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AN EQUILIBRIUM FINITE ELEMENT METHOD IN THREE-DIMENSIONAL ELASTICITY

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INTRODUCTION

The aim of the present paper is to generalize the triangular composite equilibrium element – introduced by Watwood-Hartz [6] – for the three-dimensional space and to demonstrate its applicability to solving the dual three-dimensional problem of linear elasticity (a weak version of the Castigliano-Menabrea principle [7]).

The triangular composite equilibrium element has also been analyzed by Hlaváček [8] and by Johnson-Mercier [9]. Stress tensors (2×2) of this element are defined on a triangle which is composed of three subtriangles (see Fig. 1). These tensors are symmetric and linear on any subtriangle and along any contact of subtriangles

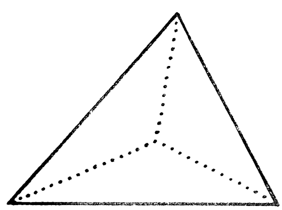

Fig. 1.

the continuity of the stress vector is demanded. We emphasize that a single (not composed) triangle with linear stresses cannot be employed, since it has a small number of independent parameters on the sides to balance an arbitrary loading which is linear on any side (see [6, 8]).

In Section 2, we introduce the composite tetrahedral stress element. Then we investigate two different types of a finite (piecewise linear) approximation of the dual elasticity problem on a polyhedral domain. For both types we establish a priori error estimates $O(h^2)$ in L_2-norm and $O(h^{1/2})$ in L_∞-norm, provided the solution is smooth enough. To obtain these estimates we also have to prove that for any polyhedron there exists a strongly regular family of decompositions into tetrahedra.

1. DUAL VARIATIONAL FORMULATION OF THE LINEAR ELASTICITY PROBLEM

First, let us introduce some definitions and notations. Let $\Omega \neq \emptyset$ be a bounded domain with a Lipschitz boundary [7, 10] in $\mathbb{R}^p$ which is equipped with the Euclidean norm $\|\cdot\|$. Note that a normal to the boundary $\partial\Omega$ exists almost everywhere, the outward unit normal being always denoted by ν. Let μ_p be the Lebesgue measure on $\mathbb{R}^p$. Denote the space of real infinitely differentiable functions with a compact support in Ω by $\mathscr{D}(\Omega)$. The Sobolev space of functions, the derivatives of which up to the order m exist (in the sense of distributions) and are square-integrable in Ω, is denoted by $H^m(\Omega)$. The usual norm and semi-norm in $H^m(\Omega)$ are denoted by $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$, respectively. The usual scalar product in $L_2(\Omega) = H^0(\Omega)$ is denoted by $(\cdot, \cdot)_{0,\Omega}$. The space of real measurable functions which are essentially bounded (i.e., except a set of measure zero) is denoted by $L_\infty(\Omega)$.

If Z is a closed domain, $Z = \bar{\Omega}$, we shall write $\mathscr{D}(Z)$, $H^m(Z)$, $L_\infty(Z)$, $\|\cdot\|_{m,Z}$, $|\cdot|_{m,Z}$ instead of $\mathscr{D}(\Omega)$, $H^m(\Omega)$, $L_\infty(\Omega)$, $\|\cdot\|_{m,\Omega}$, $|\cdot|_{m,\Omega}$ for the sake of simplicity.

If $\emptyset \neq \Delta \subset \mathbb{R}^p$ is an open or closed domain, then $C^m(\Delta)$ denotes the space of real functions, the (classical) derivatives of which up to the order m are continuous in Δ. We write $C(\Delta) = C^0(\Delta)$. The space of polynomials of the order at most j defined on the set Δ is denoted by $P_j(\Delta)$ and we write

$$V_\Delta = [P_1(\Delta)]^3 .$$

All vectors will be column vectors. Since there is no danger of ambiguity, the scalar product of $u = (u_1, \ldots, u_p)^{\mathsf{T}}$, $v = (v_1, \ldots, v_p)^{\mathsf{T}} \in [L_2(\Omega)]^p$ is denoted by $(\cdot, \cdot)_{0,\Omega}$ as in $L_2(\Omega)$ and we put

$$(u, v)_{0,\Omega} = \Big(\sum_{i=1}^{p} (u_i, v_i)_{0,\Omega}^2\Big)^{1/2} .$$

Similarly, for $v = (v_1, \ldots, v_p)^{\mathsf{T}} \in [H^m(\Omega)]^p$ we put

$$\|v\|_{m,\Omega} = \Big(\sum_{i=1}^{p} \|v_i\|_{m,\Omega}^2\Big)^{1/2} \quad \text{and} \quad |v|_{m,\Omega} = \Big(\sum_{i=1}^{p} |v_i|_{m,\Omega}^2\Big)^{1/2} .$$

For simplicity, the dual elasticity problem will be formulated only on polyhedral domains as we shall consider some finite element methods for its approximate solution later. Notice that in the mathematical literature there are many different definitions of the polyhedron. Only *the convex polyhedron* is defined almost everywhere in the same way as the intersection of a finite number of closed half-spaces in $\mathbb{R}^3$ which is bounded and has at least one interior point. In this paper we shall use the following definition of the (generally non-convex) polyhedron.

Definition. *A polyhedron is a nonempty closed bounded domain in* $\mathbb{R}^3$, *the boundary of which can be expressed as a finite union of polygons (where a polygon is a nonempty closed bounded domain in* $\mathbb{R}^2$, *the boundary of which can be expressed as a finite union of line segments).*

This definition evidently generalizes that of the convex polyhedron since it is known [1, 11] that the boundary of any convex polyhedron is composed of a finite number of convex polygons.

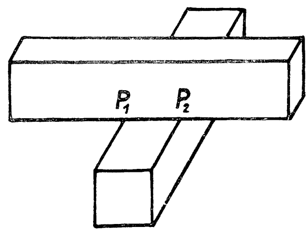

Fig. 2.

From now on, let Ω be the interior of the polyhedron on which the dual elasticity problem will be formulated. Except for Section 3, we assume that Ω has a Lipschitz boundary, since this is assumed in the theory of elasticity. Fig. 2 shows a polyhedron which has a non-Lipschitz boundary, since the boundary in any neighbourhood of the points P_i can be expressed only by a multivalued function in any coordinate system (while any Lipschitz function is one-valued).

For simplicity, we put

$$(\cdot,\cdot)_0 = (\cdot,\cdot)_{0,\Omega}, \quad \|\cdot\|_m = \|\cdot\|_{m,\Omega}, \quad |\cdot|_m = |\cdot|_{m,\Omega}.$$

Assume that the boundary $\partial\Omega$ is divided into mutually disjoint parts $\Gamma_0, \Gamma_1, \Gamma_2$ such that

$$\Gamma_0 \cup \Gamma_1 \cup \Gamma_2 = \partial\Omega, \tag{1-1}$$

where Γ_0 is the union of a finite number of line segments and Γ_1, Γ_2 are open in $\partial\Omega$, i.e., for any $x \in \Gamma_i$, $i = 1, 2$, there exists an open sphere $\mathscr{S} \subset \mathbb{R}^3$ such that $x \in \mathscr{S}$ and $\mathscr{S} \cap \partial\Omega \subset \Gamma_i$.

Henceforth, let a body force $f \in [L_2(\Omega)]^3$, a boundary force (load) $g \in [L_2(\Gamma_1)]^3$ and a displacement $u_0 \in [H^1(\Omega)]^3$ be given. In the case $\Gamma_2 = \emptyset$, we always assume that the equilibrium conditions for forces f, g and their moments are satisfied, i e.,

$$\int_\Omega f \, dx + \int_{\partial\Omega} g \, ds = 0, \quad \int_\Omega x \times f \, dx + \int_{\partial\Omega} x \times g \, ds = 0. \tag{1-2}$$

We define the space of symmetric 3×3 stress tensors on the open or closed domain $Z \subset \mathbb{R}^3$ as

$$T(Z) = \{\tau \in [L_2(Z)]^9 \mid \tau = \tau^{\mathrm{T}}\}, \quad \boldsymbol{T} = T(\Omega),$$

and the set of statically admissible stresses as

(1-3) $$E(f, g) = \{\tau \in \boldsymbol{T} \mid \int_\Omega \tau \,.\, \varepsilon(v) \,\mathrm{d}x = \int_\Omega v^{\mathsf{T}} f \,\mathrm{d}x + \int_{\Gamma_1} v^{\mathsf{T}} g \,\mathrm{d}s \;\forall v \in V\}\,,$$

where

$$V = \{v \in [H^1(\Omega)]^3 \mid v = 0 \quad \text{on} \quad \Gamma_2\}$$

is the space of virtual displacements,

$$\varepsilon(v) = \tfrac{1}{2}\left(\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial x}\right)^{\mathsf{T}}\right)$$

is the infinitesimal strain tensor, $\partial v / \partial x$ is the 3×3 matrix of the first partial derivatives of v, and

(1-4) $$\tau \,.\, \varepsilon(v) = \operatorname{tr}(\tau^{\mathsf{T}} \varepsilon(v)) = \sum_{i,j=1}^{3} \tau_{ij}\, \varepsilon_{ij}(v)\,.$$

Further, we introduce the generalized linear inverse Hook's law for a non-homogeneous and anisotropic material of the elastic body:

$$\varepsilon = \boldsymbol{A} \,.\, \tau \quad \left(\varepsilon_{ij} = \sum_{k,l=1}^{3} A_{ijkl} \tau_{kl}\right),$$

where we assume that $\boldsymbol{A} = (A_{ijkl})_{i,j,k,l=1}^{3} \in [L_\infty(\Omega)]^{81}$,

$$A_{ijkl} = A_{jikl} = A_{klij}$$

and that there exists a constant $C_A > 0$ such that

(1-5) $$\varphi \,.\, (\boldsymbol{A}(x) \,.\, \varphi) \geqq C_A \|\varphi\|^2 \quad \forall \varphi = \varphi^{\mathsf{T}} \in \mathbb{R}^9$$

holds almost everywhere in Ω.

Definition. *The dual problem of the linear elasticity consists in finding σ which minimizes the functional (of the complementary energy) $J : \boldsymbol{T} \to \mathbb{R}^1$ defined by*

(1-6) $$J(\tau) = \tfrac{1}{2} \int_\Omega \tau \,.\, (\boldsymbol{A} \,.\, \tau) \,\mathrm{d}x - \int_\Omega \tau \,.\, \varepsilon(u_0) \,\mathrm{d}x = \tfrac{1}{2}\boldsymbol{a}(\tau, \tau) - \boldsymbol{b}(\tau),\ \tau \in \boldsymbol{T}\,,$$

over the set $E(f, g)$.

It is known [4, 5, 7] that this problem has a unique solution, since the symmetric bilinear form $\boldsymbol{a}(.,.)$ is $\boldsymbol{T}$-elliptic by (1-5) and since $E(f, g)$ is nonempty, closed in $\boldsymbol{T}$ and convex. (Furthermore, $\varepsilon(u) = \boldsymbol{A} \,.\, \sigma$, where u is the solution of the primary problem.)

Definition. *Let $\bar{f} \in [L_2(Z)]^3$, where $\emptyset \neq Z \subset \mathbb{R}^3$ is a closed or open domain with a Lipschitz boundary. If*

$$\int_Z \tau \,.\, \varepsilon(v) \,\mathrm{d}x = \int_Z v^{\mathsf{T}} \bar{f} \,\mathrm{d}x \quad \forall v \in [\mathscr{D}(Z)]^3$$

holds for $\tau \in T(Z)$, *then we say that the divergence of the tensor* τ *exists in the sense of distributions in* Z *and we define* $\operatorname{div} \tau = -\bar{f}$ *in* $[L_2(Z)]^3$.

Note that the first generalized derivatives of τ need not exist. But, if $\tau \in T(Z) \cap \cap [H^1(Z)]^9$, then it is easy to show that

$$\operatorname{div} \tau = \left(\sum_{j=1}^{3} \frac{\partial \tau_{1j}}{\partial x_j}, \sum_{j=1}^{3} \frac{\partial \tau_{2j}}{\partial x_j}, \sum_{j=1}^{3} \frac{\partial \tau_{3j}}{\partial x_j} \right)^{\mathsf{T}}.$$

By the above definition, $\tau \in E(f, g)$ if and only if

$$\text{(1-7)} \qquad \operatorname{div} \tau + f = 0 \quad \text{in} \quad [L_2(\Omega)]^3 , \quad \tau \nu = g \quad \text{in} \quad [L_2(\Gamma_1)]^3 ,$$

where ν is the outward unit normal to Γ_1. (This easily follows by using the linear functional – see e.g. [5] –

$$\langle \tau \nu, w \rangle = \int_{\Omega} (\tau \,.\, \varepsilon(v) + v^{\mathsf{T}} \operatorname{div} \tau) \, \mathrm{d}x , \quad w \in [H^{1/2}(\partial\Omega)]^3 ,$$

which does not depend on the extension $v \in [H^1(\Omega)]^3$ of the trace $w = v/\partial\Omega$ and which is bounded on $[H^{1/2}(\partial\Omega)]^3$.)

In Section 4 we shall construct spaces of finite elements of stresses such that the divergence of these elements will exist in the sense of distributions in Ω. However, before that we present two important sections.

2. THE TETRAHEDRAL COMPOSITE STRESS ELEMENT

Let K be an arbitrary tetrahedron with vertices A, B, C, D and let E be an arbitrary fixed point of the interior of K. Divide K into four tetrahedra $BCDE$, $ACDE$, $ABDE$, $ABCE$ and denote them by K_1, K_2, K_3, K_4, respectively (see Fig. 3). We call these tetrahedra *the blocks* of the tetrahedron K. Now, we have 10 triangular faces in this composed tetrahedron K: 4 *external faces* BCD, ACD, ABD, ABC and 6 faces ABE, ACE, ADE, BCE, BDE, CDE, which we call *internal faces* of the composed tetrahedron K. By a normal to an external or internal face S we shall understand a normal to the plane which contains S. Thus, we can consider a normal to S also at the boundary points of S.

