \cup {}^3\bar{\boldsymbol{R}} \cup \ldots \cup {}^n\bar{\boldsymbol{R}}$ *is an n-tuply connected region.* Fig. 10 shows a 4-tuply connected region.

* * *

Fig. 19 and Fig. 20 show a more complicated example of the set operations on the regions.

7. TECHNICAL NOTES

1. In computer processing, a curve is usually aproximated by a *broken* line. Then in our case, a simply connected region is a polygon (without the boundary). The problem of finding the intersections of the straight line p and the boundary $\boldsymbol{r}$ is then reduced to the problem of finding the intersections of this straight line and any side of the polygon.

2. To simplify the computations we shall suppose that the line p is parallel to the axis x (from the coordinate system Oxy), which can always be reached by a suitable rotation of the coordinate system. Let x_i, y_i be the coordinates of the vertex V_i

of the region $\boldsymbol{R}$. The neighbouring vertices to the vertex V_i are V_{i-1} and V_{i+1}. The vertex V_1 is joined by sides with the vertices V_2 and V_n.

3. If a point P_i is an inside point of the side V_iV_{i+1} of the boundary then the ***r-value*** at this point can be determined as follows (instead of determining it by the angle φ_i):

Let the equation of the line p (which is oriented in the same sense as the axis x) be $y = y_0$;

a) if $y_i < y_0 < y_{i+1}$ then the ***r-value*** at the point P_i is 1,
b) if $y_{i+1} < y_0 < y_i$ then the ***r-value*** at the point P_i is -1 (Fig. 11).

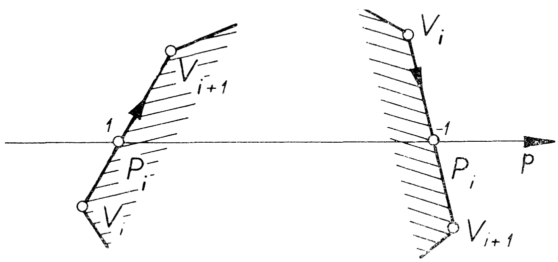

Fig. 11. To determining the ***r-value*** on the broken line boundary.

4. Let $P_i = V_i$ (neither $V_{i-1}V_i$ nor V_iV_{i+1} being parallel to the axis x). Then the ***r-value*** at this point will be (cf. Table 1):

1, if $y_{i-1} < y_i < y_{i+1}$;
-1, if $y_{i+1} < y_i < y_{i-1}$;
0, if $y_{i-1} < y_i \wedge y_{i+1} < y_i$;
0, if $y_i < y_{i-1} \wedge y_i < y_{i+1}$ (Fig. 12).

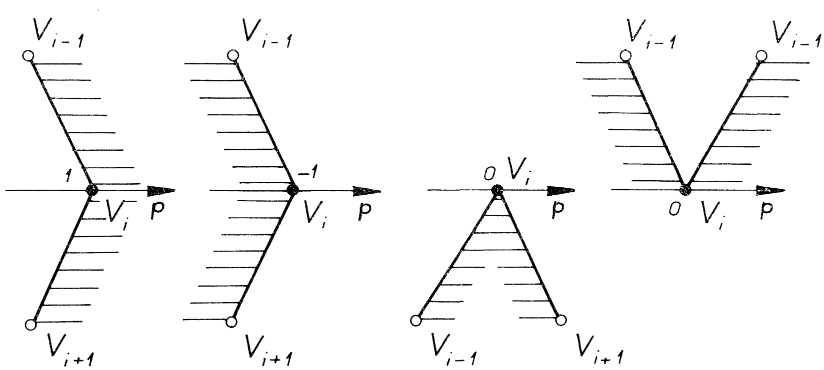

Fig. 12. The ***R-values*** at the vertices.

