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REGIONS OF STABILITY FOR ILL-POSED CONVEX PROGRAMS*)

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1. INTRODUCTION

A mathematical problem is well-posed (in the sense of Hadamard) if it has a unique solution and if the solution continuously depends on the data. If one of these two requirements is not met, the problem is ill-posed.

In this paper we will study ill-posed convex programs

$$(P, \theta) \qquad \operatorname*{Min}_{(x)} f^0(x, \theta)$$
$$\text{s.t.}$$
$$f^k(x, \theta) \leqq 0\,, \quad k \in \mathscr{P} =^{\Delta} \{1, \ldots, m\}$$

where $f^i : R^n \times R^p \to R$ are continuous functions and $f^i(\cdot, \theta) : R^n \to R$ are convex for every $\theta \in R^p$, $i \in \{0\} \cup \mathscr{P}$. The program will be studied locally around an arbitrary but fixed $\theta = \theta^*$. It is assumed that the feasible set

$$F(\theta^*) =^{\Delta} \{x \in R^n : f^k(x, \theta^*) \leqq 0, k \in \mathscr{P}\}$$

is nonempty and bounded. (If $F(\theta^*) = \emptyset$, then one may optimize $f^0(x, \theta^*)$ over the set of say, Chebyshev solutions of (P, θ^*), as has been suggested in e.g. [2], [17]. In the case of linear equality constraints this approach is closely related to the theory of best approximate solutions, see e.g. [4], [18].)

Of the two aspects of ill-posedness, stability, i.e. dependence of the optimal solutions and values on the data, is the more serious one. The stability has been well studied for the right hand side (RHS) perturbations (see e.g. [23]), linear programs

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(e.g. [5], [19], [20], [21], [25]), quadratic programs (e.g. [12]), and under the assumption that Slater's condition (e.g. [15]) is satisfied, i.e. that

$$\exists \hat{x} \in R^n \quad \text{such that} \quad f^k(\hat{x}, \theta^*) < 0 \quad \forall k \in \mathscr{P}$$

(e.g. [7]). Other results on stability are mostly of general nature and they characterize stability in terms of point-to-set-mappings [6], [8], [9], [11], [14] or dual programs [22]. Recently in [28], [31], a different approach to stability has been suggested. Rather than characterizing stability we have looked for "chunks" of space containing θ^* in which both the optimal solutions and optimal values continuously depend on the data. Such chunks are termed "regions of stability". The objective of this paper is to survey these regions, study their properties, and demonstrate their importance and usefulness.

The paper is organized as follows. The regions of stability from [27], [28], and [31] are recalled in Section 2. The main result of this section states that the problem of solving an arbitrary convex program using restricted Lagrangians (e.g. [28], [29]) is a continuous process for perturbations in a region of stability. The second aspect of ill-posedness, i.e. the non-uniqueness of an optimal solution is studied in Section 3. Using "Tihonov's regularization" (e.g. [7], [24]) a prescribed optimal solution x^*, say the one of the smallest norm, can be obtained after adding an appropriate "penalty" to the objective function, in which case x^* is obtained as the limit of a sequence $x = x(\alpha, \theta)$ as $\alpha \to 0$, $\theta \to \theta^*$. However, in order to guarantee the convergence $x \to x^*$, θ must approach θ^* through a region of stability, thus preserving continuity of the process. In Sections 4 and 5 we utilize a new region of stability and define the marginal value of (P, θ). Then explicit formulas for the marginal value are furnished in terms of the derivatives of restricted Lagrangians. In Section 4 we assume that an optimal solution of (P, θ^*) is unique; this assumption is dropped in Section 5, where essentially different results of the minimax type are derived. Section 6 relates our results to the ones from the literature [7], [23], [27]. Finally, in Section 7, we outline how the regions of stability can be used in convex multicriteria decision making to check whether a given point is Pareto optimal (e.g. [3]), and also in calculating the minimal index set of binding constraints for convex programs (e.g. [1]). The two nonlinear problems reduce to the problem of constructing a region of stability for simple linear programs.

2. REGIONS OF STABILITY

For a perturbed convex program (P, θ), at every $\theta \in R^p$, we denote by

$$F(\theta) =^{\Delta} \{x \in R^n : f^k(x, \theta) \leqq 0, k \in \mathscr{P}\}$$

the *feasible set*, by $\tilde{F}(\theta) \subset F(\theta)$ the set of all *optimal solutions*. by $\tilde{x}(\theta) \in \tilde{F}(\theta)$ an optimal solution, and by $\tilde{f}(\theta)$ the corresponding *optimal value*. Further

$$\mathscr{P}^{=}(\theta) =^{\Delta} \{k \in \mathscr{P} : x \in F(\theta) \Rightarrow f^k(x, \theta) = 0\}$$

is the *minimal index set of binding constraints* (e.g. [1], [2]),

$$\mathscr{P}^{<}(\theta) =^{\Delta} \mathscr{P} \setminus \mathscr{P}^{=}(\theta)$$

and

$$F^{=}(\theta) =^{\Delta} \{x \in R^n : f^k(x, \theta) = 0,\ k \in \mathscr{P}^{=}(\theta)\} .$$

Algorithms for calculating $\mathscr{P}^{=}$ and $F^{=}$ are suggested in [1], [30], see also [2].

Definition 2.1. Perturbed convex program (P, θ) is *stable in a region* $S \subset R^p$ *at* $\theta^* \in S$ if, for some neighbourhood N^* of θ^*, both

(i) $\theta \in N^* \cap S \Rightarrow \tilde{F}(\theta) \neq \emptyset$, and
(ii) $\theta \in N^* \cap S$ and $\theta \to \theta^* \Rightarrow \{\tilde{x}(\theta)\}$ is bounded and all its limit points are in $\tilde{F}(\theta^*)$.