Now, we define an auxiliary space

$$\tilde{T}_K = \{ \tau \in T(K) \mid \tau/K_i \in [P_1(K_i)]^9, \; i = 1, 2, 3, 4 \} .$$

If $\tau \in \tilde{T}_K$, then we denote the linear extension of τ/K_i, $i = 1, 2, 3, 4$, to the whole space $\mathbb{R}^3$ by τ_i, i.e., $\tau_i \in [P_1(\mathbb{R}^3)]^9$.

Definition. *Let* $\tau \in \tilde{T}_K$. *Then the stress vector* τn *is said to be continuous at the internal faces of* K, *if for any internal face* S *common to two different blocks*

K_i, K_j, $1 \leqq i < j \leqq 4$, *we have*

$$\text{(2-1)} \qquad \tau_i n = \tau_j n \quad on \quad S\,,$$

where n is a normal to S.

Next, we define the subspace T_K of $\tilde{T}_K$ as

$$T_K = \{\tau \in \tilde{T}_K \mid \tau n \text{ is continuous at the internal faces of } K\}\,.$$

The main purpose of this section is to show (similarly as in [9] for the triangular element) that the stress tensor $\tau \in T_K$ is uniquely determined by:

(i) the values of τn at three points, not lying in one straight line, of each external face of K, and
(ii) $\int_K \tau \,\mathrm{d}x$.

Definition. *A tetrahedral composite finite stress element is a triple* $(K, T_K, \{\Phi_p\}_{p=1}^{42})$ *where* Φ_p *are functionals defined on* T_K *in the following manner:*

For any external face $S_i = K_i \cap \partial K$, $i = 1, 2, 3, 4$, *select a normal* n_i *and three points* $M_{ij} \in S_i$, $j = 1, 2, 3$, *not lying in one straight line. Then for* $\tau \in T_K$ *put*

$$\text{(y)} \qquad \Phi_{9(i-1)+3(j-1)+k}(\tau) = (\tau_i(M_{ij})\, n_i)_k \quad for \quad i = 1, 2, 3, 4, \quad j, k = 1, 2, 3\,.$$

(*where* $(\,.\,)_k$ *denotes the k-th component of a vector from* $\mathbb{R}^3$),

$$\text{(yy)} \qquad \Phi_{36+l}(\tau) = \int_K \tau_{ll}\,\mathrm{d}x \quad for \quad l = 1, 2, 3\,,$$

$$\Phi_{40}(\tau) = \int_K \tau_{12}\,\mathrm{d}x\,, \quad \Phi_{41}(\tau) = \int_K \tau_{13}\,\mathrm{d}x\,, \quad \Phi_{42}(\tau) = \int_K \tau_{23}\,\mathrm{d}x\,.$$

The functionals $\Phi_1, \ldots, \Phi_{42}$ *are called the degrees of freedom of the tetrahedral composite finite stress element.*

The motives for this definition will be obvious from the proof of Theorem 2.1, where we show that $\{\Phi_p\}_{p=1}^{42}$ is a basis of the dual space $[T_K]'$.

Theorem 2.1. *The dimension of space* T_K *is* 42 *and for arbitrary real numbers* $\alpha_1, \ldots, \alpha_{42}$ *there exists a unique tensor* $\tau \in T_K$ *such that*

$$\text{(2-2)} \qquad \Phi_p(\tau) = \alpha_p \quad for \quad p = 1, \ldots, 42$$

(*i.e., the set* $\{\Phi_p\}_{p=1}^{42}$ *is* T_K*-unisolvent — see* [4], *p.* 78).

Proof is based on the following seven lemmas. We remark that the unisolvency of the triangular composite stress element is proved in [9]. However, the proof is based on the existence of Airy's function, the analogue of which in the three-dimensional space (Maxwell's or Morera's stress functions) has a too complicated shape for our purpose. Therefore, we have chosen a way which leans only in the geometrical properties of tetrahedra and on basic results of linear algebra.

Lemma 2.1. *Given a composed tetrahedron K, there exist such normals* $n_1, n_2, n_3 \in \mathbb{R}^3$ *of the external faces BCD, ACD, ABD, respectively, that* $n_1 - n_2$, $n_2 - n_3$, $n_3 - n_1$ *are normals of the internal faces CDE, ADE, BDE, respectively.*

Proof. Let $v_1, v_2, v_3 \in \mathbb{R}^3$ be the outward unit normals of the faces BCD, ACD, ABD, respectively, of the tetrahedron $ABCD$. Put $n_1 = v_1$. Since the planes BCD, CDE, ACD contain the straight line CD, their normal vectors are linearly dependent.

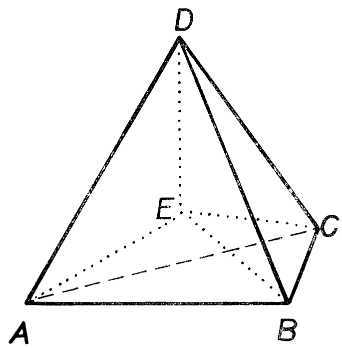

Fig. 3.

Hence, there exists a unique constant $\beta \in \mathbb{R}^1$ such that $n_1 - \beta v_2$ is the normal of CDE and we put $n_2 = \beta v_2$. By investigating analogously the planes BCD, BDE, ABD we find that there exists a unique $\gamma \in \mathbb{R}^1$ such that $\gamma v_3 - n_1$ is the normal of BDE. Therefore, we put $n_3 = \gamma v_3$.

To complete the proof, we must show that $n_2 - n_3$ is the normal of ADE. The plane ADE has a common straight line, DE, with the planes CDE (with the normal $n_1 - n_2$) and BDE (with the normal $n_3 - n_1$). Hence, there exists a unique $\alpha \in \mathbb{R}^1$ such that

$$\alpha(n_1 - n_2) - (n_3 - n_1) \tag{2-3}$$

is the normal of ADE. However, the plane ADE has a common straight line, AD, with ACD (with the normal n_2) and ABD (with the normal n_3). Therefore, the normal of ADE can be expressed as a linear combination of only n_2 and n_3 (independently of n_1). Thus, $\alpha = -1$ in (2-3) and $n_2 - n_3$ is the normal of ADE. ∎

Lemma 2.2. *Let* $n_1, n_2, n_3 \in \mathbb{R}^3$ *be linearly independent vectors and let* P_1, P_2, P_3 *be symmetric* 3×3 *matrices such that*

$$P_1 n_1 = P_2 n_2 = P_3 n_3 = 0\,,$$

$$P_1 n_2 = -P_2 n_1\,, \quad P_2 n_3 = -P_3 n_2\,, \quad P_3 n_1 = -P_1 n_3\,.$$

Then $P_1 = P_2 = P_3 = 0$.

Proof. We show that $n_i^T P_1 n_j = 0$ for any i, j, $1 \leqq j \leqq i \leqq 3$, since then certainly $P_1 = 0$. Obviously,

$$n_i^T P_1 n_1 = 0 \quad \text{for} \quad i = 1, 2, 3 .$$

Further, the assumptions imply

$$n_2^T P_1 n_2 = -n_2^T P_2 n_1 = -n_1^T P_2 n_2 = 0 ,$$

$$n_3^T P_1 n_3 = -n_3^T P_3 n_1 = -n_1^T P_3 n_3 = 0 .$$

Finally,

$$n_3^T P_1 n_2 = -n_3^T P_2 n_1 = -n_1^T P_2 n_3 = n_1^T P_3 n_2 = n_2^T P_3 n_1 = -n_2^T P_1 n_3 = -n_3^T P_1 n_2 ,$$

which yields $n_3^T P_1 n_2 = 0$. Thus, $P_1 = 0$. Similarly, it can be proved that $P_2 = P_3 = 0$. ∎

Lemma 2.3. *If $\tau \in T_K$ and $\tau n = 0$ on ∂K, then $\tau = 0$ on ∂K. (In detail: if $\tau_i n_i = 0$ on $S_i = K_i \cap \partial K$, $i = 1, 2. 3, 4$, where n_i is an arbitrary normal to S_i, then $\tau_i = 0$ on S_i for $i = 1, 2, 3, 4$.)*

Proof. Let $\tau \in T_K$ and let n_1, n_2, n_3 be the normals from Lemma 2.1, which are evidently linearly independent. Since the external faces S_1, S_2, S_3 contain the vertex D, we have from the assumptions

$$\tau_1(D)\, n_1 = \tau_2(D)\, n_2 = \tau_3(D)\, n_3 = 0 . \tag{2-4}$$

The vectors $n_1 - n_2$, $n_2 - n_3$, $n_3 - n_1$ are normals of the three internal faces, which contain the vertex D. From the continuity of the stress vector at the internal faces, we get

$$\tau_1(D)(n_1 - n_2) = \tau_2(D)(n_1 - n_2) ,$$

$$\tau_2(D)(n_2 - n_3) = \tau_3(D)(n_2 - n_3) ,$$

$$\tau_3(D)(n_3 - n_1) = \tau_1(D)(n_3 - n_1) .$$

Then by (2-4)

$$\tau_1(D)\, n_2 = -\tau_2(D)\, n_1 , \quad \tau_2(D)\, n_3 = -\tau_3(D)\, n_2 , \quad \tau_3(D)\, n_1 = -\tau_1(D)\, n_3 .$$

Applying Lemma 2.2 to the matrices $P_i = \tau_i(D)$ for $i = 1, 2, 3$, we find

$$\tau_i(D) = 0 , \quad i = 1, 2, 3 . \tag{2-5}$$

In the same way we find that

$$\tau_i(A) = 0 , \quad i = 2, 3, 4 , \tag{2-6}$$

$$\tau_i(B) = 0 , \quad i = 1, 3, 4 ,$$

$$\tau_i(C) = 0 , \quad i = 1, 2, 4 .$$

Hence, the linearity of τ_i implies that $\tau_i = 0$ on S_i, $i = 1, 2, 3, 4$. ∎

Lemma 2.4. *Given a composed tetrahedron K, there exist such normals $\bar{n}_1, \bar{n}_2, \bar{n}_3 \in \in \mathbb{R}^3$ of the internal faces BCE, ACE, ABE, respectively, that $\bar{n}_1 - \bar{n}_2$, $\bar{n}_2 - \bar{n}_3$, $\bar{n}_3 - \bar{n}_1$ are normals of the internal faces CDE, ADE, ABE, respectively.*

Proof is identical with that of Lemma 2.1 after interchanging the letters D and E. ■

Lemma 2.5. *If $\tau \in T_K$, then*

$$\tau_1(E) = \tau_2(E) = \tau_3(E) = \tau_4(E)\,. \tag{2-7}$$

Proof. Let $\tau \in T_K$ and let $\bar{n}_1, \bar{n}_2, \bar{n}_3$ be the normals from Lemma 2.4, which are linearly independent. Since all internal faces contain the point E, we have from (2-1) that

$$\tau_1(E)\,\bar{n}_1 = \tau_4(E)\,\bar{n}_1\,, \quad \tau_2(E)\,\bar{n}_2 = \tau_4(E)\,\bar{n}_2\,, \quad \tau_3(E)\,\bar{n}_3 = \tau_4(E)\,\bar{n}_3\,,$$
$$\tau_1(E)\,(\bar{n}_1 - \bar{n}_2) = \tau_2(E)\,(\bar{n}_1 - \bar{n}_2)\,,$$
$$\tau_2(E)\,(\bar{n}_2 - \bar{n}_3) = \tau_3(E)\,(\bar{n}_2 - \bar{n}_3)\,,$$
$$\tau_3(E)\,(\bar{n}_3 - \bar{n}_1) = \tau_1(E)\,(\bar{n}_3 - \bar{n}_1)\,.$$

Setting $P_i = \tau_i(E) - \tau_4(E)$ for $i = 1, 2, 3,$ we can transform the above system into the form

$$P_1\bar{n}_1 = P_2\bar{n}_2 = P_3\bar{n}_3 = 0\,,$$
$$P_1(\bar{n}_1 - \bar{n}_2) = P_2(\bar{n}_1 - \bar{n}_2)\,,$$
$$P_2(\bar{n}_2 - \bar{n}_3) = P_3(\bar{n}_2 - \bar{n}_3)\,,$$
$$P_3(\bar{n}_3 - \bar{n}_1) = P_1(\bar{n}_3 - \bar{n}_1)\,.$$

Now, from Lemma 2.2 we see that $P_1 = P_2 = P_3 = 0$ ■

Let us note that lemmas analogous to Lemma 2.3 and 2.5 could be easily proved also for the triangular composite stress element. Finally, we shall use two following well-known and simple lemmas for the proof of Theorem 2.1.

Lemma 2.6. *Let Y be a linear space of a finite dimension m and let A_p, $p = 1, \ldots, r$, $r \leqq m$, be linear functionals on Y. Then the dimension of the space*

$$\{y \in Y \,/\, A_p(y) = 0,\ p = 1, \ldots, r\}$$

is at least $m - r$ and the dimension of this space is equal to $m - r$ if and only if the functionals A_p are linearly independent. ■

Lemma 2.7. *Let K' be an arbitrary tetrahedron with vertices A_1, A_2, A_3, A_4 and a gravity center G. Then*

$$\int_{K'} p \,\mathrm{d}x = \tfrac{1}{4}\mu_3(K') \sum_{i=1}^{4} p(A_i) = \mu_3(K')\, p(G)$$

for any $p \in [P_1(K')]^m$ (m integer). ■

Proof of Theorem 2.1. First we show that $\dim T_K \leqq 42$. Let $\tau \in T_K$ and let $\Phi_p(\tau) = 0$ for any $p = 1, \ldots, 42$. Then obviously $\tau n = 0$ on ∂K and $\int_K \tau \,\mathrm{d}x = 0$. From the linearity of τ on any block, from Lemma 2.7 and (2-5), (2-6), (2-7) it and follows that

$$0 = \int_K \tau \,\mathrm{d}x = \sum_{i=1}^{4} \int_{K_i} \tau_i \,\mathrm{d}x = \tfrac{1}{4}\mu_3(K_1)\left(\tau_1(B) + \tau_1(C) + \tau_1(D) + \tau_1(E)\right) + \ldots$$

$$\ldots + \tfrac{1}{4}\mu_3(K_4)\left(\tau_4(A) + \tau_4(B) + \tau_4(C) + \tau_4(E)\right) = \tfrac{1}{4}\sum_{i=1}^{4} \mu_3(K_i)\,\tau_1(E)\,,$$

i.e., $\tau_i(E) = 0$ for $i = 1, 2, 3, 4$. Thus, we see that τ_i attains the zero value at all vertices of any block K_i, $i = 1, 2, 3, 4$. Hence, $\tau = 0$ on the whole K and by Lemma 2.6 we have

$$0 = \dim \{\tau \in T_K \mid \Phi_p(\tau) = 0, \; p = 1, \ldots, 42\} \geqq \dim T_K - 42\,. \tag{2-8}$$

Further we show that $\dim T_K \geqq 42$. Since a symmetric stress tensor has six independent components and since $\dim P_1(K_i) = 4$ for $i = 1, 2, 3, 4$, we immediately see that $\dim \tilde{T}_K = 6 \times 4 \times 4 = 96$. Let $\bar{n}$ be a normal of the internal face $S = K_1 \cap \cap K_2$ and let $N \in S$. For $\tau \in \tilde{T}_K$ we define functionals Ψ_k, $k = 1, 2, 3$, by

$$\Psi_k(\tau) = \left(\tau_1(N)\,\bar{n} - \tau_2(N)\,\bar{n}\right)_k\,,$$

where, as above, $(.)_k$ denotes the k-th component of a vector from $\mathbb{R}^3$. If we select from any of the six internal faces three points not lying in one straight line, we can analogously define $6 \times 3 \times 3 = 54$ linear functionals $\Psi_1, \ldots, \Psi_{54}$ on the space $\tilde{T}_K$. It easily follows that all these functionals vanish if and only if $\tau \in T_K$, i.e., if the stress vector is continuous at all internal faces of K (see (2-1)). Thus, by Lemma 2.6,

$$\dim T_K = \dim \{\tau \in \tilde{T}_K \mid \Psi_q(\tau) = 0, \; q = 1, \ldots, 54\} \geqq 96 - 54 = 42\,,$$

and together with (2-8) this gives $\dim T_K = 42$.