5. Let $V_iV_{i+1} \parallel x$. Then we assign no ***r-value*** to the inside points on the side V_iV_{i+1}. For the points $V_i = P_j$ and $V_{i+1} = P_k$ we first determine the so-called preliminary ***r-values***:

If $y_{i-1} < y_i$ then the preliminary ***r-value*** at V_i is 1,
if $y_i < y_{i-1}$ then the preliminary ***r-value*** at V_i is -1,
if $y_{i+2} < y_i$ then the preliminary ***r-value*** at V_{i+1} is -1,
if $y_i < y_{i-2}$ the the preliminary ***r-value*** at V_{i-1} is 1.

We determine the resulting ***r-values*** of the boundary at the points V_i and V_{i+1} from Table 2. Here we re-label the points V_i, V_{i+1} by A, B in such a way that the

preliminary r-value			resulting r-value	
A	B	⟹	A	B
-1	1	⟹	-1	1
1	-1	⟹	0	0
-1	-1	⟹	-1	0
1	1	⟹	0	1

Tab. 2.

x-coordinate of the point A is less than the x-coordinate of the point B. Fig. 13 shows an example a) of the preliminary ***r-values***, b) of the resulting ***r-values*** for the vertices of the boundary for this special case.

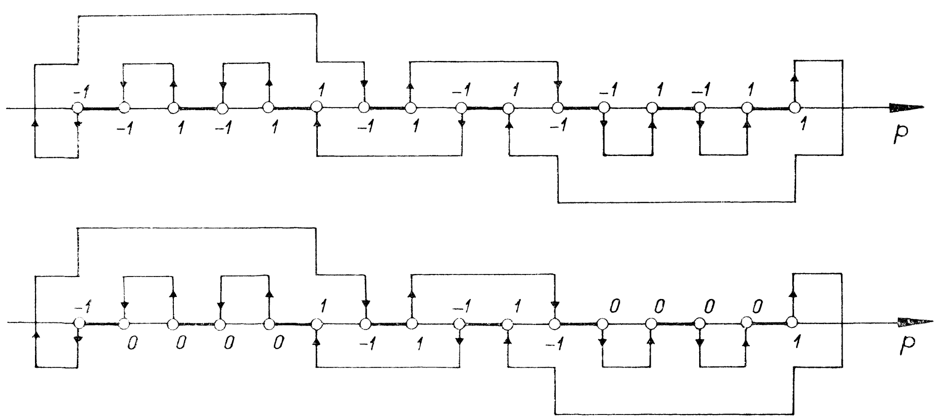

Fig. 13. Example a) of preliminary ***r-values*** and b) of resulting ***r-values***.

6. The ***R**-values* of the k-tuply connected region, the ***R**-values* of the intersection, the union and the difference of these regions are determined, as in the foregoing sections. In the case that one of a region ${}^i\boldsymbol{R}$ belongs to the line p and this side contains the point P_{ji} of intersection of the boundary of another region ${}^j\boldsymbol{R}$ and the line p (Fig. 14), then the resulting ***r**-value* at this point is just the ***r**-value* at the point P_{ji} with

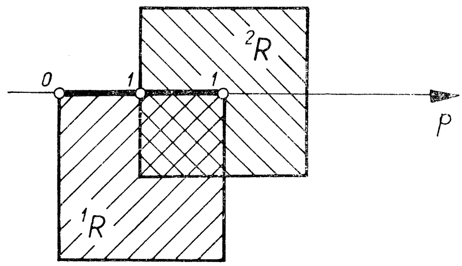

Fig. 14. The ***r**-value* at a point that belongs to the side.

regard to the region ${}^j\boldsymbol{R}$ (cf. Note 5 in this section). If sides of more regions lie on the line p and they have a common intersection then we determine by using Note 5 of this section only the ***r**-values* at the endpoints of these sides (in Fig. 15 there are two cases of such two regions).

