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SOME FAST FINITE-DIFFERENCE SOLVERS FOR DIRICHLET PROBLEMS ON SPECIAL DOMAINS

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Our aim is to prove the existence of asymptotic error expansions to some simple finite-difference schemes for Dirichlet problems on the so-called uniform domains. The Richardson extrapolation [1] then leads to fast finite-difference solvers for the problems mentioned.

1. UNIFORM AND NEARLY UNIFORM DOMAINS

In order to simplify the notation we shall consider only the two-dimensional geometry; the result can be generalized to the n-dimensional case. Let D be a bounded domain in the (x, y)-plane with a boundary G. For some real numbers x_0, y_0 let us consider a uniform grid over the (x, y)-plane:

$$(1)\qquad \begin{aligned}(x_i, y_j)\,, \quad x_i &= x_0 + ih\,, \quad h = \text{const} > 0\,,\\ y_j &= y_0 + jk\,, \quad k = \text{const} > 0\,,\\ 0 &< \text{const} < h/k < \text{const}\,.\end{aligned}$$

The domain D will be called uniform if there exist two values x_0, y_0 and two sequences of positive numbers $\{h\}$ and $\{k\}$ tending simultaneously to zero so that the grid lines $x = x_i$ and $y = y_j$ cut the boundary G only at the points of the form (x_m, y_n). Then the points (1) cover D with a uniform grid which consists of the set D_h of interior grid points (x_i, y_j) which belong to the interior of D and the set G_h of boundary grid points (x_i, y_j) lying just on G. The domain D will ve called nearly uniform if there exist four real numbers a, b, c, d, a sequence of positive numbers $\{h\}$ tending to zero and two strictly increasing and smooth functions $x(t)$, $(a \leqq t \leqq c)$, $y(t)$, $(b \leqq t \leqq d)$, such that D lies in the rectangle $x(a) \leqq x \leqq x(c)$, $y(b) \leqq y \leqq y(d)$ and the lines $x = x_i = x(a + ih)$ and $y = y_j = y(b + jh)$, i, j integers, cut the boundary G only at the points of the form (x_m, y_n), m, n integers. So we can cover D with a grid (x_i, y_j), $x_i = x(a + ih)$, $y_j = y(b + jh)$, i, $j = 0, 1, 2, 3, \ldots$, which consists of the set D_h of interior grid points (x_i, y_j) which belong to the interior of D and the set G_h of boundary grid points (x_i, y_j) lying just on G. This grid is not uniform but depends uniformly on one parameter h.

2. THE DIFFERENTIAL PROBLEM

On a uniform domain D consider the differential problem

$$(2) \qquad Lu = \frac{\partial}{\partial x}\left(\left(p(x, y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\left(q(x, y)\frac{\partial u}{\partial y}\right) - c(x, y)\, u = \right.\right.$$
$$= f(x, y), \quad (x, y) \in D,$$
$$u(x, y) = g(x, y), \quad (x, y) \in G,$$

where p, q, c, f, g are given smooth enough functions with $p \geqq p_0 = \text{const} > 0$, $q \geqq q_0 = \text{const} > 0$, $c \geqq 0$.

3. THE DISCRETE PROBLEM

We cover D with a uniform grid $D_h \cup G_h$ as described above and consider the following discrete problem with respect to the unknown $v(x_i, y_j)$:

$$L_h v = (1/h^2)\left[p(x_i + 0{\cdot}5h, y_j)\left(v(x_{i+1}, y_j) - v(x_i, y_j)\right) -\right.$$
$$\left. - p(x_i - 0{\cdot}5h, y_j)\left(v(x_i, y_j) - v(x_{i-1}, y_j)\right)\right] +$$
$$+ (1/k^2)\left[q(x_i, y_j + 0{\cdot}5k)\left(v(x_i, y_{j+1}) - v(x_i, y_j)\right) -\right.$$
$$\left. - q(x_i, y_j - 0{\cdot}5k)\left(v(x_i, y_j) - v(x_i, y_{j-1})\right)\right] -$$
$$- c(x_i, y_j)\, v(x_i, y_j) = f(x_i, y_j), (x_i, y_j) \in D_h,$$
$$v(x_i, y_j) = g(x_i, y_j), \quad (x_i, y_j) \in G_h.$$

It is clear that the operator L_h satisfies the maximum principle.