Using Lemma 2.6 again, we observe that the functionals Φ_p are linearly independent, since now (2-8) turns to equality. The existence and unicity of the tensor $\tau \in T_K$ satisfying (2-2) are now obvious. ∎

3. EXISTENCE OF A STRONGLY REGULAR FAMILY OF DECOMPOSITIONS OF A POLYHEDRON INTO TETRAHEDRA

The results of this section will be used not only to construct the space of the finite elements, but also mainly for the convergence proofs of Sections 6, 7 and 8.

Definition. *A finite set of tetrahedra is said to be a decomposition of the polyhedron $\bar{\Omega}$ into tetrahedra if*

(i) *the union of all these tetrahedra is $\bar{\Omega}$,*

(ii) *the interiors of these tetrahedra are mutually disjoint,*

Theorem 3.1. *For any polyhedron there exists a decomposition into tetrahedra.*

The proof is based on an auxiliary lemma.

Lemma 3.1. *For any polyhedron $\bar{\Omega}$ there exist convex polyhedra $\bar{\Omega}_1, \ldots, \bar{\Omega}_r$* such *that*

(y) *the union of all $\bar{\Omega}_p$, $p = 1, \ldots, r$,* is $\bar{\Omega}$,

(yy) *the interiors of these convex polyhedra are mutually disjoint,*

(yyy) *any face of any polyhedron $\bar{\Omega}_p$, $p \in \{1, \ldots, r\}$, is either a face of another polyhedron $\bar{\Omega}_q$, $q \neq p$, or a subset of the boundary $\partial\Omega$.*

Proof of Lemma 3.1. Let $\bar{\Omega}$ be an arbitrary polyhedron and let $P^1, \ldots, P^k$ be polygons the union of which is $\partial\Omega$. Let $R^1, \ldots, R^k$ be planes such that $P^i \subset R^i$, $i = 1, \ldots, k$. Finally, let $\Omega_1, \ldots, \Omega_r \subset \mathbb{R}^3$ be all components of the set $\bar{\Omega} \setminus \bigcup_{i=1}^{k} R^i$ (i.e., the components which arise by "cutting" $\bar{\Omega}$ by the planes R^i). We show that $\bar{\Omega}_p$, $p = 1, \ldots, r$, are the convex polyhedra sought (their number is finite, because k planes divide the space $\mathbb{R}^3$ into 2^k parts at most).

Since $\partial\Omega \subset \bigcup_{i=1}^{k} R^i$, it follows that

$$\bar{\Omega} \setminus \bigcup_{i=1}^{k} R^i = \Omega \setminus \bigcup_{i=1}^{k} R^i .$$

This set is open since Ω is open and $\bigcup_{i=1}^{k} R^i$ is closed, i.e., Ω_p are open connected sets.

Let $p \in \{1, \ldots, r\}$ be fixed. Any plane R^i, $i = 1, \ldots, k$, splits the space $\mathbb{R}^3$ into two halfspaces. Denoting by Q^i that closed halfspace with the boundary plane R^i which contains $\bar{\Omega}_p$, it is easy to show

$$\bar{\Omega}_p = \bigcap_{i=1}^{k} Q^i .$$

Hence, $\bar{\Omega}_p$ is a convex polyhedron, since the set $\bar{\Omega}_p$ is bounded and contains at least one interior point.

Using the definition formula $\bigcup_{p=1}^{r} \Omega_p = \bar{\Omega} \setminus \bigcup_{i=1}^{k} R^i$, we find that the condition (y) holds. Since any two components Ω_p, Ω_q, $p \neq q$, are separated by at least one plane R^i, (yy) holds. It remains to verify (yyy).

Let x be an interior point of a face S of the convex polyhedron $\bar{\Omega}_p$ and let $x \in \partial\Omega_q$, $q \neq p$. Suppose, for the moment, that x lies on an edge of the convex polyhedron $\bar{\Omega}_q$. Then x must lie in at least two different planes R^s, R^t, $s, t \in \{1, \ldots, k\}$, But this is a contradiction, since x is an interior point of S. Hence, x is also an interior point of a face S' of the polyhedron $\bar{\Omega}_q$ and we deduce that S and S' have common interior points, i.e., $S = S'$. If the face S of $\bar{\Omega}_p$ does not coincide with a face of any other polyhedron $\bar{\Omega}_q$, $q \neq p$, then it is easy to see that $S \subset \partial\Omega$. ∎

Proof of Theorem 3.1. Let $\bar{\Omega} \subset R^3$ be an arbitrary polyhedron and let $\bar{\Omega}_1, \ldots$ $\ldots, \bar{\Omega}_r$ be the convex polyhedra from Lemma 3.1. We shall divide all these convex polyhedra into tetrahedra in the way described in [11]. Let $p \in \{1, \ldots, r\}$ be arbitrary. As has been said, all faces of the convex polyhedron $\bar{\Omega}_p$ are convex polygons. Denoting by $B_1, \ldots, B_j$ (for instance, counter-clockwise) the vertices of any face we can divide this face into the triangles $B_1B_2B_3, B_1B_3B_4, \ldots, B_1B_{j-1}B_j$. Let $\{S_p^u\}_{u=1}^{m_p}$ be the set of all triangles which are obtained in this way on the surface of the polyhedron $\bar{\Omega}_p$ (see Fig. 4). In addition, we require that common faces of two convex polyhedra

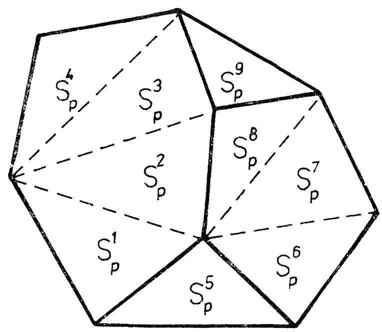

Fig. 4.

(i.e., faces $S \subset \bar{\Omega}_p$, $S' \subset \bar{\Omega}_q$, $p \neq q$, such that $S = S'$) be divided into triangles "in the same manner". Let A_p be an arbitrary interior point of the polyhedron $\bar{\Omega}_p$. The convex hull K_p^u of the triangle S_p^u and the point A_p is a tetrahedron. Introduce the set

$$\mathscr{D} = \{K_p^u \mid p = 1, \ldots, r, \ u = 1, \ldots, m_p\} .$$

Utilizing the conditions (y), (yy), (yyy) from Lemma 3.1 as well as the fact that a possible common face of two convex polyhedra $\bar{\Omega}_p$, $\bar{\Omega}_q$, $p \neq q$, is divided into triangles in the same manner, it is easy to show that the finite set $\mathscr{D}$ satisfies (i), (ii), (iii). ∎

Denote by h_K (= diam K) the length of the largest edge of a tetrahedron K. To any decomposition $\mathscr{D}$ if the polyhedron $\bar{\Omega}$ into tetrahedra we assign the real number

$$h_{\mathscr{D}} = \max_{K \in \mathscr{D}} h_K . \tag{3-1}$$

The number $h_{\mathscr{D}}$ is called *the norm of the decomposition* $\mathscr{D}$.

Definition. *A set of decompositions* $\mathfrak{M}$ *of the polyhedron* $\bar{\Omega}$ *into tetrahedra is called a family decompositions if for any* $\varepsilon > 0$ *there exists a decomposition* $\mathscr{D} \in \mathfrak{M}$ *such that* $h_{\mathscr{D}} \leqq \varepsilon$.

Definition. *A family of decompositions* $\mathfrak{M}$ *of the polyhedron* $\bar{\Omega}$ *into tetrahedra is said to be regular (strongly regular) if there exists a constant* $\varkappa > 0$ *such that for any decomposition* $\mathscr{D} \in \mathfrak{M}$ *and for any tetrahedron* $K \in \mathscr{D}$ *there exists a sphere* $\mathscr{S}_K$ *with a radius* ϱ_K *such that* $\mathscr{S}_K \subset K$ *and*

$$\varkappa h_K \leqq \varrho_K \quad (\varkappa \boldsymbol{h}_{\mathscr{D}} \leqq \varrho_K) . \tag{3-2}$$

(iii) *any face of any tetrahedron in the decomposition is either a face of another tetrahedron in the decomposition, or a subset of the boundary* $\partial\Omega$.

The constant $\varkappa$ *is said to be coefficient of the regular (strongly regular) family* $\mathfrak{M}$.

Obviously, any strongly regular family is regular. Note that a strongly regular family of triangulations of a polygon (the definition is analogous) is easy to obtain due to the fact that any triangle in the triangulation is divided by midlines into four coinciding triangles, which are similar to the original one. In three-dimensional space the situation is considerably more complicated, since it may not be possible to divide any tetrahedron into more coinciding tetrahedra which would be similar to the original tetrahedron.

Theorem 3.2. *For any polyhedron there exists a strongly regular family of decompositions into tetrahedra.*

Proof will be composed of three parts — a), b), c).

a) First, we prove the theorem for the simplest polyhedron — tetrahedron, which will be particularly selected. So let $\bar{\Omega} = \tilde{K}$, where $\tilde{K}$ is the tetrahedron with the vertices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ having the coordinates $(\frac{1}{2}, 0, 0)^{\mathsf{T}}$, $(-\frac{1}{2}, 0, 0)^{\mathsf{T}}$, $(0, \frac{1}{2}, \frac{1}{2})^{\mathsf{T}}$, $(0, -\frac{1}{2}, \frac{1}{2})^{\mathsf{T}}$, respectively (see Fig. 5). The length of the opposite edges $\tilde{A}\tilde{B}$ and $\tilde{C}\tilde{D}$ is equal to 1 and the length of all other edges is $\sqrt{3/2}$. Denote by $\tilde{M}_1, \tilde{M}_2, \tilde{M}_3, \tilde{M}_4, \tilde{M}_5, \tilde{M}_6$ the midpoints of $\tilde{A}\tilde{B}, \tilde{A}\tilde{C}, \tilde{A}\tilde{D}, \tilde{B}\tilde{C}, \tilde{B}\tilde{D}, \tilde{C}\tilde{D}$, respectively and divide the tetrahedron $\tilde{K}$ into eight tetrahedra (see Fig. 5):

$$\tilde{A}\tilde{M}_1\tilde{M}_2\tilde{M}_3, \quad \tilde{B}\tilde{M}_1\tilde{M}_4\tilde{M}_5\,, \quad \tilde{C}\tilde{M}_6\tilde{M}_2\tilde{M}_4\,, \quad \tilde{D}\tilde{M}_6\tilde{M}_3\tilde{M}_5\,,$$
$$\tilde{M}_1\tilde{M}_6\tilde{M}_2\tilde{M}_3\,, \quad \tilde{M}_1\tilde{M}_6\tilde{M}_2\tilde{M}_4\,, \quad \tilde{M}_1\tilde{M}_6\tilde{M}_3\tilde{M}_5\,, \quad \tilde{M}_1\tilde{M}_6\tilde{M}_4\tilde{M}_5\,.$$

It is easily seen that all these tetrahedra form a decomposition of $\tilde{K}$. We denote by $\tilde{\mathscr{D}}_1$ this decomposition and put $\tilde{\mathscr{D}}_0 = \{\tilde{K}\}$. The length of the edge $\tilde{M}_1\tilde{M}_6$ is $\frac{1}{2}$ since the coordinates of the end points are $(0, 0, 0)^{\mathsf{T}}$, $(0, 0, \frac{1}{2})^{\mathsf{T}}$. The length of all the edges which are the midlines of the external faces of $\tilde{K}$ and which are parallel with $\tilde{A}\tilde{B}$ or $\tilde{C}\tilde{D}$ is $\frac{1}{2}$ as weel. Therefore, the length of the edges

$$\tilde{A}\tilde{M}_1,\ \tilde{B}\tilde{M}_1,\ \tilde{M}_2\tilde{M}_4,\ \tilde{M}_3\tilde{M}_5,\ \tilde{C}\tilde{M}_6,\ \tilde{D}\tilde{M}_6,\ \tilde{M}_2\tilde{M}_3,\ \tilde{M}_4\tilde{M}_5,\ \tilde{M}_1\tilde{M}_6$$

is equal to $\frac{1}{2}$ and for any tetrahedron of $\tilde{\mathscr{D}}_1$ precisely two of these edges are opposite. The length of all remaining edges of the tetrahedra of $\tilde{\mathscr{D}}_1$ is $\sqrt{3/4}$. Hence, all tetrahedra of $\tilde{\mathscr{D}}_1$ are coincident and similar to the original tetrahedron $\tilde{K}$. If $\tilde{\varrho}$ is the