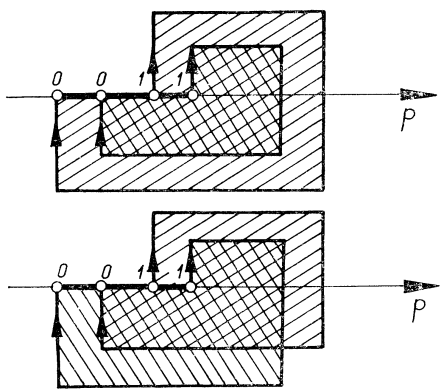

Fig. 15. The ***r**-value* at the endpoint of the intersection of the sides.

8. EXAMPLES

We present some suggestions for the application of our method.

1. When solving the visibility of solid objects on the plotter, display or alphanumeric high-speed printer it is necessary first to detect whether the given point is occluded

(in projection) by some object. In Fig. 16(a) and 17(a) the point M does not lie in the region that is given by the projection of the object, in Fig. 16(b) and 17(b) the point M lies in this region.

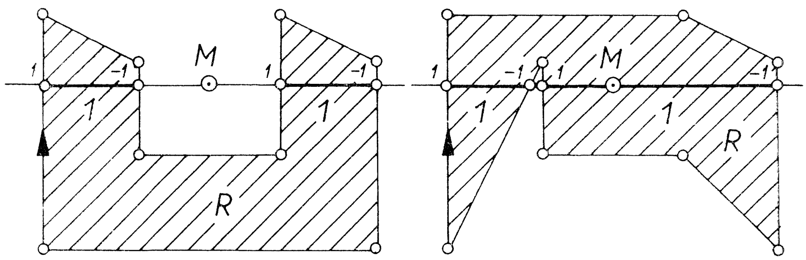

Fig. 16 (a) and (b). The point M and the region.

2. The relative visibility of a face of a polyhedron can be determined by means of the position of the outside notmal vector of this face to the center of projection (see e.g. [3]). An other method is as follows: To the outside of the boundary of the face let us assign the negative by oriented boundary. If the projection of the poly-

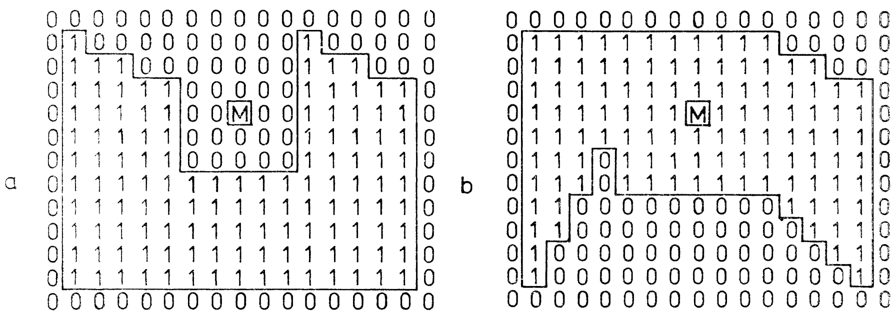

Fig. 17 (a) and (b) The point M and the region.

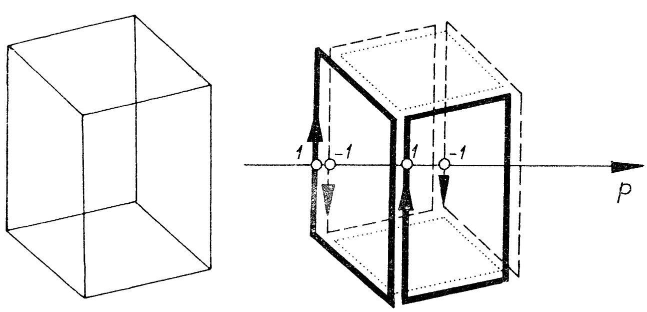

Fig. 18. To the detection of visibility of the face.

hedron face has a positive by oriented boundary then the face is occluded. It is sufficient to detect the *r-value* of the first point of intersection of the boundary and any line p (Fig. 18).