It has been shown (see e.g. [28], [31]) that (P, θ) is stable in the followin regions at θ^*:

$$M(\theta^*) = \{\theta : F(\theta^*) \subset F(\theta)\} ;$$

$$V(\theta^*) = \{\theta : F^{=}(\theta^*) \subset F^{=}(\theta) \text{ and } f^k(x, \theta) \leqq 0\ \forall x \in F(\theta^*),\ k \in \mathscr{P}^{=}(\theta^*),\ k \notin \mathscr{P}^{=}(\theta)\}$$

$$W(\theta^*) = \{\theta : F^{=}(\theta^*) \subset F^{=}(\theta) \text{ and } \mathscr{P}^{=}(\theta^*) = \mathscr{P}^{=}(\theta)\} .$$

While $W(\theta^*) \subset V(\theta^*)$, the two regions $M(\theta^*)$ and $V(\theta^*)$ are not comparable (see [31] for examples). In order to derive formulas for the marginal value, one should consider subsets of the stability regions. Thus in [27] the subset

$$Z_1(\theta^*) = \{\theta : F(\theta^*) \subset F(\theta) \subset F^{=}(\theta^*)\}$$

of $M(\theta^*)$ is used. In this paper we will derive marginal value formulas for a subset of the stability region $V(\theta^*)$, namely.

$$Z_2(\theta^*) = \{\theta : F^{=}(\theta^*) = F^{=}(\theta) \text{ and } f^k(x, \theta) \leqq 0\ \forall x \in F(\theta^*),\ k \in \mathscr{P}^{=}(\theta^*),\ k \notin \mathscr{P}^{=}(\theta)\} .$$

Note that under Slater's condition, $Z_2(\theta^*)$ becomes a neighbourhood of θ^*, in which case the familiar results on stability and marginal values from e.g. [7], [23] will be recovered. This is not the case with the region $Z_1(\theta^*)$; if Slater's condition holds, $Z_1(\theta^*)$ is not necessarily a neighbourhood of θ^*. (See Section 6.)

In the study of perturbed convex program, the standard Lagrangian function is defined as

$$L(x, u; \theta) = f^0(x, \theta) + \sum_{k \in \mathscr{P}} u_k f^k(x, \theta) .$$

However, the following *restricted Lagrangians* (see e.g. [27], [28], [29]) are often found more suitable:

$$L_\theta(x, u) = f^0(x, \theta) + \sum_{k \in \mathscr{P}^{<}(\theta)} u_k f^k(x, \theta)$$

and

$$L_{\theta*}(x, u) = f^0(x, \theta) + \sum_{k \in \mathscr{P}^<(\theta^*)} u_k f^k(x, \theta).$$

The restricted Lagrangian $L_\theta(x, u)$ is used in characterizing optimal solutions of convex programs. Indeed, a point $\tilde{x} \in F^=(\theta^*)$ is optimal for (P, θ^*) if, and only if, there exists a $\tilde{u} \in R_+^{q(\theta^*)}$, the nonnegative orthant of $R^{q(\theta^*)}$, where $q(\theta^*) = \text{card}\, \mathscr{P}^<(\theta^*)$, such that $(\tilde{x}, \tilde{u})$ is a saddle point of L_{θ^*}, i.e.

$$(2.1) \qquad L_{\theta^*}(\tilde{x}, u) \leqq L_{\theta^*}(\tilde{x}, \tilde{u}) \leqq L_{\theta^*}(x, \tilde{u})$$

for all $x \in F^=(\theta^*)$ and $u \in R_+^{q(\theta^*)}$ (e.g. [29]). For a fixed θ we denote by $\tilde{U}^<(\theta)$ the set of multipliers $\tilde{u}_k = \tilde{u}_k(\theta)$, $k \in \mathscr{P}^<(\theta^*)$ which correspond to an optimal solution $\tilde{x}(\theta)$ of (P, θ). Some important properties of this set are summarized in Theorem 2.2 below. We denote by $S(\theta^*)$ the union of the five regions of stability:

$$\begin{aligned} S(\theta^*) &= M(\theta^*) \cup V(\theta^*) \cup W(\theta^*) \cup Z_1(\theta^*) \cup Z_2(\theta^*) \\ &= M(\theta^*) \cup V(\theta^*). \end{aligned}$$

Theorem 2.2. *Let $\tilde{F}(\theta^*) \neq \emptyset$ be bounded and let $\mathscr{P}^<(\theta^*) \neq \emptyset$. Then there exists a neighbourhood N^* of θ^* such that*

(i) $\tilde{U}^<(\theta) \neq \emptyset$ *for every* $\theta \in N^* \cap S(\theta^*)$;
(ii) *The set* $\cup\{\tilde{U}^<(\theta) : \theta \in N^* \cap S(\theta^*)\}$ *is bounded;*
(iii) $\tilde{F}(\theta) \times \tilde{U}^<(\theta) \neq \emptyset$ *for every* $\theta \in N^* \cap S(\theta^*)$;
(iv) *If* $\theta \in N^* \cap S(\theta^*)$, $(\tilde{x}(\theta), \tilde{u}(\theta)) \in \tilde{F}(\theta) \times \tilde{U}^<(\theta)$ *and* $\theta \to \theta^*$, *then the sequence* $\{(\tilde{x}(\theta), \tilde{u}(\theta))\}$ *is bounded and every one of its limit points is in* $\tilde{F}(\theta^*) \times \tilde{U}^<(\theta^*)$.

Proof. This result is proved in [28, Lemma 3 and Theorem 3] for the set $V(\theta^*)$. The same proof works for $M(\theta^*)$, and thus for $S(\theta^*)$. □

The last statement in Theorem 2.2 shows that the problem of solving (P, θ^*) is a continuous process for perturbations $\theta \to \theta^*$ if they pass through a region of stability. The restricted Lagrangian L_θ^* will be used to derive the formulas for marginal values.

If Slater's condition holds for (P, θ^*), then $\mathscr{P}^=(\theta^*) = \emptyset, F^=(\theta^*) = R^n$ and, by the continuity of f^k, $k \in \mathscr{P}$, $S(\theta^*)$ becomes a neighbourhood of θ^*. Theorem 2.2 reduces in this case to the results obtained in e.g. [7, Lemma 26.2 and Theorem 26.4].