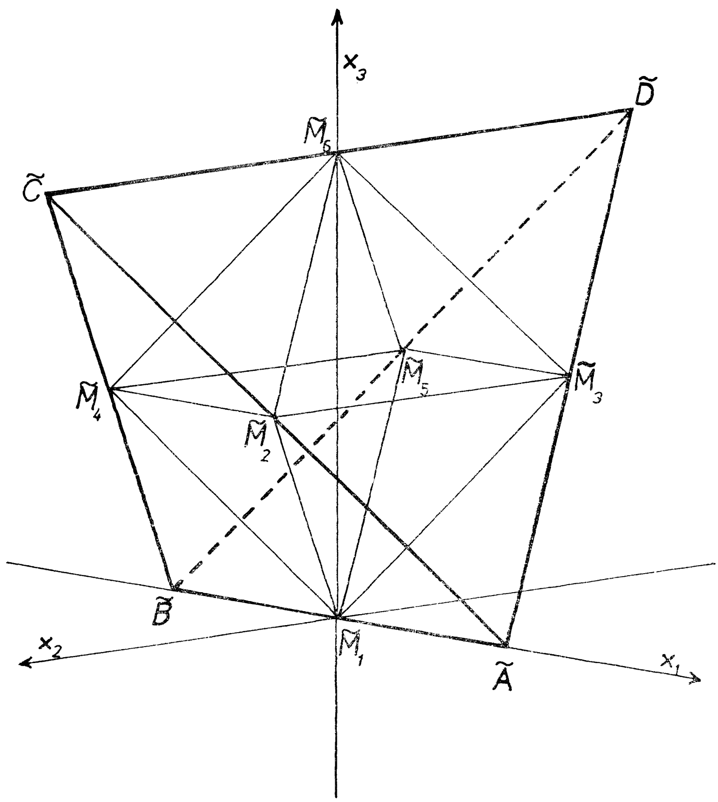

Fig. 5.

radius of the inscribed sphere of $\tilde{K}$ then the radius of the inscribed sphere of any tetrahedron of $\tilde{\mathscr{D}}_1$ is evidently $\frac{1}{2}\tilde{\varrho}$. Any tetrahedron of $\tilde{\mathscr{D}}_1$ can be divided in the same way into eight coincident tetrahedra similar to $\tilde{K}$ again and we obtain the next decomposition $\tilde{\mathscr{D}}_2$. Repeating this process to infinity, we get the family of decompositions $\{\tilde{\mathscr{D}}_m\}_{m=0}^{\infty}$, since the norm of $\tilde{\mathscr{D}}_m$ is 2^{-m}. Moreover, this family is strongly regular since the corresponding coefficient $\tilde{\varkappa}$ can be chosen as $\tilde{\varkappa} = 2^{-m}\tilde{\varrho}/2^{-m} = \tilde{\varrho}$.

b) Consider $\bar{\Omega} = K$, where K is an arbitrary tetrahedron with vertices A, B, C, D, the coordinates of which are

(3-3) $$a = (a_1, a_2, a_3)^{\mathsf{T}}, \quad b = (b_1, b_2, b_3)^{\mathsf{T}}, \quad c = (c_1, c_2, c_3)^{\mathsf{T}},$$
$$d = (d_1, d_2, d_3)^{\mathsf{T}}.$$

We introduce an affine one-to-one mapping $F : \tilde{K} \to K$ given by

(3-4) $$F(\tilde{x}) = Q\tilde{x} + q, \quad \tilde{x} \in \tilde{K},$$

where $q = (a + b)/2$ and $Q = (a - b, c - d, c + d - a - b)$ is a regular matrix as $F(\tilde{A}) = A, \ldots, F(\tilde{D}) = D$. This mapping transforms the edges of $\tilde{K}$ onto the corresponding edges of K, the midlines of the faces of $\tilde{K}$ onto the midlines of the corresponding faces of K and so on. Therefore, for $m = 0, 1, 2, \ldots$ the set

$$\mathscr{D}_m = \{F(\tilde{K}') \mid \tilde{K}' \in \tilde{\mathscr{D}}_m\}$$

is also a decomposition of K. Let e_m be the largest edge of all tetrahedra from $\mathscr{D}_m$, its length being $h_{\mathscr{D}_m}$, and let $\tilde{h}_m$ be the length of the corresponding edge $\tilde{e}_m = F^{-1}(e_m)$. According to (3-4) and part a) we have

$$h_{\mathscr{D}_m} \leqq \|Q\| \, \tilde{h}_m \leqq \|Q\| 2^{-m}, \quad m = 0, 1, 2, \ldots \tag{3-5}$$

Thus, $\{\mathscr{D}_m\}_{m=0}^{\infty}$ is a family of decompositions of K.

Denote by $\tilde{\mathscr{S}}$ the inscribed sphere of $\tilde{K}$, i.e.,

$$\tilde{\mathscr{S}} = \{\tilde{x} \mid \|\tilde{x} - \tilde{x}_0\| \leqq \tilde{\varrho}\} \subset \tilde{K},$$

where $\tilde{x}_0 = (0, 0, \frac{1}{4})^{\mathsf{T}}$ is the centre. Then

$$F(\tilde{\mathscr{S}}) = \mathscr{E} = \{x \mid \|Q^{-1}(x - q) - \tilde{x}_0\| \leqq \tilde{\varrho}\} \subset K$$

is the ellipsoid with the centre $x_0 = Q^{-1}q + \tilde{x}_0$ which is inscribed in K. Denote by ϱ the length of the shortest semi-axis of $\mathscr{E}$ and let $\mathscr{S}$ be the sphere with the centre x_0 and the radius ϱ. Then $\mathscr{S} \subseteq \mathscr{E} \subset K$. According to a), the radius of all the inscribed spheres of the tetrahedral from $\tilde{\mathscr{D}}_m$ is $2^{-m}\tilde{\varrho}$. Hence, the mapping F transforms all these spheres onto ellipsoids that are coinciding and similar to $\mathscr{E}$. These ellipsoids will be inscribed in the corresponding tetrahedra from $\mathscr{D}_m$ and, obviously, the length of their shortest semi-axes will be $2^{-m}\varrho$. One immediately sees that for any $K' \in \mathscr{D}_m$ there exists a sphere $\mathscr{S}'$ with the radius $2^{-m}\varrho$ such that $\mathscr{S}' \subset K'$. Consequently, the family $\{\mathscr{D}_m\}$ is strongly regular and since $(3-5)$ implies

$$\frac{2^{-m}\varrho}{h_{\mathscr{D}_m}} \geqq \frac{2^{-m}\varrho}{2^{-m}\|Q\|} = \frac{\varrho}{\|Q\|},$$

the corresponding coefficient $\varkappa$ can be chosen as $\varkappa = \varrho/\|Q\|$.

c) Let $\bar{\Omega}$ be an arbitrary polyhedron and let $\mathscr{D}$ be an arbitrary decomposition of $\bar{\Omega}$ into tetrahedra. According to b), a strongly regular family $\{\mathscr{D}_m(K)\}_{m=0}^{\infty}$ with a coefficient $\varkappa_K$ corresponds to any $K \in \mathscr{D}$. Setting

$$\bar{\mathscr{D}}_m = \bigcup_{K \in \mathscr{D}} \mathscr{D}_m(K), \quad m = 0, 1, 2, \ldots,$$

we can easily verify that $\bar{\mathscr{D}}_m$ is a decomposition of $\bar{\Omega}$. The norm of $\bar{\mathscr{D}}_m$ is evidently $\max_{K \in \mathscr{D}} h_{\mathscr{D}_m(K)}$. Thus, $\{\bar{\mathscr{D}}_m\}_{m=0}^{\infty}$ is a family of decompositions of $\bar{\Omega}$, since according to b)

$$h_{\mathscr{D}_m(K)} \to 0 \quad \text{as} \quad m \to \infty$$

for any K from the finite set $\mathscr{D}$. This family is strongly regular, since the corresponding coefficient $\bar{\varkappa}$ can be chosen as $\bar{\varkappa} = \min_{K \in \mathscr{D}} \varkappa_K$. ■

Now, we investigate the question whether there exists a decomposition of $\bar{\Omega}$ such that the parts Γ_1, Γ_2 satisfying (1-1) are covered by whole faces of tetrahedra of the decomposition (see Fig. 6).

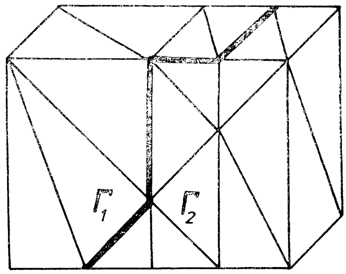

Fig. 6.

Definition. *Let the parts $\Gamma_0, \Gamma_1, \Gamma_2$ of the boundary of $\bar{\Omega}$ satisfy (1-1). Then a decomposition $\mathscr{D}$ of $\bar{\Omega}$ is said to be consistent with Γ_1 and Γ_2 if the interior of any face of any $K \in \mathscr{D}$ is disjoint with Γ_0.*

Theorem 3.3. *Let the parts $\Gamma_0, \Gamma_1, \Gamma_2$ of the boundary of $\bar{\Omega}$ satisfy (1-1). Then there exists a strongly regular family of decompositions of $\bar{\Omega}$ (into tetrahedra) consistent with Γ_1 and Γ_2.*

Proof. Let P^i and R^i, $i = 1, \ldots, k$, be the polygons and the planes, respectively, from the proof of Lemma 3.1. Let $\Gamma_0 \neq \emptyset$ (the case $\Gamma_0 = \emptyset$ was proved in Theorem 3.2) and let $p^1, \ldots, p^l$ be the line segments the union of which is Γ_0. Evidently, for any $p^j, j = 1, \ldots, l$, there exists a plane R^{s_j}, $s_j \in \{1, \ldots, k\}$ such that $p^j \subset R^{s_j}$. Denote by R^{k+j} the plane such that $p^j \subset R^{k+j}$, which is perpendicular to R^{s_j}. Let $\Omega_1, \ldots, \Omega_r$ be all components of the set $\bar{\Omega} \setminus \bigcup_{i=1}^{k+l} R^i$. Now, proceeding as in the proofs of Lemma 3.1, Theorems 3.1 and 3.2, we obtain the strongly regular family $\{\mathscr{D}_m\}_{m=0}^{\infty}$ of decompositions of $\bar{\Omega}$. Since any line segment p^j, $j = 1, \ldots, l$, is included in at least two different planes R^s, R^t, $1 \leqq s, t \leqq k + l$, the interior of any face of any $K \in \mathscr{D}_m$, $m = 0, 1, 2, \ldots$, is disjoint with Γ_0. ∎

4. FINITE ELEMENT SPACES

Let $\mathscr{D}$ be an arbitrary decomposition of $\bar{\Omega}$ into tetrahedra. We define the finite element spaces corresponding to the decomposition $\mathscr{D}$ similarly as in [9]. *The space of the finite elements of stresses* is the space

$$\boldsymbol{T}_{\mathscr{D}} = \{\tau \in \bar{\boldsymbol{T}}_{\mathscr{D}} \mid \operatorname{div} \tau \in [L_2(\Omega)]^3\},$$

where

$$\overline{T}_{\mathscr{D}} = \{\tau \in T \mid \tau/K \in T_K, \ K \in \mathscr{D}\} ,$$

and *the space of finite elements of displacements* is

$$V_{\mathscr{D}} = \{v \in [L_2(\Omega)]^3 \mid v/K \in V_K, \ K \in \mathscr{D}\} .$$

Obviously, $T_{\mathscr{D}} \subset T$, but the analogous inclusion between $V_{\mathscr{D}}$ and V does not hold in general. Now, we describe the character of tensors of $T_{\mathscr{D}}$.

Definition. Let $\tau \in \overline{T}_{\mathscr{D}}$, *Then the stress vector τn is said to be continuous at the external faces of the tetrahedra of $\mathscr{D}$, if for any face S common to two different tetrahedra K, $K' \in \mathscr{D}$ we have*

$$\text{(4-1)} \qquad \tau_i n = \tau'_j n \quad \text{on} \quad S ,$$

where n is a normal to S, τ_i and τ'_j are the linear extensions of τ/K_i and τ'/K'_j, respectively, to the whole space $\mathbb{R}^3$ and $K_i \subset K$, $K'_j \subset K'$ are blocks such that $S = K_i \cap K'_j$.

Lemma 4.1. *Let $\tau \in \overline{T}_{\mathscr{D}}$. Then $\tau \in T_{\mathscr{D}}$ if and only if τn is continuous at the external faces of the tetrahedra of $\mathscr{D}$.*

Proof. Let $\tau \in T_{\mathscr{D}}$ and let $S = K_i \cap K'_j$, $K_i \subset K, K'_j \subset K'$, $K, K' \in \mathscr{D}$, $K \neq K'$. Using Green's theorem, we have for $v \in [\mathscr{D}(K_i \cup K'_j)]^3$

$$\int_{K_i} (\tau \,.\, \varepsilon(v) + v^{\mathsf{T}} \operatorname{div} \tau) \, dx = \int_{K_i} v^{\mathsf{T}} \tau \nu \, ds = \int_S v^{\mathsf{T}} \tau_i \nu_i \, ds ,$$

$$\int_{K_j'} (\tau \,.\, \varepsilon(v) + v^{\mathsf{T}} \operatorname{div} \tau) \, dx = \int_{K_j'} v^{\mathsf{T}} \tau \nu \, ds = \int_S v^{\mathsf{T}} \tau'_j \nu'_j \, ds ,$$

where $\nu_i = -\nu'_j$ is the unit normal to S (outward to K_i). Since the divergence of τ exists in the sense of distributions also on the subset $K_i \cup K'_j \subset \overline{\Omega}$, summing both the above identities gives

$$0 = \int_S v^{\mathsf{T}} (\tau_i \nu_i - \tau'_j \nu_i) \, ds \quad \forall v \in [\mathscr{D}(K_i \cup K'_j)]^3 ,$$

i.e., particularly for all $v \in [\mathscr{D}(S)]^3$. Thus, $\tau_i \nu_i - \tau'_j \nu_i = 0$ on S.