4. MAIN RESULT

Theorem 1. *Assume that the problem* (2) *has a unique solution* $u(x, y) \in C^{2n+4}(\bar{D})$, p *and* $q \in C^{2n+3}(\bar{D})$, *and that the problem*

$$Lw = F(x, y) \in C^m(\bar{D}), \quad (x, y) \in D,$$
$$w(x, y) = 0, \quad (x, y) \in G,$$

has a unique solution $w \in C^{m+2}(\bar{D})$. *Then for* h *and* k *small enough there exist* $n(n+1)/2 - 1$ *functions* $w_{ij}(x, y)$ *independent of* h *and* k *so that*

$$v(x_i, y_j) - u(x_i, y_j) - h^2 w_{10}(x_i, y_j) - k^2 w_{01}(x_i, y_j) -$$
$$- h^4 w_{20}(x_i, y_j) - h^2 k^2 w_{11}(x_i, y_j) - k^4 w_{02}(x_i, y_j) - \dots -$$
$$- h^{2n} w_{n0}(x_i, y_j) - h^{2n-2} k^2 w_{n-1,1}(x_i, y_j) - \dots - k^{2n} w_{0n}(x_i, y_j) =$$
$$= O(h^{2n+2} + k^{2n+2}), \quad (x_i, y_j) \in D_h .$$

Proof. For any $w \in C^{2p+4}(\bar{D})$ we have by Taylor's formula:

$$L_h w = Lw + h^2 G_{10}(w) + k^2 G_{01}(w) + h^4 G_{20}(w) + k^4 G_{02}(w) + \dots +$$
$$+ h^{2p} G_{p0}(w) + k^{2p} G_{0p}(w) + O(h^{2p+2} + k^{2p+2}),$$

where $G_{ij}(w)$ depend only on w and its derivatives and belong to $C^{2p+2-2(i+j)}(\bar{D})$. Now for $w_{ij} \in C^{2n+4-2(i+j)}(\bar{D})$ we put

$$z = v - u - h^2 w_{10} - k^2 w_{01} - h^4 w_{20} - h^2 k^2 w_{11} -$$
$$- k^4 w_{02} - \dots - h^{2n} w_{n0} - h^{2n-2} k^2 w_{0n-1,1} - \dots - k^{2n} w_{0n} .$$

Then we have

$$L_h z = h^2(-Lw_{10} + F_{10}) + k^2(-Lw_{01} + F_{01}) +$$
$$+ h^4(-Lw_{20} + F_{20}) + \dots + h^{2n}(-Lw_{n0} + F_{n0}) +$$
$$+ h^{2n-2} k^2(-Lw_{n-1,1} + F_{n-1,1}) + \dots + k^{2n}(-Lw_{0n} + F_{0n}) +$$
$$+ O(h^{2n+2} + k^{2n+2}),$$

where F_{ij} depend only on u and w_{rs} with $r + s < i + j$ and $F_{ij} \in C^{2n+2-2(i+j)}(\bar{D})$. Now we choose w_{ij} recursively by

$$Lw_{ij} = F_{ij}, \quad (x, y) \in D, \quad w_{ij} = 0, \quad (x, y) \in G, \quad i + j = 1, \dots, n,$$

which exist by assumption and satisfy $w_{ij} \in C^{2n+4-2(i+j)}(\bar{D})$. Then we have

$$L_h z = \varphi \quad \text{on} \quad D_h, \quad z = 0 \quad \text{on} \quad G_h,$$

where

$$\varphi = O(h^{2n+2} + k^{2n+2}).$$

To evaluate z we consider the problem

$$LB(x, y) = -2 \quad \text{on} \quad D, \quad B(x, y) = 0 \quad \text{on} \quad G.$$

We deduce

$$B \geqq 0, \quad B(x, y) \leqq M = \text{const}$$

and, by Taylor's formula,

$$L_h B = LB + O(h^2 + k^2) \quad \text{on} \quad D_h .$$

Then for h and k small enough we have

$$L_h B \leqq -1 .$$

Now we consider the problem

$$L\,A(x, y) = -2K \quad \text{on} \quad D\,, \quad A(x, y) = 0 \quad \text{on } G\,,$$

where $K = \max |\varphi|$ on D_h. Then we have

$$A = KB\,, \quad 0 \leqq A = KB \leqq M \max |\varphi| \quad \text{on} \quad D_h\,,$$

and at the same time

$$L_h A = K L_h B \leqq -K .$$

Hence

$$L_h(A \pm z) \leqq 0 \quad \text{on} \quad D_h\,, \quad A \pm z = 0 \quad \text{on} \quad G_h\,.$$

Then by the maximum principle we have $A \pm z \geqq 0$, that is

$$|z| \leqq A \leqq M \max |\varphi| \quad \text{on} \quad D_h\,.$$

Theorem 1 is proved.

Note 1. If $p = \text{const} > 0$ and $q = \text{const} > 0$, the theorem is true without assuming that h and k are small enough.

Note 2. The result is still available if the term cu in the differential equation is replaced by $c(x, y, u)$ with $\partial c/\partial u \geqq 0$.