3. TIHONOV'S REGULARIZATION

If (P, θ^*) does not have a unique solution, one may still obtain an arbitrary prescribed optimal solution as the limit of an infinite sequence. In Tihonov's regularization one considers the auxiliary problem

$$(T, \theta) \qquad \begin{aligned} &\operatorname*{Min}_{(x)} f^0(x, \theta) + \varepsilon \mathscr{J}(x) \\ &\text{s.t.} \\ &f^k(x, \theta) \leqq 0, \quad k \in \mathscr{P} \end{aligned}$$

where $\varepsilon > 0$ and $\mathscr{J}(x)$ is a strictly convex function such that, for an arbitrary but fixed δ_0, the set $\{x : \mathscr{J}(x) \leqq \delta_0\}$ is nonempty and bounded. Let $x(\varepsilon; \theta)$ denote the (unique) optimal solution of (T, θ) for a given $\varepsilon > 0$ and let $\tilde{x}$ denote the (unique) optimal solution of

$$\begin{aligned} &\text{Min } \mathscr{J}(x) \\ &\text{s.t.} \\ &x \in \tilde{F}(\theta^*) . \end{aligned}$$

Then the convergence is established by the following result, which is included here for the sake of completeness.

Theorem 3.1. *If $\tilde{F}(\theta^*) \neq \emptyset$ then, for every fixed $\varepsilon > 0$, there is a neighbourhood N^* of θ^* such that*

$$\lim_{\substack{\theta \in N^* \cap S(\theta^*) \\ \theta \to \theta^*}} x(\varepsilon; \theta) = x(\varepsilon; \theta^*) .$$

Moreover,

$$\lim_{\varepsilon \to 0^+} x(\varepsilon; \theta^*) = \tilde{x} .$$

Proof. For a proof of this result and for related results the reader is referred to [31]. □

4. MARGINAL VALUE

In this section we will derive formulas for the marginal value of perturbations in $Z_2(\theta^*)$. The marginal value will be expressed in terms of the partial derivaties of the restricted Lagrangian L_θ^* with respect to the parameter θ.

First we need the following result.

Lemma 4.1. *Consider (P, θ^*). Then $\mathscr{P}^<(\theta^*) \subset \mathscr{P}^<(\theta)$ for every $\theta \in V(\theta^*)$.*

Proof. Take a $\theta \in V(\theta^*)$. We claim that

$$\mathscr{P}^=(\theta) \subset \mathscr{P}^=(\theta^*) . \tag{4.1}$$

If (4.1) were not true, there would exist an index k_0 such that $k_0 \in \mathscr{P}^=(\theta)$ but $k_0 \notin \mathscr{P}^=(\theta^*)$. Now

$$\begin{aligned} F^=(\theta^*) &\subset F^=(\theta) , \quad \text{since} \quad \theta \in V(\theta^*) \\ &\subset \{x : f^{k_0}(x, \theta) = 0\} , \quad \text{since} \quad k_0 \in \mathscr{P}^=(\theta) . \end{aligned}$$

But this means that $f^{k_0}(x, \theta) = 0$ for every $x \in F^=(\theta^*)$, i.e. $k_0 \in \mathscr{P}^=(\theta^*)$; a contradiction. The inclusion (4.1) implies $\mathscr{P} \setminus \mathscr{P}^<(\theta) \subset \mathscr{P} \setminus \mathscr{P}^<(\theta^*)$ and hence $\mathscr{P}^<(\theta^*) \subset \subset \mathscr{P}^<(\theta)$.

We will also need two inequalities.

Lemma 4.2. *If $\tilde{F}(\theta^*) \neq \emptyset$ is bounded, and if $\mathscr{P}^<(\theta^*) \neq \emptyset$, then there exists a neighbourhood N^* of θ^* such that for every $\theta \in N^* \cap V(\theta^*)$:*

$$(4.2) \qquad \tilde{f}(\theta) - \tilde{f}(\theta^*) \geqq L_\theta(\tilde{x}(\theta), u) - L_{\theta^*}(x, \tilde{u}(\theta^*)) \quad \forall u \in R_+^{q(\theta)}, \quad \forall x \in F^=(\theta^*)$$

and

$$(4.3) \qquad \tilde{f}(\theta) - \tilde{f}(\theta^*) \leqq L_\theta(x, \tilde{u}(\theta)) - L_{\theta^*}(\tilde{x}(\theta^*), u) \quad \forall u \in R_+^{q(\theta^*)}, \quad \forall x \in F^=(\theta).$$

Proof. The existence of a neighbourhood N^* of θ^* such that $\tilde{F}(\theta) \times \tilde{U}^<(\theta) \neq \emptyset$ for every $\theta \in N^* \cap V(\theta^*)$ is guaranteed by Theorem 2.2. The inequalities (4.2) and (4.3) now follow, after appropriate subtractions, from the saddle point optimality condition (2.1), written separately for $\theta \in N^* \cap V(\theta^*)$ and θ^*. □

A marginal value formula for perturbations in $Z_2(\theta^*)$ follows. The same formula is derived in [27] for perturbations in $Z_1(\theta^*)$, but different arguments are used there in the proof. The formula holds for the convex program (P, θ) with some additional assumptions.

Theorem 4.3. *Let $\tilde{F}(\theta^*) \neq \emptyset$ be bounded. Suppose also that $\tilde{F}(\theta^*) \times \tilde{U}^<(\theta^*)$ consists of a unique point $(\tilde{x}(\theta^*), \tilde{u}(\theta^*))$. If $f^k(x, \cdot)$, $k \in \{0\} \cup \mathscr{P}^<(\theta^*)$ are convex and differentiable in $N^* \cap Z_2(\theta^*)$, where N^* is a neighbourhood of θ^*, and if $[f^k(x, \theta)]'_\theta$ is continuous in x and θ at $(\tilde{x}(\theta^*), \theta^*)$, then, for every fixed path $\theta \in Z_2(\theta^*)$, $\theta \to \theta^*$,*

$$(4.4) \qquad \lim_{\substack{\theta \in Z_2(\theta^*) \\ \theta \to \theta^*}} \frac{\tilde{f}(\theta) - \tilde{f}(\theta^*)}{\|\theta - \theta^*\|} = ([L_\theta^*(\tilde{x}(\theta^*), \tilde{u}(\theta^*))]'_{\theta=\theta^*}, l)$$

where

$$(4.5) \qquad l = \lim_{\substack{\theta \in Z_2(\theta^*) \\ \theta \to \theta^*}} \frac{\theta - \theta^*}{\|\theta - \theta^*\|}.$$