Conversely, let $\tau \in \overline{T}_{\mathscr{D}}$ and let τn be continuous at the external faces. Since $\operatorname{div} \tau$ exists in any block,

$$\text{(4-2)} \qquad \int_{K_i} \tau \,.\, \varepsilon(v) \, dx = \int_{\partial K_i} v^{\mathsf{T}} \tau \nu \, ds - \int_{K_i} v^{\mathsf{T}} \operatorname{div} \tau \, dx \quad \forall v \in [\mathscr{D}(\Omega)]^3 .$$

Define $\bar{f} \in [L_2(\Omega)]^3$ by

$$\bar{f}/K_i = - \operatorname{div} \tau \quad \forall K_i \subset K , \quad i = 1, 2, 3, 4 , \quad \forall K \in \mathscr{D} .$$

Summing (4-2) for all $i = 1, 2, 3, 4$ and all $K \in \mathscr{D}$, we see that

$$\int_\Omega \tau \,.\, \varepsilon(v)\,\mathrm{d}x = \sum_{K \in \mathscr{D}} \sum_{K_i \subset K} \left(\int_{\partial K_i} v^{\mathsf{T}} \tau v\,\mathrm{d}s + \int_{K_i} v^{\mathsf{T}} \bar{f}\,\mathrm{d}x \right) = \int_\Omega v^{\mathsf{T}} \bar{f}\,\mathrm{d}x \quad \forall v \in [\mathscr{D}(\Omega)]^3\,,$$

since (2-1) and (4-1) hold. Thus, $-\bar{f}$ is the divergence of τ in the sense of distributions on the whole Ω and $\tau \in \boldsymbol{T}_{\mathscr{D}}$. ∎

Now, one sees that Green's formula

$$\text{(4-3)} \qquad \int_K \tau \,.\, \varepsilon(v)\,\mathrm{d}x + \int_K v^{\mathsf{T}} \operatorname{div} \tau\,\mathrm{d}x = \int_{\partial K} v^{\mathsf{T}} \tau v\,\mathrm{d}s\,, \quad v \in V_K\,,$$

holds also for $\tau \in T_K$ (the components of which are not from $H^1(K)$).

We denote by $\Gamma(\mathscr{D})$ the set of all external faces of all $K \in \mathscr{D}$, i.e.,

$$\Gamma(\mathscr{D}) = \{S \subset \mathbb{R}^3 \mid S \text{ is an external face of } K \in \mathscr{D}\}\,.$$

It is clear from Section 2 that $\tau \in \boldsymbol{T}_{\mathscr{D}}$ is uniquely determined by:

(i) the values of τn at three points, not lying in one straight line, of each $S \in \Gamma(\mathscr{D})$,

(ii) $\int_K \tau\,\mathrm{d}x$ for each $K \in \mathscr{D}$.

Analogously as in Section 2, we could define the degrees of freedom $\Phi^1, \ldots, \Phi^r$ of the space $\boldsymbol{T}_{\mathscr{D}}$, where $r = \dim \boldsymbol{T}_{\mathscr{D}} = 9 \operatorname{card} \Gamma(\mathscr{D}) + 6 \operatorname{card} \mathscr{D}$. Now, we show that the definite element spaces of stresses and displacements have the so-called equilibrium property (see [9]).

Lemma 4.2. *If* $\tau \in \boldsymbol{T}_{\mathscr{D}}$ *and* $(v, \operatorname{div} \tau)_0 = 0$ *for all* $v \in V_{\mathscr{D}}$, *then* $\operatorname{div} \tau = 0$ in $[L_2(\Omega)]^3$.

Proof. Let $K \in \mathscr{D}$ be arbitrary and let G_i, $i = 1, 2, 3, 4$, be the gravity centers of the corresponding blocks K_i. Clearly, $G_1 G_2 G_3 G_4$ cannot be a degenerate tetrahedron, since it is similar to the tetrahedron $G'_1 G'_2 G'_3 G'_4$, where G'_i are the gravity centers of the faces $S_i = K_i \cap \partial K$, and $G'_1 G'_2 G'_3 G'_4$ is similar to the mirror image of K. Thus, we choose $v_{jk} \in V_{\mathscr{D}}$, $j = 1, 2, 3, 4$, $k = 1, 2, 3$, such that $v_{jk}/K' = 0$ for any $K' \in \mathscr{D}$, $K' \neq K$, and we can define v_{jk} linear on K so that

$$v_{jk}(G_j) = (\delta_{1k}, \delta_{2k}, \delta_{3k})^{\mathsf{T}}\,, \quad v_{jk}(G_l) = (0, 0, 0)^{\mathsf{T}} \quad \text{for} \quad l \neq j\,, \quad l \in \{1, 2, 3, 4\}\,,$$

where δ_{mk} is Kronecker's symbol. Using Lemma 2.7, we obtain

$$0 = \int_\Omega v_{jk}^{\mathsf{T}} \operatorname{div} \tau\,\mathrm{d}x = \sum_{i=1}^{4} \int_{K_i} v_{jk}^{\mathsf{T}} \operatorname{div} \tau_i\,\mathrm{d}x = \sum_{i=1}^{4} \mu_3(K_i)\, v_{jk}^{\mathsf{T}}(G_i) \operatorname{div} \tau_i =$$

$$= \mu_3(K_j)\, v_{jk}^{\mathsf{T}}(G_j) \operatorname{div} \tau_j = \mu_3(K_j)\,(\operatorname{div} \tau_j)_k\,,$$

where $(\operatorname{div} \tau_j)_k$ denotes the k-th component of the constant vector $\operatorname{div} \tau_j$. Therefore, $\operatorname{div} \tau_j = (0, 0, 0)^{\mathsf{T}}$ for $j = 1, 2, 3, 4$. ∎

5. EXTERNAL APPROXIMATION OF THE DUAL PROBLEM

As in Section 1, let $f \in [L_2(\Omega)]^3$, $g \in [L_2(\Gamma_1)]^3$ and let (1-2) hold if $\Gamma_2 = \emptyset$. For an arbitrary decomposition $\mathscr{D}$ of $\bar{\Omega}$ consistent with Γ_1 and Γ_2 define the set of statically admissible approximations of stresses (cf. (1-7))

$$E_{\mathscr{D}}(f, g) = \left\{\tau \in \boldsymbol{T}_{\mathscr{D}} \,\middle|\, \int_\Omega v^{\mathsf{T}} \operatorname{div} \tau \,\mathrm{d}x = -\int_\Omega v^{\mathsf{T}} f \,\mathrm{d}x \quad \forall v \in \boldsymbol{V}_{\mathscr{D}}, \right.$$
$$\left. \int_S v^{\mathsf{T}} \tau \nu \,\mathrm{d}s = \int_S v^{\mathsf{T}} g \,\mathrm{d}s \quad \forall v \in V_S, \quad \forall S \in \Gamma(\mathscr{D}), \quad S \subset \Gamma_1 \right\}.$$

Definition. *An external approximation of the dual problem (corresponding to the decomposition* $\mathscr{D}$*) consists in finding* $\sigma_{\mathscr{D}}$ *which minimizes the functional* (1-6) *over the set* $E_{\mathscr{D}}(f, g)$.

Theorem 5.1. *There exists a unique solution* $\sigma_{\mathscr{D}}$ *of the external approximation of the dual problem.*

The proof is based on two auxiliary lemmas.

Lemma 5.1. *Let* $K \in \mathscr{D}$, $\bar{g} \in [L_2(\partial K)]^3$ *and let*

$$\int_K w^{\mathsf{T}} f \,\mathrm{d}x + \int_{\partial K} w^{\mathsf{T}} \bar{g} \,\mathrm{d}s = 0 \quad \forall w \in W_K = \{v \in V_K \mid \varepsilon(v) = 0\} \tag{5-1}$$

(i.e., the equilibrium conditions for the forces f, $\bar{g}$ *and their moments are satisfied on* K*). Then there a unique* $\tau \in T_K$ *such that*

$$\int_K v^{\mathsf{T}} \operatorname{div} \tau \,\mathrm{d}x = -\int_K v^{\mathsf{T}} f \,\mathrm{d}x \quad \forall v \in V_K, \tag{5-2}$$

$$\int_S v^{\mathsf{T}} \tau \nu \,\mathrm{d}s = \int_S v^{\mathsf{T}} \bar{g} \,\mathrm{d}s \qquad \forall v \in V_S \tag{5-3}$$

and for all external faces S *of* K *with the outward unit normal* ν.

Proof. Existence. By Riesz Theorem, for any external face S of K there exists a unique $t \in V_S$ such that

$$\int_S v^{\mathsf{T}} t \,\mathrm{d}s = \int_S v^{\mathsf{T}} \bar{g} \,\mathrm{d}s \quad \forall v \in V_S. \tag{5-4}$$

We choose $\tau \in T_K$ such that (see Section 2)

(i) $\tau \nu = t$ on ∂K,
(ii) $\int_K \tau \,\mathrm{d}x = \frac{1}{2}\left(\int_K (x f^{\mathsf{T}} + f x^{\mathsf{T}}) \,\mathrm{d}x + \int_{\partial K} (x \bar{g}^{\mathsf{T}} + \bar{g} x^{\mathsf{T}}) \,\mathrm{d}s\right)$.

Then evidently (5-3) holds. Let $v \in V_K$ be arbitrary and let

$$v = u + w, \quad u \in U_K, \quad w \in W_K, \tag{5-5}$$

where U_K is the orthocomplement of W_K in V_K. Using (1-4) and the symmetry of the constant tensor $\varepsilon(v)$, we see that

$$\int_K \tau \,.\, \varepsilon(v) \,\mathrm{d}x = \int_K \tau \,.\, \varepsilon(u) \,\mathrm{d}x = \tfrac{1}{2}\left(\int_K (x f^{\mathsf{T}} + f x^{\mathsf{T}}) \,\mathrm{d}x + \int_{\partial K} (x \bar{g}^{\mathsf{T}} + \bar{g} x^{\mathsf{T}}) \,\mathrm{d}s\right) . \,\varepsilon(u) =$$

$$= \int_K \varepsilon(u) \cdot (xf^{\mathsf{T}})\,\mathrm{d}x + \int_{\partial K} \varepsilon(u) \cdot (x\bar{g}^{\mathsf{T}})\,\mathrm{d}s = \operatorname{tr}\left(\int_K \varepsilon(u)\, xf^{\mathsf{T}}\,\mathrm{d}x + \int_{\partial K} \varepsilon(u) x\bar{g}^{\mathsf{T}}\,\mathrm{d}s\right) =$$

$$= \operatorname{tr}\left(\int_K uf^{\mathsf{T}}\mathrm{d}x + \int_{\partial K} u\bar{g}^{\mathsf{T}}\,\mathrm{d}s\right) = \int_K u^{\mathsf{T}} f\,\mathrm{d}x + \int_{\partial K} u^{\mathsf{T}}\bar{g}\,\mathrm{d}s\,.$$

Hence, (4-3), (5-1), (5-4) and (5-5) imply

$$\int_K v^{\mathsf{T}} \operatorname{div} \tau\,\mathrm{d}x = -\int_K \tau \cdot \varepsilon(v)\,\mathrm{d}x + \int_{\partial K} u^{\mathsf{T}}\tau\nu\,\mathrm{d}s + \int_{\partial K} w^{\mathsf{T}}\tau\nu\,\mathrm{d}s =$$

$$= -\int_K u^{\mathsf{T}} f\,\mathrm{d}x - \int_{\partial K} u^{\mathsf{T}}\bar{g}\,\mathrm{d}s + \int_{\partial K} u^{\mathsf{T}}\bar{g}\,\mathrm{d}s - \int_K w^{\mathsf{T}} f\,\mathrm{d}x = -\int_K v^{\mathsf{T}} f\,\mathrm{d}x\,.$$

Unicity. Let $\tau', \tau'' \in T_K$ satisfy (5-2) and (5-3). By (4-3), we have $\int_K(\tau' - \tau'') \cdot \varepsilon(v)\,\mathrm{d}x = 0$ for any $v \in V_K$. Thus, $\int_K(\tau' - \tau'')\,\mathrm{d}x = 0$ and since $(\tau' - \tau'')\,\nu = 0$ on ∂K, we get $\tau' - \tau'' = 0$ on K. ∎

Lemma 5.2. *The set $E_{\mathscr{D}}(f, g)$ is nonempty.*

Proof. Let us number the tetrahedra of $\mathscr{D}$ in this manner: Let $K^1 \in \mathscr{D}$ be arbitrary. Successively, we denote by $K^j \in \mathscr{D}$, $j = 2, \ldots, m$ ($=$ card $\mathscr{D}$), an arbitrary tetrahedron which is different from $K^1, \ldots, K^{j-1}$ and which has a common face with at least one tetrahedron $K^1, \ldots, K^{j-1}$. Let

$$\bar{\Omega}^k = \bigcup_{i=1}^{k} K^i\,, \quad k = 1, \ldots, m\,.$$

We shall define a certain extension g^* of g to all faces $S \in \Gamma(\mathscr{D})$ in such a way that the equilibrium conditions for the forces f, g^* and their moments will be satisfied on any $\bar{\Omega}^k$, which is evidently a connected set (a polyhedron).

First, we define g^* on $\partial\Omega^m = \partial\Omega$. We put $g^* = g$ on Γ_1. In the case $\mu_2(\Gamma_2) > 0$, we put in addition

$$g^* = 0 \quad \text{on any} \quad S \subset \Gamma_2\,, \quad S \neq S'\,, \quad S \in \Gamma(\mathscr{D})\,,$$

where $S' \in \Gamma(\mathscr{D})$ is a fixed face in Γ_2. It is easy to see that on this remaining face $S' \subset \partial\Omega^m$, g^* can be defined (e.g., as a linear function) so that

$$\int_{\Omega^m} w^{\mathsf{T}} f\,\mathrm{d}x + \int_{\partial\Omega^m} w^{\mathsf{T}} g^*\,\mathrm{d}s = 0 \quad \forall w \in W_\Omega = \{v \in V_\Omega \mid \varepsilon(v) = 0\}\,.$$

Next, we define g^* step by step on $S \in \Gamma(\mathscr{D})$, $S \not\subset \partial\Omega$. Let j successively attain the values $m, m - 1, \ldots, 2$.