Note 3. The result is still available if the domain D is nearly uniform. Then we use the grid $D_h \cup G_h$ as described in Section 1. This grid is not uniform but depends uniformly on one parameter h and has all the boundary grid points just on the boundary G. We put $h_i = x(a + ih) - x(a + (i - 1)\,h)$, $k_j = y(b + jh) - y(b + (j - 1)\,h)$ and consider the discrete problem

$$\begin{aligned} L_h v = {} & [2/(h_i + h_{i+1})]\,[p(x_i + 0{\cdot}5h_{i+1}, y_j)\,(v(x_{i+1}, y_j) - v(x_i, y_j))/h_{i+1} - \\ & - p(x_i - 0{\cdot}5h_i, y_j)\,(v(x_i, y_j) - v(x_{i-1}, y_j))/h_i] + \\ & + [2/(k_j + k_{j+1})]\,[q(x_i, y_j + 0{\cdot}5k_{j+1})\,(v(x_i, y_{j+1}) - v(x_i, y_j))/k_{j+1} - \\ & - q(x_i, y_j - 0{\cdot}5k_j)\,(v(x_i, y_j) - v(x_i, y_{j-1}))/k_j] - c(x_i, y_j)\,v(x_i, y_j) = \\ & = f(x_i, y_j)\,, \quad (x_i, y_j) \in D_h\,, \quad v(x_i, y_j) = g(x_i, y_j)\,, \quad (x_i, y_j) \in G_h\,. \end{aligned}$$

The result can be stated as follows:

Theorem 2. *Assume that the problem* (2) *has a unique solution* $u(x, y) \in C^{2n+4}(\bar{D})$ *and* $p, q \in C^{2n+3}(\bar{D})$, $x(t) \in C^{2n+2}([a, c])$, $y(t) \in C^{2n+2}([b, d])$, *and that the problem*

$$Lw = F(x, y) \in C^m(D)\,, \quad (x, y) \in D\,,$$
$$w(x, y) = 0\,, \quad (x, y) \in G\,,$$

has a unique solution $w(x, y) \in C^{m+2}(\bar{D})$. *Then for* h *small enough there exist* n *functions* $w_i(x, y)$ *independent of* h *so that*

$$v(x_i, y_j) - u(x_i, y_j) - h^2 w_1(x_i, y_j) - h^4 w_2(x_i, y_j) - \ldots - h^{2n} w_n(x_i, y_j) = = O(h^{2n+2}) .$$

5. A NUMERICAL EXAMPLE

Let D be a circle $x^2 + y^2 < 1$ with the boundary G. Consider the differential problem

$$\Delta u = f(x, y) , \quad (x, y) \in D , \quad u(x, y) = g(x, y) , \quad (x, y) \in G ,$$

where

$$f(x, y) = -\sin x - \cos y , \quad g(x, y) = \sin x + \cos y .$$

The solution is $u = \sin x + \cos y$. Because the circle clearly is a nearly uniform domain, we use a one-parameter grid

$$x_i = \cos \pi(1 - ih) , \quad y_j = \cos \pi(1 - jh) ,$$

$h = 1/N$, N being an even integer >0, $i, j = \overline{0, N}$ as in Section 1.

We consider the discrete problem described in Section 3 and denote the approximate value of $u(x_P, y_P)$ calculated on this grid at a grid point P by $v(P; h)$. From Theorem 2 we deduce

$$v(P; h; h/2) \equiv \tfrac{4}{3}\, v(P; h/2) - \tfrac{1}{3}\, v(P; h) = u(x_P, y_P) + O(h^4) ,$$

where P denotes a grid point common for the two grids with grid spacings h and $h/2$. The numerical results at the point $0(0, 0)$ are presented in Table 1.

Table 1

$N = 1/h$	Number of equations	$v(0; h)$	$v(0; h; h/2)$	$u(0, 0)$
2	1	1·02015	1·00049	1.
4	5	1·00541		

These results show the effectiveness of our algorithm.

Reference

[1] *O. V. Widlund:* Some recent applications of asymptotic error expansions to finite difference schemes. Proc. Royal Soc. London, A 323, N. 1553 (1971), 167—177.

Souhrn

RYCHLÉ ŘEŠENÍ DIRICHLETOVA PROBLÉMU NA SPECIÁLNÍ OBLASTI METODOU KONEČNÝCH DIFERENCÍ

TA VAN DINH

Autor dokazuje existenci mnohoparametrického asymptotického rozvoje pro chybu obvyklého pětibodového diferenčního schématu pro Dirichletův problém pro lineární a semilineární eliptickou parciální rovnici na jistých speciálních (tzv. uniformních) oblastech. Tento rozvoj dává s použitím Richardsonovy extrapolace jednoduchý způsob zrychlení konvergence dané metody. Postup je ilustrován na numerickém příkladě.

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