Proof. By the assumptions, $L_\theta^*(x, u)$ is convex and differentiable in $\theta \in N^* \cap Z_2(\theta^*)$. This is in particular true for $L_\theta^*(\tilde{x}(\theta^*), u)$. Now

$$([L_\theta^*(\tilde{x}(\theta^*), \tilde{u}(\theta)]'_\theta, \theta - \theta^*) \geqq L_\theta^*(\tilde{x}(\theta^*), \tilde{u}(\theta)) - L_{\theta^*}^*(\tilde{x}(\theta^*), \tilde{u}(\theta)),$$

by the gradient inequality;

$$= L_\theta(\tilde{x}(\theta^*), \tilde{u}(\theta)) - L_{\theta^*}^*(\tilde{x}(\theta^*), \tilde{u}(\theta)) - \sum_{k \in \mathscr{P}^<(\theta)/\mathscr{P}^<(\theta^*)} u_k(\theta) f^k(\tilde{x}(\theta^*), \theta),$$

since $L_{\theta^*}^* = L_{\theta^*}$ and $\mathscr{P}^<(\theta^*) \subset \mathscr{P}^<(\theta)$, by Lemma 4.1;

$$\geqq L_\theta(\tilde{x}(\theta^*), \tilde{u}(\theta)) - L_{\theta^*}(\tilde{x}(\theta^*), \tilde{u}(\theta));$$

since $u_k \geqq 0$, $k \in \mathscr{P}^<(\theta)$, and hence $u_k(\theta) f^k(\tilde{x}(\theta^*), \theta) \leqq 0$, $k \in \mathscr{P}^<(\theta) \setminus \mathscr{P}^<(\theta^*)$; (The latter is true because $\tilde{x}(\theta^*) \in F(\theta^*)$ and hence $f^k(\tilde{x}(\theta^*), \theta) \leqq 0$ for every $k \in \in \mathscr{P}^<(\theta) \setminus \mathscr{P}^<(\theta^*) = \mathscr{P}^=(\theta^*) \setminus \mathscr{P}^=(\theta)$, by the definition of $Z_2(\theta^*)$.)

$$\geqq \tilde{f}(\theta) - \tilde{f}(\theta^*),$$ by (4.3) after specifying $u = \tilde{u}(\theta^*)$ and $x = \tilde{x}(\theta^*)$.

(Note that

$$\tilde{x}(\theta^*) \in F(\theta^*) \subset F^=(\theta^*) = F^=(\theta)\,; \tag{4.6}$$

the inclusion follows by the definition of $F^=$ and the equality by the definition of $Z_2(\theta^*)$.)

$$\geqq L_\theta(\tilde{x}(\theta), u) - L_{\theta^*}(x, \tilde{u}(\theta^*)),$$ by (4.2) for every $u \in R_+^{q(\theta)}$ and every $x \in F^=(\theta)$; in particular for $u = (u_k)$, where

$$u_k = \begin{cases} [\tilde{u}(\theta^*)]_k & \text{if } \; k \in \mathscr{P}^<(\theta^*) \\ 0 & \text{if } \; k \in \mathscr{P}^<(\theta) \setminus \mathscr{P}^<(\theta^*)\,, \end{cases}$$

and $x = \tilde{x}(\theta)$. (Note that $\tilde{x}(\theta) \in F(\theta) \subset F^=(\theta)$.) With these specifications:

$$= L^*_\theta(\tilde{x}(\theta), \tilde{u}(\theta^*)) - L^*_{\theta^*}(\tilde{x}(\theta), \tilde{u}(\theta^*))$$
$$\geqq ([L^*_\theta(\tilde{x}(\theta), \tilde{u}(\theta^*))]'_{\theta=\theta^*}, \theta - \theta^*)\,,$$

by the gradient inequality.

Thus we have proved that for all $\theta \in N^* \cap Z_2(\theta^*)$:

$$\begin{aligned}([L^*_\theta(\tilde{x}(\theta^*), \tilde{u}(\theta)]'_\theta, \theta - \theta^*) &\geqq \tilde{f}(\theta) - \tilde{f}(\theta^*) \geqq \\ &\geqq [(L^*_\theta(\tilde{x}(\theta), \tilde{u}(\theta^*))]'_{\theta=\theta^*}, \theta - \theta^*)\,.\end{aligned} \tag{4.7}$$

Now we divide (4.7) by $\|\theta - \theta^*\|$ and set $\theta \to \theta^*$ over the stability region $Z_2(\theta^*)$. Since $(\tilde{x}(\theta^*), \tilde{u}(\theta^*))$ is a unique point in $\tilde{F}(\theta^*) \times \tilde{U}^<(\theta^*)$, $(\tilde{x}(\theta), \tilde{u}(\theta)) \to (\tilde{x}(\theta^*), \tilde{x}(\theta^*))$, by Theorem 2.2 (iv). Furthermore, by the continuity of the derivative in both x and θ at $x = \tilde{x}(\theta^*)$ and $\theta = \theta^*$, it follows that

$$\lim_{\substack{\theta \in Z_2(\theta^*) \\ \theta \to \theta^*}} [L^*_\theta(\tilde{x}(\theta), \tilde{u}(\theta^*))]'_\theta = [L^*_\theta(\tilde{x}(\theta^*), \tilde{u}(\theta^*))]'_{\theta=\theta^*}\,. \tag{4.8}$$

Also, by convexity and differentiability (and hence continuous differentiability, see e.g. [7, Exercise 4, p. 125]) assumption on $f^k(x, \cdot)$,

$$\lim_{\substack{\theta \in Z_2(\theta^*) \\ \theta \to \theta}} [L^*_\theta(\tilde{x}(\theta^*), \tilde{u}(\theta))]'_\theta = [L^*_\theta(\tilde{x}(\theta^*), \tilde{u}(\theta^*))]'_{\theta=\theta^*}\,. \tag{4.9}$$

This completes the proof.

Example 4.4 Consider the program

$$\text{Min } f^0 = -x$$
s.t.

$$(E, \theta) \qquad \begin{aligned} f^1 &= (x - \theta_1)(x - \theta_1 - 1) \leqq 0 \\ f^2 &= (\theta_1 - \theta_2)^2 x \leqq 0 \end{aligned}$$

perturbed at $\theta^* = (0, 0)^T$.