Let $S^j_1, \ldots, S^j_{k_j}$, $k_j \in \{1, 2, 3, 4\}$, be all external faces of K^j which belong to $\partial\Omega^{j-1}$. Then we define an auxiliary function $g^j \in [L_2(\partial K^j)]^3$ by

$$g^j = 0 \quad \text{on} \quad S^j_2, \quad \ldots, S^j_{k_j}\,,$$

$$g^j = g^* \text{ on} \quad S^j_{k_j+1}, \ldots, S^j_4\,,$$

and we choose g^j linear on the remaining face S_1^j so that

(5-6) $$\int_{K^j} w^{\mathsf{T}} f \,\mathrm{d}x + \int_{\partial K^j} w^{\mathsf{T}} g^j \,\mathrm{d}s = 0 \quad \forall w \in W_\Omega .$$

Setting now

(5-7) $$g^* = -g^j \quad \text{on} \quad S_1^j, \ldots, S_{k_j}^j ,$$

we see that the forces f, g^* and their moments are equilibrate on $\bar{\Omega}^{j-1}$, i.e.,

$$\int_{\Omega^{j-1}} w^{\mathsf{T}} f \,\mathrm{d}x + \int_{\partial\Omega^{j-1}} w^{\mathsf{T}} g^* \,\mathrm{d}s = 0 \quad \forall w \in W_\Omega ,$$

since

$$\int_{\Omega^j} w^{\mathsf{T}} f \,\mathrm{d}x + \int_{\partial\Omega^j} w^{\mathsf{T}} g^* \,\mathrm{d}s = 0 \quad \forall w \in W_\Omega .$$

Finally, for $j = 1$ we put $g^1 = g^*$ on ∂K^1.

Let $\tau^j \in T_{K^j}$, $j = 1, \ldots, m$, be the tensor from Lemma 5.1 which corresponds to the forces f, g^j satisfying (5-6) on K^j. Define $\bar{\tau} \in \overline{\boldsymbol{T}}_{\mathscr{D}}$ by

$$\bar{\tau} = \tau^j \quad \text{on} \quad K_j , \quad j = 1, \ldots, m .$$

Using (5-3) and (5-7), it is easy to show that $\bar{\tau}n$ is continuous at the external faces of the tetrahedra of $\mathscr{D}$. Therefore, by Lemma 4.1, $\bar{\tau} \in \boldsymbol{T}_{\mathscr{D}}$ and evidently $\bar{\tau} \in E_{\mathscr{D}}(f, g)$. ∎

Proof of Theorem 5.1. The set $E_{\mathscr{D}}(0, 0)$ is closed in $\boldsymbol{T}$, since it is a finite-dimensional subspace of $\boldsymbol{T}$. Thus, the nonempty set

$$E_{\mathscr{D}}(f, g) = E_{\mathscr{D}}(0, 0) + \bar{\tau} ,$$

where $\bar{\tau} \in E_{\mathscr{D}}(f, g)$, is closed in $\boldsymbol{T}$ and obviously convex. Using the $\boldsymbol{T}$-ellipticity of the symmetric bilinear form $\boldsymbol{a}(.,.)$ in (1-6), we get that the solution $\sigma_{\mathscr{D}} \in E_{\mathscr{D}}(f, g)$ exists and is unique (see [4], Theorem 1.1.1). ∎

Lemma 5.3. $E_{\mathscr{D}}(0, 0) \subset E(0, 0)$.

Proof. Let $\tau \in E_{\mathscr{D}}(0, 0) \subset \boldsymbol{T}$. Since $\tau v/S \in V_S$ for $S \in \Gamma(\mathscr{D})$ and

$$\int_S v^{\mathsf{T}} \tau v \,\mathrm{d}s = 0 \quad \forall v \in V_S , \quad S \subset \Gamma_1 ,$$

we get $\tau v = 0$ on Γ_1. Together with $\operatorname{div} \tau = 0$, which follows from Lemma 4.2, we have by (1-3) and (1-7) $\int_\Omega \tau \,.\, \varepsilon(v) \,\mathrm{d}x = 0 \;\forall v \in V$. ∎

Theorem 5.2. *There exists a constant* $C(\boldsymbol{a})$ *such that*

$$\|\sigma - \sigma_{\mathscr{D}}\|_0 \leqq C(\boldsymbol{a}) \inf \{ \|\sigma - \tau_{\mathscr{D}}\|_0 \mid \tau_{\mathscr{D}} \in E_{\mathscr{D}}(f, g) \} .$$

Proof. The functional (1-6) attains its minimum over the set $E(f, g)$ at σ if and only if (see [4], Theorem 1.1.2)

$$\boldsymbol{a}(\sigma, \tau - \sigma) \geqq \boldsymbol{b}(\tau - \sigma) \quad \forall \tau \in E(f, g) .$$

Therefore,

$$a(\sigma, \chi) = b(\chi) \quad \forall \chi \in E(0, 0)$$

and analogously we obtain

$$a(\sigma_{\mathscr{D}}, \chi_{\mathscr{D}}) = b(\chi_{\mathscr{D}}) \quad \forall \chi_{\mathscr{D}} \in E_{\mathscr{D}}(0, 0)\,.$$

Thus, by Lemma 5.3,

$$a(\sigma - \sigma_{\mathscr{D}}, \sigma_{\mathscr{D}} - \tau_{\mathscr{D}}) = 0 \quad \forall \tau_{\mathscr{D}} \in E_{\mathscr{D}}(f, g)\,.$$

Using (1-5), (1-6) and the Schwarz inequality, we obtain

$$C_A \|\sigma - \sigma_{\mathscr{D}}\|_0^2 \leqq a(\sigma - \sigma_{\mathscr{D}}, \sigma - \sigma_{\mathscr{D}}) = a(\sigma - \sigma_{\mathscr{D}}, \sigma - \tau_{\mathscr{D}}) \leqq$$
$$\leqq C_S \|\sigma - \sigma_{\mathscr{D}}\|_0 \, \|\sigma_{\mathscr{D}} - \tau_{\mathscr{D}}\|_0$$

for any $\tau_{\mathscr{D}} \in E_{\mathscr{D}}(f, g)$. ∎

Note that this theorem has been obtained by modifying Cea's lemma (see [4], Theorem 2.4.1), where the infimum is taken over the whole space of finite elements and not only over a subset, as was the case in the present proof.

6. L_2-ESTIMATE FOR THE EXTERNAL APPROXIMATION

In this section we use a standard convergence technique (see [2, 4, 8, 9]). First, we introduce *the composed reference tetrahedron* $\hat{K}$ with the vertices $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ the coordinates of which are $(1, 0, 0)^{\mathsf{T}}$, $(0, 1, 0)^{\mathsf{T}}$, $(0, 0, 1)^{\mathsf{T}}$, $(0, 0, 0)^{\mathsf{T}}$, respectively, and with an interior point $\hat{E}$ which coincides with the gravity center, i.e., $\hat{E} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})^{\mathsf{T}}$.

In the following, $\mathfrak{M}$ denotes a fixed regular family of decomposition of $\bar{\Omega}$ (with a coefficient $\varkappa$) consistent with Γ_1 and Γ_2. Suppose, for simplicity, that the interior point E of any $K \in \mathscr{D} \in \mathfrak{M}$ is the gravity center. Let A, B, C, D be vertices of some $K \in \mathscr{D} \in \mathfrak{M}$, their coordinates being given by (3-3). Define an affine one-to-one mapping $F_K : \hat{K} \to K$ by

(6-1) $$F_K(\hat{x}) = B_K \hat{x} + d\,, \quad \hat{x} \in \hat{K}\,,$$

where $B_K = (a - d, b - d, c - d)$ is a regular matrix as $F_K(\hat{A}) = A, \ldots, F_K(\hat{D}) = D$. Furthermore, $F_K(\hat{E}) = E$. Clearly,

(6-2) $$\|B_K\| \leqq 3 \max(\|a - d\|, \|b - d\|, \|c - d\|) \leqq 3h_K\,.$$

Since it is known that $\mu_3(K) = \frac{1}{6}|\det B_K|$ (see e.g. [11]), it follows from (3-2) that

(6-3) $$\tfrac{4}{3}\pi\varkappa^3 h_K^3 \leqq \tfrac{1}{6}|\det B_K|\,.$$

Denoting by B_K^* the matrix of the algebraic adjoints of B_K, the components of which can be evidently estimated by $2h_K^2$, we arrive at

(6-4) $$\|B_K^{-1}\| = \frac{1}{|\det B_K|}\,\|B_K^*\| \leqq \frac{3}{24\pi\varkappa^3 h_K^3}\,6h_K^2 \leqq \varkappa^{-3} h_K^{-1}\,.$$

For $\tau \in T(K)$ define $\hat{\tau} \in T(\hat{K})$ by

(6-5) $$\hat{\tau}(\hat{x}) = B_K^{-1} \tau(F_K(\hat{x})) (B_K^{-1})^{\mathsf{T}}, \quad \hat{x} \in \hat{K}.$$

Further, for $m = 0, 1, 2, \ldots$ and a nonempty bounded domain $Z \subset \mathbb{R}^3$ with a Lipschitz boundary define the space

$$\mathscr{H}^m(Z) = [H^m(Z)]^9 \cap T(Z)$$

with the norm $\|\cdot\|_{m,Z}$. First, we shall approximate stresses $\hat{\tau} \in \mathscr{H}^1(\hat{K})$ as in [9] for the two-dimensional problem:

To any $\hat{\tau} \in \mathscr{H}^1(\hat{K})$ assign $\hat{\Pi}\hat{\tau} \in T_{\hat{K}}$ satisfying

(6-6) $$\int_{\hat{S}} \hat{v}^{\mathsf{T}} \hat{\Pi}\hat{\tau}\hat{\nu} \, \mathrm{d}\hat{s} = \int_{\hat{S}} \hat{v}^{\mathsf{T}} \hat{\tau}\hat{\nu} \, \mathrm{d}\hat{s} \quad \forall \hat{v} \in V_{\hat{S}}$$

for any external face $\hat{S}$ of $\hat{K}$ with the outward unit normal $\hat{\nu}$, and

(6-7) $$\int_{\hat{K}} \hat{\Pi}\hat{\tau} \, \mathrm{d}\hat{x} = \int_{\hat{K}} \hat{\tau} \, \mathrm{d}\hat{x}.$$

By Theorem 2.1 it follows that $\hat{\Pi}\hat{\tau}$ is uniquely determined by these requirements, since for any $\hat{S}$ there exists a unique linear vector function $\hat{t}_{\hat{S}} \in V_{\hat{S}}$ such that

$$\int_{\hat{S}} \hat{v}^{\mathsf{T}} \hat{t}_{\hat{S}} \, \mathrm{d}\hat{s} = \int_{\hat{S}} \hat{v}^{\mathsf{T}} \hat{\tau}\hat{\nu} \, \mathrm{d}\hat{s} \quad \forall \hat{v} \in V_{\hat{S}}$$

and thus, $\hat{\Pi}\hat{\tau}\hat{\nu}/\hat{S} = \hat{t}_{\hat{S}}$.

Lemma 6.1. *There exists a constant $\hat{C}$ such that*

$$\|\hat{\Pi}\hat{\tau}\|_{0,\hat{K}} \leqq \hat{C}\|\hat{\tau}\|_{1,\hat{K}} \quad \forall \hat{\tau} \in \mathscr{H}^1(\hat{K}).$$

Proof. By Theorem 2.1 there exists a linear operator $\mathscr{B} : \mathbb{R}^{42} \to T_{\hat{K}}$, which assigns to any vector $\alpha = (\alpha_1, \ldots, \alpha_{42})^{\mathsf{T}}$ an element $\hat{\varphi} \in T_{\hat{K}}$ such that

$$\hat{\Phi}_p(\hat{\varphi}) = \alpha_p, \quad p = 1, \ldots, 42,$$

where the degrees of freedom $\hat{\Phi}_p$ are defined, for simplicity, by the outward unit normals $\hat{\nu}_i$ of $\hat{K}$. Thus, there exists a constant $C_{\mathscr{B}} > 0$ such that

$$\frac{1}{C_{\mathscr{B}}} \|\hat{\varphi}\|_{0,\hat{K}} \leqq \|\alpha\| \leqq (\alpha_1^2 + \ldots + \alpha_{36}^2)^{1/2} + (\alpha_{37}^2 + \ldots + \alpha_{42}^2)^{1/2} =$$
$$= (\sum_{i=1}^{4} \sum_{j=1}^{3} \|\hat{\varphi}_i(\hat{M}_{ij})\, \hat{\nu}_i\|^2)^{1/2} + \left\| \int_{\hat{K}} \hat{\varphi} \, \mathrm{d}\hat{x} \right\| \quad \forall \hat{\varphi} \in T_{\hat{K}}.$$

Since all norms in a finite-dimensional space are equivalent and since

$$\hat{t}_i \in V_{\hat{S}_i} \mapsto (\sum_{j=1}^{3} \|\hat{t}_i(\hat{M}_{ij})\|)^{1/2}$$

is the norm in $V_{\hat{S}_i}$ for all external faces $\hat{S}_i \subset \hat{K}$, $i = 1, 2, 3, 4$, we arrive at

$$\|\hat{\varphi}\|_{0,\hat{K}} \leqq C_e \left(\|\hat{\varphi}\hat{v}\|_{0,\partial\hat{K}} + \left\| \int_{\hat{K}} \hat{\varphi} \, d\hat{x} \right\| \right), \tag{6-8}$$

where C_e does not depend on $\hat{\varphi} \in T_{\hat{K}}$. Let $\hat{\tau} \in \mathscr{H}^1(\hat{K})$. Applying $\hat{v} = \hat{\Pi}\hat{\tau}\hat{v}/_{\hat{S}} \in V_{\hat{S}}$ to (6-6), we obtain by the Schwarz inequality

$$\|\hat{\Pi}\hat{\tau}\hat{v}\|_{0,\hat{S}}^2 \leqq \|\hat{\Pi}\hat{\tau}\hat{v}\|_{0,\hat{S}} \, \|\hat{\tau}\hat{v}\|_{0,\hat{S}} \, . \tag{6-9}$$