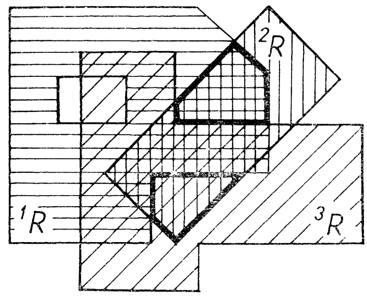

Fig. 19. To the detection of the union, intersection or difference of regions.

3. Hatching (or indication by a sign) of the region that is given by intersection, union and difference of more regions can be another example for using the given method. Fig. 19 shows a full two times hatched region $({}^1\boldsymbol{R} \setminus {}^3\boldsymbol{R}) \cap {}^2\boldsymbol{R}$ and the region $({}^3\boldsymbol{R} \setminus {}^1\boldsymbol{R}) \cap {}^2\boldsymbol{R}$. In Fig. 20 these regions are labelled by D and I, respectively.

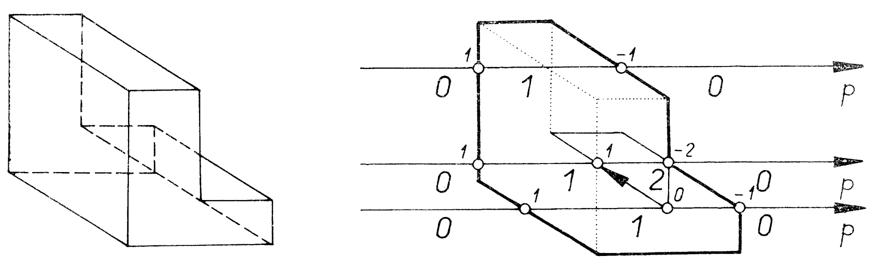

Fig. 20. To the detection of the regions that are labelled by characters.

```
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 A A A A A A A A A 0 B B 0 0 0 0 0
0 0 A A A A A A A A A D B B B 0 0 0 0
0 0 A A A E E E E A D D D B B B 0 0 0
0 0 A A 0 C C E E D D D D B B 0 0 0 0
0 0 A A 0 C C E H D D D D B 0 0 0 0 0
0 0 A A A E E H H H H H H C C C C 0 0
0 0 A A A E H H H H H H E C C C C 0 0
0 0 A A A E E H I I I C C C C C C 0 0
0 0 A A A E E E I I C C C C C C C 0 0
0 0 A A A E E E C C C C C C C C C 0 0
0 0 0 0 0 C C C C C 0 0 0 0 0 0 0 0 0
0 0 0 0 0 C C C C C 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
```

Fig. 21. Detecting the boundary of the projection of an object.

4. The boundary of the union viewed as the outline of the projection of a polyhedron [3]. In fact, it is the line composed of the points between the region with the ***R**-value* 0 and the ***R**-value* that is greater than 0 (Fig. 21). By out method differing from the at used in [3] we can determine the ***r**-value* at an arbitrary point of the boundary by (just) one line p that goes through this point.

References

[1] *P. Lemkin:* The Run Length Map: A Representation of Contours and Regions for Efficient Search & Low Level Semantic Encoding Comp. Science Technical Report Series, Univ. of Maryland, Apr. 1978.

[2] *M. Minsky* and *S. Papert:* Perceptrons — an Introduction to Comupational Geometry. MIT Press, Cambridge, Mass, 1969.

[3] *D. Nitzan* and *G. J. Agin:* Fast Methods for Finding Object Outlines. Computer Graphics and Image Processing 1 (1979), 22—39.

[4] *A. Rosenfeld:* Survey — Picture Processing: 1978, Computer graphics and Image Processing 4 (1979), 354—393.

Súhrn

POČÍTAČOVÁ IDENTIFIKÁCIA ROVINNEJ OBLASTI

JOZEF ZÁMOŽÍK

Pomocou ***R**-hodnôt* bodov rovinnej oblasti, resp. ***r**-hodnôt* hranice oblasti možno identifikovať body oblasti a ich hranice, pričom výsledná oblasť môže byť produktom (množinových) operácií na ľubovoľnom množstve k-násobne súvislých oblastí.

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