Since

$$F(\theta) = \begin{cases} [\theta_1, 1 + \theta_1] & \text{if} \quad \theta_1 = \theta_2 \\ [\theta_1, 1 + \theta_1] \cap (-\infty, 0] & \text{if} \quad \theta_1 \neq \theta_2 , \end{cases} \tag{4.10}$$

the path $\theta \to \theta^*$ along

$$\theta \in \pi(\theta^*) \overset{\Delta}{=} \left\{ \begin{pmatrix} 0 \\ \theta_2 \end{pmatrix} : \theta_2 \to {}^+0 \right\}$$

gives

$$\tilde{x}(\theta) = \begin{cases} 0 & \text{if} \quad \theta_2 \neq 0 \\ 1 & \text{if} \quad \theta_2 = 0 \end{cases} \quad \text{and} \quad \tilde{f}(\theta) = \begin{cases} 0 & \text{if} \quad \theta_2 \neq 0 \\ -1 & \text{if} \quad \theta_2 = 0 . \end{cases}$$

The marginal value blows to infinity along this path:

$$\frac{\tilde{f}(\theta) - \tilde{f}(\theta^*)}{\|\theta - \theta^*\|} = \frac{1}{\theta_2} \to \infty ,$$

i.e. the program is unstable.

In order to find a region of stability, in which the marginal value is finite and the formulas (4.4)–(4.5) apply, let us first attempt $Z_1(\theta^*)$. It is obvious from (4.10) that $F(\theta^*) = [0, 1] \subset F(\theta)$ only if $\theta_1 = 0 = \theta_2$, implying that $Z_1(\theta^*)$ consists only of the single point $Z_1(\theta^*) = \theta^* = 0$. Thus $Z_1(\theta^*)$ does not provide, in this particular extreme case, any information on how to perturb the program to perserve continuity or how to calculate the corresponding marginal value.

However, $Z_2(\theta^*)$ is here more useful. Since $\mathscr{P}^=(\theta) = \mathscr{P}^=(\theta^*) = \{2\}$, whenever $\theta_1 = \theta_2 \in R$, and for such θ's we have $F^=(\theta) = F^=(\theta^*) = R$, it follows that

$$Z_2(\theta^*) = \left\{ \begin{pmatrix} \theta_1 \\ \theta_1 \end{pmatrix} : \theta_1 \in R \right\} .$$

The conditions of Theorem 4.3 are satisfied and the marginal value formula (4.4) is applicable. Since (E, θ^*) has the unique saddle point $\tilde{x}(\theta^*) = 1$, $\tilde{u}(\theta^*) = 1$ of the restricted Lagrangian

$$L_{\theta^*}(x, u) = f^0(x, \theta^*) + u_1 f^1(x, \theta^*) ,$$

we calculate

$$L_\theta^*(\tilde{x}(\theta^*), \tilde{u}(\theta^*)) = -1 + \theta_1^2 - \theta_1$$

and find that

$$[L_\theta^*(\tilde{x}(\theta^*), \tilde{u}(\theta^*))]'_{\theta=\theta^*} = \begin{pmatrix} 2\theta_1 - 1 \\ 0 \end{pmatrix}_{\theta=\theta^*} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} .$$

Also, since $\theta^* = 0$,

$$l = \lim_{\substack{\theta \in Z_2(0) \\ \theta \to {}^+0}} \frac{\theta}{\|\theta\|} = \frac{\sqrt{2}}{2}\begin{pmatrix}1\\1\end{pmatrix}$$

and the marginal value for perturbations $\theta \in Z_2(\theta^*)$, $\theta \to {}^+\theta$ (θ approaching zero through the positive orthant) is

$$[(L_\theta^*(\tilde{x}(\theta^*), \tilde{u}(\theta^*))]'_{\theta=\theta^*}, l) = \frac{\sqrt{2}}{2}(-1, 0)\begin{pmatrix}1\\1\end{pmatrix} = -\frac{\sqrt{2}}{2}.$$

The above result is confirmed by the direct calculation: For every $\theta \in Z_2(\theta^*)$, $\tilde{x}(\theta) = 1 + \theta_1$ and $\tilde{f}(\theta) = -1 - \theta_1$. Therefore, for such θ's,

$$\frac{\tilde{f}(\theta) - \tilde{f}(\theta^*)}{\|\theta - \theta^*\|} = \frac{-\theta_1}{\sqrt{(2)}\,|\theta|} \to -\frac{\sqrt{(2)}}{2}$$

as $\theta \to {}^+0$. (For perturbations $\theta \in Z_2(\theta^*)$, $\theta \to \theta^*$ through the negative orthant, the marginal value formula gives $+\sqrt{(2)}/2$.)

5. MARGINAL VALUE: UNIQUENESS NOT REQUIRED

The marginal value formula in Theorem 4.3 is derived under the assumption that the saddle point of the restricted Lagrangian L_θ^* is unique. In this section we will drop this assumption.

Following Eremin and Astafiev [7, Section 28] we assume that $f^k(x, \cdot)$, $k \in \{0\} \cup \mathscr{P}$ are differentiable in θ at $\theta = \theta^*$ for every x in some metric ε-extention of $\tilde{F}(\theta^*)$, i.e.

$$\forall x \in \tilde{F}_\varepsilon(\theta^*) =^\Delta \{x \in R^n : \inf_{y \in \tilde{F}(\theta^*)} \|x - y\| \leqq \varepsilon\}$$

for some $\varepsilon > 0$. As before, $[f^k(x, \theta)]'_{\theta=\theta^*}$ denotes the gradient of $f^k(x, \cdot)$ with respect to θ at θ^*. Thus by differentiability of $f^k(x, \cdot)$,

$$f^k(x, \theta) - f^k(x, \theta^*) = ([f^k(x, \theta)]'_{\theta=\theta^*}, \theta - \theta^*) + r^k(x, \theta) \tag{5.1}$$

where

$$\lim_{\theta \to \theta^*} \frac{r^k(x, \theta)}{\|\theta - \theta^*\|} = 0 \quad \forall x \in \tilde{F}_\varepsilon(\theta^*), \quad k \in \{0\} \cup \mathscr{P}. \tag{5.2}$$

A stronger condition than (5.2) is

$$\lim_{\theta \to \theta^*} \frac{r^k(x, \theta)}{\|\theta - \theta^*\|} = 0 \text{ uniformly in } \quad x \in \tilde{F}_\varepsilon(\theta^*), \quad k \in \{0\} \cup \mathscr{P} \tag{5.3}$$

i.e.

$$\{x^i\} \subset \tilde{F}_\varepsilon(\theta^*), \quad \theta^i \to \theta^* \Rightarrow \frac{r^k(x^i, \theta^i)}{\|\theta^i - \theta^*\|} \to 0, \quad k \in \{0\} \cup \mathscr{P}.$$

As noted in [7], the assumption (5.3) implies continuity of $[f^k(x, \theta)]'_{\theta=\theta^*}$ in the variable x on $\tilde{F}_\varepsilon(\theta^*)$, Clearly, (5.3) trivially holds if $f^k(x, \cdot)$ are linear functions.