Referring to (6-7), (6-8) and (6-9), we see that

$$\|\hat{\Pi}\hat{\tau}\|_{0,\hat{K}} \leqq C_e \left(\|\hat{\Pi}\hat{\tau}\hat{v}\|_{0,\partial\hat{K}} + \left\| \int_{\hat{K}} \hat{\Pi}\hat{\tau} \, d\hat{x} \right\| \right) \leqq \tag{6-10}$$

$$\leqq C_e \left((\|\hat{\tau}\hat{v}\|_{0,\partial\hat{K}} + \left\| \int_{\hat{K}} \hat{\tau} \, d\hat{x} \right\| \right) \leqq C_e \left(\|\hat{\tau}\|_{0,\partial\hat{K}} + \int_{\hat{K}} \|\hat{\tau}\| \, d\hat{x} \right).$$

Using the trace theorem [7, 10] and the Hölder inequality, we get

$$\|\hat{\Pi}\hat{\tau}\|_{0,\hat{K}} \leqq C_e(C_t\|\hat{\tau}\|_{1,\hat{K}} + \sqrt{(\mu_3(\hat{K}))} \, \|\hat{\tau}\|_{0,\hat{K}}) \leqq \hat{C}\|\hat{\tau}\|_{1,\hat{K}} \, . \quad \blacksquare$$

Lemma 6.2. *There exists a constant $\bar{C}$ such that*

$$\|\hat{\tau} - \hat{\Pi}\hat{\tau}\|_{0,\hat{K}} \leqq \bar{C}|\hat{\tau}|_{m,\hat{K}} \quad \forall \hat{\tau} \in \mathscr{H}^m(\hat{K}) \, , \quad m = 1, 2 \, .$$

Proof. For $m = 1, 2$ and $\hat{\psi} \in \mathscr{H}^0(\hat{K})$ define the linear functional Ξ^m by

$$\Xi^m(\hat{\tau}) = (\hat{\tau} - \hat{\Pi}\hat{\tau}, \hat{\psi})_{0,\hat{K}} \, , \quad \hat{\tau} \in \mathscr{H}^m(\hat{K}) \, . \tag{6-11}$$

Applying the Schwarz inequality and Lemma 6.1, we obtain

$$|\Xi^m(\hat{\tau})| \leqq \|\hat{\tau} - \hat{\Pi}\hat{\tau}\|_{0,\hat{K}} \, \|\hat{\psi}\|_{0,\hat{K}} \leqq (1 + \hat{C}) \, \|\hat{\tau}\|_{1,\hat{K}} \, \|\hat{\psi}\|_{0,\hat{K}} \leqq$$
$$\leqq (1 + \hat{C}) \, \|\hat{\tau}\|_{m,\hat{K}} \, \|\hat{\psi}\|_{0,\hat{K}} \, .$$

Since $\hat{\Pi}\hat{\varphi} = \hat{\varphi}$ for $\hat{\varphi} \in [P_1(\hat{K})]^9 \cap T(\hat{K})$, we see that $\Xi^m \equiv 0$ on $[P_{m-1}(\hat{K})]^9 \cap T(\hat{K})$, $m = 1, 2$. Using Bramble-Hilbert Lemma [2, 4, 8], we get

$$|\Xi^m(\hat{\tau})| \leqq \bar{C}|\hat{\tau}|_{m,\hat{K}} \, \|\hat{\psi}\|_{0,\hat{K}} \quad \forall \hat{\tau} \in \mathscr{H}^m(\hat{K}) \, ,$$

which together with (6-11) proves the lemma. $\blacksquare$

Now, we formulate an analogous lemma for $\tau \in \mathscr{H}^1(K)$, $K \in \mathscr{D} \in \mathfrak{M}$. Similarly as for $\hat{\Pi}$, we define the operator $\Pi_K : \mathscr{H}^1(K) \to T_K$ by

$$\int_S v^{\mathrm{T}}(\tau - \Pi_K\tau) \, n \, ds = 0 \quad \forall v \in V_S \tag{6-12}$$

for any external face S of K with the normal n, and

$$\int_K (\tau - \Pi_K\tau) \, dx = 0 \, . \tag{6-13}$$

It is worth noticing that $\Pi_K\tau$ does not depend on the size of n.

Lemma 6.3. *There exists a constant C' such that for any tetrahedron K of any decomposition $\mathscr{D} \in \mathfrak{M}$ we have*

$$\|\tau - \Pi_K \tau\|_{0,K} \leqq C' h_K^m |\tau|_{m,K} \quad \forall \tau \in \mathscr{H}^m(K), \quad m = 1, 2.$$

Proof. Using the previous notation, we assign to any $v \in V_S$ a vector $\hat{v} \in V_{\hat{S}}$ by

$$\hat{v}(\hat{s}) = B_K^{\mathsf{T}} v(F_K(\hat{s})), \quad \hat{s} \in \hat{S}.$$

It is easy to show that $n = (B_K^{-1})^{\mathsf{T}} \hat{v}$ is the normal of the face $S = F_K(\hat{S})$. Then (6-5), (6-12) and (6-13) imply that

$$0 = \int_S v^{\mathsf{T}}(\tau - \Pi_K \tau)\, n \,\mathrm{d}s = \int_{\hat{S}} \hat{v}^{\mathsf{T}} B_K^{-1} B_K(\hat{\tau} - \widehat{\Pi_K \tau})\, B_K^{\mathsf{T}}(B_K^{-1})^{\mathsf{T}}\, \hat{v} |\det B_K| \,\mathrm{d}\hat{s},$$

$$0 = \int_K (\tau - \Pi_K \tau)\,\mathrm{d}x = \int_{\hat{K}} B_K(\hat{\tau} - \widehat{\Pi_K \tau})\, B_K^{\mathsf{T}} |\det B_K| \,\mathrm{d}\hat{x}.$$

Therefore,

$$\int_{\hat{S}} \hat{v}^{\mathsf{T}}(\hat{\tau} - \widehat{\Pi_K \tau})\, \hat{v} \,\mathrm{d}\hat{s} = 0, \quad \int_{\hat{K}} (\hat{\tau} - \widehat{\Pi_K \tau})\,\mathrm{d}\hat{x} = 0$$

for any $\hat{v} \in V_{\hat{S}}$ and any external face $\hat{S}$ of $\hat{K}$. Comparing this with (6-6) and (6-7), we get

$$\widehat{\Pi_K \tau} = \hat{\Pi} \hat{\tau} \tag{6-14}$$

for any $\tau \in \mathscr{H}^m(K)$, $m = 1, 2$, and the corresponding $\hat{\tau} \in \mathscr{H}^m(\hat{K})$. By (6-5), (6-14) and Lemma 6.2,

$$\|\tau - \Pi_K \tau\|_{0,K} \leqq \|B_K\|^2 |\det B_K|^{1/2} \|\hat{\tau} - \hat{\Pi}\hat{\tau}\|_{0,\hat{K}} \leqq \bar{C} \|B_K\|^2 |\det B_K|^{1/2} |\hat{\tau}|_{m,\hat{K}} \tag{6-15}$$

for $m = 1, 2$. It is known [3, 4] that any component $\tau_{ij} \in H^m(K)$ fulfils

$$|\tau_{ij} \circ F_K|_{m,\hat{K}} \leqq \|B_K\|^m |\det B_K|^{-1/2} |\tau_{ij}|_{m,K}, \quad m = 0, 1, 2, \ldots,$$

where F_K is defined by (6-1), "$\circ$" denotes the composition of the functions τ_{ij} and F_K. Therefore, by (6-5),

$$|\hat{\tau}|_{m,\hat{K}} \leqq 9 \|B_K^{-1}\|^2 \|B_K\|^m |\det B_K|^{-1/2} |\tau|_{m,K}, \quad m = 0, 1, 2, \ldots \tag{6-16}$$

Referring to (6-15), (6-2) and (6-4), we see that

$$\|\tau - \Pi_K \tau\|_{0,K} \leqq 9\bar{C} \|B_K\|^{m+2} \|B_K^{-1}\|^2 |\tau|_{m,K} \leqq 3^{m+4} \bar{C} \varkappa^{-6} h_K^m |\tau|_{m,K}$$

for $m = 1, 2$. ■

Theorem 6.1. *There exists a constant C such that for any $\mathscr{D} \in \mathfrak{M}$*

$$\|\sigma - \sigma_{\mathscr{D}}\|_0 \leqq C h_{\mathscr{D}} |\sigma|_1 \quad \text{if} \quad \sigma \in \mathscr{H}^1(\Omega),$$

$$\|\sigma - \sigma_{\mathscr{D}}\|_0 \leqq C h_{\mathscr{D}}^2 |\sigma|_2 \quad \text{if} \quad \sigma \in \mathscr{H}^2(\Omega),$$

where σ and $\sigma_{\mathscr{D}}$ are the solution of the dual problem and the solution of its external approximation, respectively.

Proof. Let $\mathscr{D} \in \mathfrak{M}$ be arbitrary. For $\tau \in \mathscr{H}^1(\Omega)$ define $\boldsymbol{\Pi}_{\mathscr{D}}\tau \in \overline{T}_{\mathscr{D}}$ so that

$$\boldsymbol{\Pi}_{\mathscr{D}}\tau/_K = \Pi_K\tau\,, \quad K \in \mathscr{D}\,.$$

Using Lemma 4.1 and (6-12) for each $S \in \Gamma(\mathscr{D})$, it is easy to see that $\boldsymbol{\Pi}_{\mathscr{D}}\tau \in \boldsymbol{T}_{\mathscr{D}}$. By Lemma 6.3 and (3-1) we obtain

$$(6\text{-}17) \qquad \|\tau - \boldsymbol{\Pi}_{\mathscr{D}}\tau\|_0 = \big(\sum_{K\in\mathscr{D}} \|\tau - \Pi_K\tau\|^2_{0,K}\big)^{1/2} \leqq \big(\sum_{K\in\mathscr{D}} C' h_K^{2m}|\tau|^2_{m,K}\big)^{1/2} \leqq C' \boldsymbol{h}^m_{\mathscr{D}}|\tau|_m$$

for $\tau \in \mathscr{H}^m(\Omega)$ and $m = 1, 2$. If $\tau \in \mathscr{H}^1(\Omega) \cap E(f, g)$ then by Green's formula and (1-7), (6-12), (6-13), (4-3) we get that for any $K \in \mathscr{D}$

$$\int_K v^{\mathsf{T}} f \,\mathrm{d}x = -\int_K v^{\mathsf{T}} \operatorname{div} \tau \,\mathrm{d}r\, x = \int_K \tau\,.\,\varepsilon(v)\,\mathrm{d}x - \int_{\partial K} v^{\mathsf{T}}\tau\nu\,\mathrm{d}s =$$

$$= \int_K \Pi_K\tau\,.\,\varepsilon(v)\,\mathrm{d}x - \int_{\partial K} v^{\mathsf{T}}\Pi_K\tau\nu\,\mathrm{d}s = -\int_K v^{\mathsf{T}} \operatorname{div} \Pi_K\tau\,\mathrm{d}x \quad \forall v \in V_K$$

and

$$\int_S v^{\mathsf{T}} g \,\mathrm{d}s = \int_S v^{\mathsf{T}}\tau\nu\,\mathrm{d}s = \int_S v^{\mathsf{T}}\boldsymbol{\Pi}_{\mathscr{D}}\tau\nu\,\mathrm{d}s \quad \forall v \in V_S\,,$$

where $S \subset \Gamma_1$, $S \in \Gamma(\mathscr{D})$. Thus $\boldsymbol{\Pi}_{\mathscr{D}}\tau \in E_{\mathscr{D}}(f, g)$. Using (6-17) for $\tau = \sigma$ and Theorem 5.2 we obtain

$$\|\sigma - \sigma_{\mathscr{D}}\|_0 \leqq C(\boldsymbol{a})\,\|\sigma - \boldsymbol{\Pi}_{\mathscr{D}}\sigma\|_0 \leqq C\boldsymbol{h}^m_{\mathscr{D}}|\sigma|_m\,, \quad m = 1, 2,. \quad \blacksquare$$

7. L_∞-ESTIMATE FOR THE EXTERNAL APPROXIMATION

Given a closed domain $\emptyset \neq Z \subset \mathbb{R}^3$, we define the norm

$$\|\tau\|_{\infty,Z} = \operatorname*{ess\,sup}_{z\in Z} \|\tau(z)\|\,, \quad \tau \in [L_\infty(Z)]^9\,,$$

and set $\|.\|_\infty = \|.\|_{\infty,\bar{\Omega}}$. For $m = 0, 1, 2, \ldots$, put

$$\mathscr{C}^m(Z) = [C^m(Z)]^9 \cap T(Z)\,, \quad \mathscr{C}(Z) = \mathscr{C}^0(Z)\,.$$

Note that we can extend the domains of the operators $\hat{\Pi}$, Π_K, $\boldsymbol{\Pi}_{\mathscr{D}}$ also by tensors of the spaces $\mathscr{C}(\hat{K})$, $\mathscr{C}(K)$, $\mathscr{C}(\bar{\Omega})$, respectively.

Lemma 7.1. *There is a constant $\bar{c}$ such that for any $K \in \mathscr{D} \in \mathfrak{M}$*

$$\|\Pi_K\tau\|_{\infty,K} \leqq \bar{c}\|\tau\|_{\infty,K} \quad \forall \tau \in \mathscr{C}(K)\,.$$

Proof. Since all norms in a finite-dimensional space are equivalent and since (6-10) holds also for $\hat{\tau} \in \mathscr{C}(\hat{K})$, we have

$$(7\text{-}1) \qquad \|\hat{\Pi}\hat{\tau}\|_{\infty,\hat{K}} \leqq c_e\|\hat{\Pi}\hat{\tau}\|_{0,\hat{K}} \leqq c_e C_e\Big(\|\hat{\tau}\|_{0,\partial\hat{K}} + \Big\|\int_{\hat{K}} \hat{\tau}\,\mathrm{d}\hat{x}\Big\|\Big) \leqq \hat{c}\,\|\hat{\tau}\|_{\infty,\hat{K}}\,,$$

where $\hat{c}$ does not depend on $\hat{\tau} \in \mathscr{C}(\hat{K})$.