Theorem 5.1. *Let $\tilde{F}(\theta^*) \neq \emptyset$ be bounded. Suppose also that the functions $f^k(x, \theta)$, $k \in \{0\} \cup \mathscr{P}$ are differentiable in θ at $\theta = \theta^*$ for every $x \in \tilde{F}_\varepsilon(\theta^*)$ and such that the uniformity condition* (5.3) *holds. Then, for every fixed path $\theta \in Z_2(\theta^*)$, $\theta \to \theta^*$*

$$\lim_{\substack{\theta \in Z_2(\theta^*) \\ \theta \to \theta^*}} \frac{\tilde{f}(\theta) - \tilde{f}(\theta^*)}{\|\theta - \theta^*\|} = \min_{\tilde{x} \in \tilde{F}(\theta^*)} \max_{\tilde{u} \in \tilde{U}^<(\theta^*)} \left([L^*_\theta(\tilde{x}, \tilde{u})]'_{\theta=\theta^*}, l\right) \tag{5.4}$$

where

$$l = \lim_{\substack{\theta \in Z_2(\theta^*) \\ \theta \to \theta^*}} \frac{\theta - \theta^*}{\|\theta - \theta^*\|}.$$

Proof. Denote

$$R(\theta, \tilde{u}(\theta^*)) =^\Delta r^0(\tilde{x}(\theta), \theta) + \sum_{k \in \mathscr{P}^<(\theta^*)} \tilde{u}_k(\theta^*)\, r^k(\tilde{x}(\theta), \theta). \tag{5.5}$$

Let $\tilde{x}(\theta^*)$ be an arbitrary fixed limit point in $\tilde{F}(\theta^*)$ of the sequence $\theta \in Z_2(\theta^*)$, $\theta \to \theta^*$. Now, after reexamining the proof of Theorem 4.3. we conclude that

$$\begin{aligned}\tilde{f}(\theta) - \tilde{f}(\theta^*) &\geqq L^*_\theta(\tilde{x}(\theta), \tilde{u}(\theta^*)) - L^*_{\theta^*}(\tilde{x}(\theta), \tilde{u}(\theta^*)) \quad \text{for every} \quad \tilde{u}(\theta^*) \in \tilde{U}^<(\theta^*) \\ &= ([L^*_\theta(\tilde{x}(\theta), \tilde{u}(\theta^*))]'_{\theta=\theta^*}, \theta - \theta^*) + R(\theta, \tilde{u}(\theta^*)),\end{aligned}$$

using (5.1) and (5.5).

Hence,

$$\lim_{\substack{\theta \in Z_2(\theta^*) \\ \theta \to \theta^*}} \frac{\tilde{f}(\theta) - \tilde{f}(\theta^*)}{\|\theta - \theta^*\|} \geqq \lim_{\substack{\theta \in Z_2(\theta^*) \\ \theta \to \theta^*}} \left([L^*_\theta(\tilde{x}(\theta), \tilde{u}(\theta^*))]'_{\theta=\theta^*}, \frac{\theta - \theta^*}{\|\theta - \theta^*\|}\right), \tag{5.6}$$

by (5.5) and (5.2).

The limit on the left hand side in (5.6) does not depend on how θ approaches θ^* (see e.g. [31, Theorem 3.5]). In particular, we can choose a subsequence $\{\theta^i\} \subset \{\theta\}$ such that $\tilde{x}(\theta^i) \to \tilde{x}(\theta^*)$. Hence

$$\lim_{\substack{\theta \in Z_2(\theta^*) \\ \theta \to \theta^*}} \frac{\tilde{f}(\theta) - \tilde{f}(\theta^*)}{\|\theta - \theta^*\|} \geqq ([L^*_\theta(\tilde{x}(\theta^*), \tilde{u}(\theta^*))]'_{\theta=\theta^*}, l) \tag{5.7}$$

for all $\tilde{u}(\theta^*) \in \tilde{U}^<(\theta^*)$.

Also, using (4.3), (5.1) and (5.3) we find that

$$\begin{aligned}\lim_{\substack{\theta \in Z_2(\theta^*) \\ \theta \to \theta^*}} \frac{\tilde{f}(\theta) - \tilde{f}(\theta^*)}{\|\theta - \theta^*\|} &\leqq \lim_{\substack{\theta \in Z_2(\theta^*) \\ \theta \to \theta^*}} \left([L^*_\theta(\tilde{x}(\theta^*), \tilde{u}(\theta))]'_{\theta=\theta^*}, \frac{\theta - \theta^*}{\|\theta - \theta^*\|}\right) \\ &= ([L^*_\theta(\tilde{x}(\theta^*), \tilde{u}(\theta^*))]'_{\theta=\theta^*}, l)\end{aligned} \tag{5.8}$$

for an arbitrary fixed $\tilde{u}(\theta^*) \in \tilde{U}^<(\theta^*)$ and all $\tilde{x}(\theta^*) \in \tilde{F}(\theta^*)$.

The inequalities (5.7) and (5.8) thus give

$$([L_\theta^*(x, \tilde{u}(\theta^*))]'_{\theta=\theta^*}, l) \geqq ([L_\theta^*(\tilde{x}(\theta^*), u)]'_{\theta=\theta^*}, l)$$

for every $x \in \tilde{F}(\theta^*)$ and every $u \in \tilde{U}^<(\theta^*)$, i.e. $(\tilde{x}(\theta^*), \tilde{u}(\theta^*))$ is a saddle point of

$$\Phi(x, u) =^\Delta ([L_\theta^*(x, u)]'_{\theta=\theta^*}, l)$$

on the set $\tilde{F}(\theta^*) \times \tilde{U}^<(\theta^*)$. Therefore

$$\max_{u \in \tilde{U}^<(\theta^*)} \min_{x \in \tilde{F}(\theta^*)} \Phi(x, u) = \min_{x \in \tilde{F}(\theta^*)} \max_{u \in \tilde{U}^<(\theta^*)} \Phi(x, u) = \lim_{\substack{\theta \in Z_2(\theta^*) \\ \theta \to \theta^*}} \frac{\tilde{f}(\theta) - \tilde{f}(\theta^*)}{\|\theta - \theta^*\|} . \qquad \square$$

6. SETS Z_1 AND Z_2 ARE DIFFERENT

If Slater's condition holds for (P, θ^*), then $Z_2(\theta^*)$ is a neighbourhood of θ^* and hence $Z_1(\theta^*) \subset Z_2(\theta^*)$. The example below shows that $Z_1(\theta^*)$ may be properly contained in $Z_2(\theta^*)$.

Example 6.1. Consider a program (P, θ) with the constraints

$$\begin{aligned} f^1 &= x^2 - 1 \leqq 0 , \\ f^2 &= x - \theta \leqq 0 , \\ f^3 &= x \qquad \leqq 0 . \end{aligned}$$

Here

$$F(\theta) = \begin{cases} \emptyset & \text{if} \quad \theta < -1 \\ [-1, \theta] & \text{if} \quad -1 \leqq \theta < 0 \\ [-1, 0] & \text{if} \quad \theta \geqq 0 \end{cases}$$

and Slater's condition is obviously satisfied at $\theta^* = 0$. This implies that $Z_2(\theta^*) = N^*$, a neighbourhood of θ^*. Moreover, $\mathscr{P}^=(\theta) = \emptyset$ and $F^=(\theta) = R$ for every $\theta \in N^*$. Hence $Z_1(\theta^*) = N^* \cap [0, \infty)$, clearly a smaller set than $Z_2(\theta^*)$.

In the absence of Slater's condition the opposite is the case: $Z_2(\theta^*)$ may be properly contained in $Z_1(\theta^*)$. Such a situation is described in the following example.

Example 6.2. Consider the constraints

$$\begin{aligned} f^1 &= -x \leqq 0 , \\ f^2 &= \max\{0, -\theta^2 \operatorname{sgn} \theta\} \, . \max\{0, -x^2 \operatorname{sgn} x\} \leqq 0 , \\ f^3 &= \max\{0, \quad \theta^2 \operatorname{sgn} \theta\} \, . \max\{0, -x^2 \operatorname{sgn} x\} \leqq 0 . \end{aligned}$$

Here $F(\theta) = [0, \infty)$ and $\mathscr{P}^=(\theta) = \{2, 3\}$ for every θ. But

$$F^=(\theta) = \begin{cases} R & \text{if} \quad \theta = 0 \\ [0, \infty) & \text{if} \quad \theta \neq 0 . \end{cases}$$

Hence, at $\theta^* = 0$, $Z_1(\theta^*) = R$ while $Z_2(\theta^*) = \{\theta^*\}$, just the point θ^* itself.

Combining our results with those from [27], we conclude that the marginal value formulas derived in Sections 4 and 5 hold for $\theta \in Z(\theta^*) =^{\Delta} Z_1(\theta^*) \cup Z_2(\theta^*)$, $\theta \to \theta^*$. When Slater's condition holds, $Z(\theta^*)$ is a neighbourhood of θ^*; when it does not, $Z(\theta^*)$ consists of two "chunks of space" joined together at θ^* and surrounded by the region of stability $S(\theta^*)$. In the first case the well-known results (e.g. [7], [23]) are recovered; in the latter, essentially new results are obtained. For special cases, such as the RHS perturbations or linear constraints, and for a connection with directional derivatives, the reader is referred to [27]. The results derived there for $Z_1(\theta^*)$ extend to $Z_2(\theta^*)$ and thus to their union $Z(\theta^*)$.

7. MISCELLANEOUS APPLICATIONS

We will now outline how two important convex programming problems can be studied by utilizing a region of stability for a simple linear program.

A. **Minimal Index Set of Binding Constraints.** Consider the convex program

$$
\text{(C)} \qquad \begin{aligned} &\operatorname{Min} f^0(x) \\ &\text{s.t.} \\ &f^k(x) \leqq 0 \qquad k \in \mathscr{P}\,. \end{aligned}
$$

Its *feasible set* is

$$F =^{\Delta} \{x \in R^n : f^k(x) \leqq 0\}\,.$$

At an $x \in F$, the *binding constraints* are

$$\mathscr{P}(x) =^{\Delta} \{k \in \mathscr{P} : f^k(x) = 0\}$$

and the *minimal index set of binding constraints* is

$$\mathscr{P}^{=} =^{\Delta} \{k \in \mathscr{P} : x \in F \Rightarrow f^k(x) = 0\} = \bigcap_{x \in F} \mathscr{P}(x)\,.$$

An algorithm for calculating $\mathscr{P}^{=}$ was suggested in [1], see also [2] and [30].

When $\mathscr{P}^{=}$ is known, the program (C) can be rewritten in the form

$$
\text{(7.1)} \qquad \begin{aligned} &\operatorname{Min} f^0(x) \\ &\text{s.t.} \\ &f^k(x) \leqq 0\,, \quad k \in \mathscr{P} \setminus \mathscr{P}^{=}\,. \\ &x \in F^{=} =^{\Delta} \{x : f^k(x) = 0,\ k \in \mathscr{P}^{=}\}\,. \end{aligned}
$$

Since Slater's condition holds for (7.1), i.e.

$$\exists \hat{x} \in F^{=} \quad \text{such that} \quad f^k(\hat{x}) < 0 \quad \forall k \in \mathscr{P} \setminus \mathscr{P}^{=}\,,$$

we know that (7.1) is stable with respect to the RHS perturbations. Thus an arbitrary convex program can be "stabilized" after calculating $\mathscr{P}^{=}$. It is important to identify the set $\mathscr{P}^{=}$ also in other contexts, such as duality and numerical methods (see e.g. [29]).

For any $f^k : R^n \to R$ and $x \in R^n$ we recall that

$$D_k(x) = \{d \in R^n : \exists \bar{\alpha} > 0 \text{ such that } f^k(x + \alpha d) = f^k(x)\ \forall \alpha \in [0, \bar{\alpha}]\}$$

is the *cone of directions of constancy of f^k at x*, see e.g. [2]. In order to avoid the well-known technicalities (see e.g. [23], [2]) we assume that all functions in (C) are differentiable.

Take an arbitrary $x \in F$, specify $\theta \geqq 0$, and consider the (essentially linear, see e.g. [2]) program

$$(L, \theta) \qquad \begin{aligned} &\operatorname{Min} \sum_{k \in \mathscr{P}(x)} \nabla f^k(x)^{\mathsf{T}} d \\ &\text{s.t.} \\ &\nabla f^k(x)^{\mathsf{T}} d + \theta \|d - \delta^k\|_1 \leqq 0 \\ &\delta^k \in D_k(x) \\ &\|\delta^k\|_1 \leqq 1,\ k \in \mathscr{P} \\ &\|d\|_1 \leqq 1 . \end{aligned}$$

Here the vector variables are d, δ^k, $k \in \mathscr{P}$ and for $u = (u_i)$, $\|u\|_1 =^{\Delta} \sum_{i=1} |u_i|$. Note that the optimal value $\bar{f}(\theta)$ of (L, θ) is nonpositive. If it is equal to zero for all sufficiently small $\theta > 0$, then one can prove that $\mathscr{P}^{=} = \mathscr{P}(x)$. If $\bar{f}(\theta) < 0$ at $\theta = \theta^* = 0$, then one may invoke a region of stability to check whether $\mathscr{P}^{=} \neq \mathscr{P}(x)$ (in which case $\mathscr{P}^{=}$ is properly contained in $\mathscr{P}(x)$).

Theorem 7.1. *Suppose that for an arbitrary fixed $x \in F$ the optimal value of $(L, 0)$ is negative. If there is a region of stability for (L, θ) at $\theta^* = 0$, emanating in the positive direction, their $\mathscr{P}^{=} \neq \mathscr{P}(x)$.*

Proof. If such a region of stability existed, one would have

$$\bar{f}(\theta) < 0 . \quad \text{for all small} \quad \theta \geqq 0 .$$

Hence $\mathscr{P}^{=} \neq \mathscr{P}(x)$ by [30, Theorem 2]. □

If $\bar{f}(0) < 0$ and such a region of stability cannot be found, then one should use analytic arguments to determine whether $\bar{f}(\theta) = 0$ for all small $\theta > 0$. A method for calculating $\mathscr{P}^{=}$, which uses the properties of (L, θ), has been recently suggested in [30].

B. **Multicriteria Decision Making.** For a finite number of "objectives" $f^k : R^n \to \to R$, $k \in \mathscr{P}$, a point x^* is a *Pareto minimum* if there is no point x such that

$$\begin{cases} f^k(x) \leqq f^k(x^*) , \quad k \in \mathscr{P} \\ \text{with at least one strict inequality.} \end{cases}$$

An alternative way of characterizing Pareto minimality is via mathematical programming (e.g. [2], [3]). Indeed, x^* is Pareto minimal if and only if x^* is an optimal solution of the program

$$\begin{aligned}
&\operatorname{Min} \sum_{k \in \mathscr{P}} f^k(x) \\
(7.2) \qquad &\text{s.t.} \\
&f^k(x) \leqq f^k(x^*), \quad k \in \mathscr{P}.
\end{aligned}$$

From now on we assume that f^k, $k \in \mathscr{P}$ are convex functions. Then it is easy to show that

(7.3) $\qquad x^*$ is Pareto minimal if and only if $\mathscr{P} = \mathscr{P}^=$ in (7.2).

In order to check whether x^* is Pareto minimal one may study the program (L, θ) which, when written for (7.2), becomes

$$\begin{aligned}
&\operatorname{Min} \sum_{k \in \mathscr{P}} \nabla f^k(x^*)^{\mathrm{T}} d \\
&\text{s.t.} \\
(M, \theta) \qquad &\nabla f^k(x^*)^{\mathrm{T}} d + \theta \|d - \delta^k\|_1 \leqq 0 \\
&\delta^k \in D_k(x^*) \\
&\|\delta^k\|_1 \leqq 1, \quad k \in \mathscr{P} \\
&\|d\|_1 \leqq 1.
\end{aligned}$$

In view of (7.3) and the observations preceding Theorem 7.1, it is clear that a point x^* is Pareto minimal if the optimal value $\tilde{f}(\theta)$ of (M, θ) is equal to zero for all sufficiently small $\theta > 0$. In fact, this condition is also necessary (see [30, Theorem 3]). If $\tilde{f}(0) <$ then, as in (A), one may invoke a region of stability to check whether $\tilde{f}(\theta) < 0$ for some $\theta > 0$.

Theorem 7.2. *Let x^* be an arbitrary point and suppose that the optimal value of $(M, 0)$ is negative. If there is a region of stability for (M, θ) at $\theta^* = 0$, emanating in the positive direction, then x^* is not Pareto minimal.*

Prof. If such a region of stability existed, we would conclude (see the proof of Theorem 7.1) that $\mathscr{P}^= + \mathscr{P}(x^*) = \mathscr{P}$, i.e. x^* is not Pareto minimal, by (7.3). □

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Souhrn

OBLASTI STABILITY PRO NEKOREKTNĚ ZFORMULOVANÉ PROBLÉMY KONVEXNÍHO PROGRAMOVÁNÍ

SANJO ZLOBEC

Oblasti stability jsou části prostoru parametrů, v nichž optimální řešení a optimální hodnota spojitě závisejí na datech. V takových oblastech lze chápat řešení libovolné úlohy konvexního programování jako spojitý proces a je možné provést Tichonovskou regularizaci.

V práci se zavádí nová oblast stability, která zachovává vlastnosti spojitosti. Pro tuto oblast udáváme dále formule pro marginální hodnotu. Význam oblastí stability demonstrujeme na multikriteriálních rozhodovacích problémech a na výpočtu minimální množiny indexů vázaných podmínek v konvexním programování. Tyto dvě nelineární úlohy mohou být převedeny na problém výpočtu oblasti stability jednoduchého problému lineárního programování. V případě, že jsou splněny Slaterovy podmínky stejně jako i v případě poruchy pravých stran, výsledky vedou ke známým vztahům.

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