Now, let $K \in \mathscr{D} \in \mathfrak{M}$ be arbitrary. Similarly as in (6-14), we can prove that $\widehat{\Pi_K \tau} = \hat{\Pi} \hat{\tau}$ for $\tau \in \mathscr{C}(K)$ and the corresponding $\hat{\tau} \in \mathscr{C}(\hat{K})$. Thus, (6-5), (7-1), (6-2) and (6-4) yield

$$\|\Pi_K \tau\|_{\infty,\hat{K}} \leqq \|B_K\|^2 \, \|\widehat{\Pi_K \tau}\|_{\infty,\hat{K}} = \|B_K\|^2 \, \|\hat{\Pi}\hat{\tau}\|_{\infty,K} \leqq \hat{c}\|B_K\|^2 \, \|\hat{\tau}\|_{\infty,\hat{K}} \leqq$$
$$\leqq \hat{c}\|B_K\|^2 \, \|B_K^{-1}\|^2 \, \|\tau\|_{\infty,K} \leqq \bar{c}\|\tau\|_{\infty,K} . \quad \blacksquare$$

Lemma 7.2. *For any $\tau \in \mathscr{C}^2(\bar{\Omega})$ there exists a constant c_τ such that*

$$\|\tau - \Pi_{\mathscr{D}}\tau\|_\infty \leqq c_\tau \boldsymbol{h}_{\mathscr{D}}^2 \quad \forall \mathscr{D} \in \mathfrak{M} . \tag{7-2}$$

Proof. Let $\tau \in \mathscr{C}^2(\bar{\Omega})$ and let $K \in \mathscr{D} \in \mathfrak{M}$ be arbitrary. Let $x^0 = (x_1^0, x_2^0, x_3^0)^{\mathsf{T}} \in K$ be fixed. Then, for any $x = (x_1, x_2, x_3)^{\mathsf{T}} \in K$ Taylor's formula gives

$$\tau(x) = \tau(x^0) + \tau'(x) + \tfrac{1}{2}\tau''(x) , \tag{7-3}$$

where the matrices $\tau'(x)$ and $\tau''(x)$ have

$$\sum_{k=1}^{3} \frac{\partial \tau_{ij}(x^0)}{\partial x_k} (x_k - x_k^0) \quad \text{and} \quad \sum_{k,l=1}^{3} \frac{\partial^2 \tau_{ij}(\theta_{ij}(x))}{\partial x_k \, \partial x_l} (x_k - x_k^0)(x_l - x_l^0)$$

in the position (i, j), respectively, and $\theta_{ij}(x) = \theta_{ji}(x)$, $i, j = 1, 2, 3$, lie on the line segment $\overline{x^0 x}$. Since $\Pi_K \varphi = \varphi$ for $\varphi \in [P_1(K)]^9 \cap T(K)$, we have from (7-3)

$$(\Pi_K \tau)(x) = \tau(x^0) + \tau(x) + \tfrac{1}{2}(\Pi_K \tau'')(x) .$$

We again use (7-3) to obtain by Lemma 7.1 and (3-1) that

$$\|\tau - \Pi_K \tau\|_{\infty,K} = \tfrac{1}{2}\|\tau'' - \Pi_K \tau''\|_{\infty,K} \leqq \frac{1 + \bar{c}}{2} \|\tau''\|_{\infty,K} \leqq$$
$$\leqq \frac{1 + \bar{c}}{2} \operatorname*{ess\,sup}_{z \in \bar{\Omega}} \left(\sum_{i,j=1}^{3} \left(\sum_{k,l=1}^{3} \left| \frac{\partial^2 \tau_{ij}(z)}{\partial x_k \, \partial x_l} \right| \right)^2 \right)^{1/2} h_K^2 \leqq c_\tau \boldsymbol{h}_{\mathscr{D}}^2$$

for any $K \in \mathscr{D} \in \mathfrak{M}$. Hence, (7-2) is valid. $\blacksquare$

Theorem 7.1. *Let $\mathfrak{M}$ be strongly regular. If $\sigma \in \mathscr{C}^2(\bar{\Omega})$, then*

$$\|\sigma - \sigma_{\mathscr{D}}\|_\infty = O(\boldsymbol{h}_{\mathscr{D}}^{1/2}) \quad as \quad \boldsymbol{h}_{\mathscr{D}} \to 0 , \quad \mathscr{D} \in \mathfrak{M} .$$

Proof. Let $K \in \mathscr{D} \in \mathfrak{M}$ be arbitrary. Then by (6-5), (6-16), (6-2), (6-3) and (6-4) we obtain for $\tau \in T_K$ that

$$\|\tau\|_{\infty,K} \leqq \|B_K\|^2 \, \|\hat{\tau}\|_{\infty,\hat{K}} \leqq c_e \|B_K\|^2 \, \|\hat{\tau}\|_{0,\hat{K}} \leqq \tag{7-4}$$
$$\leqq c_e \|B_K\|^2 \, 9 \|B_K^{-1}\|^2 \, |\det B_K|^{-1/2} \, \|\tau\|_{0,K} \leqq c' h_K^{-3/2} \|\tau\|_{0,K} ,$$

where c_e is from (7-1) and c' does not depend $K \in \mathscr{D} \in \mathfrak{M}$ and $\tau \in T_K$. Since $\mathfrak{M}$ is strongly regular — see (3-2) — we get

$$\|\tau\|_{\infty,K} \leqq c' h_K^{-3/2} \|\tau\|_{0,K} \leqq c' \varrho_K^{-3/2} \|\tau\|_{0,K} \leqq c' \varkappa^{-3/2} \boldsymbol{h}_{\mathscr{D}}^{-3/2} \|\tau\|_{0,K} \leqq c \boldsymbol{h}_{\mathscr{D}}^{-3/2} \|\tau\|_0$$

for any $\tau \in \boldsymbol{T}_{\mathscr{D}}$ and $K \in \mathscr{D} \in \mathfrak{M}$. Consequently,

$$\|\tau\|_\infty \leqq c\boldsymbol{h}_{\mathscr{D}}^{-3/2}\|\tau\|_0 \quad \forall \tau \in \boldsymbol{T}_{\mathscr{D}}\,. \tag{7-5}$$

Now, by Theorem 6.1, (7-2) and (6-17) we have

$$\begin{aligned}\|\sigma - \sigma_{\mathscr{D}}\|_\infty &\leqq \|\sigma - \boldsymbol{\Pi}_{\mathscr{D}}\sigma\|_\infty + c\boldsymbol{h}_{\mathscr{D}}^{-3/2}\|\boldsymbol{\Pi}_{\mathscr{D}}\sigma - \sigma_{\mathscr{D}}\|_0 \leqq \\ &\leqq c_\sigma \boldsymbol{h}_{\mathscr{D}}^2 + c\boldsymbol{h}_{\mathscr{D}}^{-3/2}(\|\sigma - \boldsymbol{\Pi}_{\mathscr{D}}\sigma\|_0 + \|\sigma - \sigma_{\mathscr{D}}\|_0) \leqq \\ &\leqq c_\sigma \boldsymbol{h}_{\mathscr{D}}^2 + c\boldsymbol{h}_{\mathscr{D}}^{-3/2}(C'\boldsymbol{h}_{\mathscr{D}}^2|\sigma|_2 + C\boldsymbol{h}_{\mathscr{D}}^2|\sigma|_2) = O(\boldsymbol{h}_{\mathscr{D}}^{1/2})\end{aligned}$$

for $\sigma \in \mathscr{C}^2(\bar{\Omega})$. ■

Remark. We have chosen the assumption of strong regularity of the family $\mathfrak{M}$, because we cannot estimate $\|\sigma - \sigma_{\mathscr{D}}\|_{0,K}$ locally in terms of h_K^2, but only globally in terms of $\boldsymbol{h}_{\mathscr{D}}^2$. Furthermore, note that the L_∞-estimate of the analogous two-dimensional problem will be even $O(\boldsymbol{h}_{\mathscr{D}})$, since we can bound the absolute value of the determinant of the affine mapping (from the reference triangle to an arbitrary triangle) from below in terms of h_K^2-compare with (6-3) and (7-4).

8. INTERNAL APPROXIMATION OF THE DUAL PROBLEM

We introduce another type of the approximation of the dual problem which has been studied by Hlaváček [8] in the two-dimensional elasticity.

Let $\bar{\tau} \in E(f, g)$ be fixed. Using the substitution $\tau = \tau^0 + \bar{\tau}$, we can formulate the dual problem of Section 1 equivalently in the following way:

Find σ^0 which minimizes the functional $J^0 : \boldsymbol{T} \to \mathbb{R}^1$ defined by

$$J^0(\tau^0) = \tfrac{1}{2}\boldsymbol{a}(\tau^0, \tau^0) - \boldsymbol{b}(\tau^0) + \boldsymbol{a}(\bar{\tau}, \tau^0)\,, \quad \tau^0 \in \boldsymbol{T}\,,$$

over the set $E(0, 0)$. (One easily sees that $\sigma^0 + \bar{\tau} = \sigma$.)

Definition. *An internal approximation of the dual problem consists in finding $\sigma_{\mathscr{D}}^0$ which minimizes J^0 over the set $E_{\mathscr{D}}(0, 0)$. The tensor $\sigma_{\mathscr{D}}^0 + \bar{\tau}$ is called the solution of the internal approximation.*

As in Section 5, we can show that $\sigma_{\mathscr{D}}^0$ exists and is unique. The previous problem is called the internal approximation, since $E_{\mathscr{D}}(0, 0) \subset E(0, 0)$ by Lemma 5.3, i.e., we approximate the set $E(0, 0)$ internally. Obviously, $J^0(\sigma^0) \leqq J^0(\sigma_{\mathscr{D}}^0)$, while the analogous inequality for the external approximation does not hold in the general case, since $E_{\mathscr{D}}(f, g)$ is not a subset of $E(f, g)$ in general. Using the above inequality and knowing, moreover, an approximation of the primary elasticity problem, we could obtain a posteriori error estimates and two-sided bounds of energy [7]. Since $\sigma - (\sigma_{\mathscr{D}}^0 + \bar{\tau}) = \sigma^0 - \sigma_{\mathscr{D}}^0$, it is sufficient to study the convergence of the difference $\sigma^0 - \sigma_{\mathscr{D}}^0$.

Theorem 8.1. *There exists a constant C independent od $\mathcal{D} \in \mathfrak{M}$ such that*

$$\|\sigma^0 - \sigma^0_{\mathcal{D}}\|_0 \leqq C h^m_{\mathcal{D}} |\sigma^0|_m \quad \text{if} \quad \sigma^0 \in \mathcal{H}^m(\Omega)\,, \quad m = 1, 2\,.$$

Proof. It can be proved as in Theorem 5.2 that

$$(8\text{-}1) \qquad \|\sigma^0 - \sigma^0_{\mathcal{D}}\|_0 \leqq C(\boldsymbol{a}) \inf \{ \|\sigma^0 - \tau^0_{\mathcal{D}}\|_0 \mid \tau^0_{\mathcal{D}} \in E_{\mathcal{D}}(0, 0)\}\,.$$

If $\sigma^0 \in \mathcal{H}^1(\Omega)$, then we know from the proof of Theorem 6.1 that $\Pi_{\mathcal{D}}\sigma^0 \in E_{\mathcal{D}}(0, 0)$. Thus, by (6-17),

$$\|\sigma^0 - \sigma^0_{\mathcal{D}}\|_0 \leqq C(\boldsymbol{a}) \|\sigma^0 - \Pi_{\mathcal{D}}\sigma^0\|_0 \leqq C h^m_{\mathcal{D}} |\sigma^0|_m$$

for $\sigma^0 \in \mathcal{H}^m(\Omega)$, $m = 1, 2$. ∎

Remark. A question arises about the convergence of the method when σ^0 is not smooth enough. Thus let $\sigma^0 \notin \mathcal{H}^1(\bar{\Omega})$ and let $\varphi^0 \in E(0, 0) \cap \mathcal{H}^1(\Omega)$. Then from (8-1),

$$\|\sigma^0 - \sigma^0_{\mathcal{D}}\|_0 \leqq C(\boldsymbol{a}) (\|\sigma^0 - \varphi^0\|_0 + \inf \{\|\varphi^0 - \tau^0_{\mathcal{D}}\|_0) \mid \tau^0_{\mathcal{D}} \in E_{\mathcal{D}}(0,0)\}) \leqq$$
$$\leqq C(\boldsymbol{a}) \|\sigma^0 - \varphi^0\|_0 + C(\boldsymbol{a}) \|\varphi^0 - \Pi_{\mathcal{D}}\varphi^0\|_0\,.$$

Since the second term is $O(h_{\mathcal{D}})$, in order to obtain convergence it is necessary to find $\varphi^0 \in E(0, 0) \cap \mathcal{H}^1(\Omega)$ sufficiently close to $\sigma^0 \in E(0, 0)$ in the L_2-norm. This can be done in the same way as for the plane problem in [8] (Theorem 4.3). where it is proved that the set $E(0, 0) \cap [C^\infty(\bar{\Omega})]^4$ is dense in $E(0, 0)$ (with the topology of $[L_2(\Omega)]^4$) provided certain assumptions on the domain $\Omega \subset \mathbb{R}^2$ are satisfied; e.g., if $\Gamma_1 = \partial\Omega$, the author supposes Ω to be starlike.

Theorem 8.2. *Let $\mathfrak{M}$ be strongly regular. Then for $\sigma^0 \in \mathcal{C}^2(\bar{\Omega})$,*

$$\|\sigma^0 - \sigma^0_{\mathcal{D}}\|_\infty = O(h^{1/2}_{\mathcal{D}}) \quad as \quad h_{\mathcal{D}} \to 0\,, \quad \mathcal{D} \in \mathfrak{M}\,.$$

Proof. Referring to (7-5), (7-2), (6-17) and Theorem 8.1, we see that

$$\|\sigma^0 - \sigma^0_{\mathcal{D}}\|_\infty \leqq \|\sigma^0 - \Pi_{\mathcal{D}}\sigma^0\|_\infty + c h^{-3/2}_{\mathcal{D}} \|\Pi_{\mathcal{D}}\sigma^0 - \sigma^0_{\mathcal{D}}\|_0 \leqq$$
$$\leqq c_{\sigma^0} h^2_{\mathcal{D}} + c h^{-3/2}_{\mathcal{D}} (\|\sigma^0 - \Pi_{\mathcal{D}}\sigma^0\|_0 + \|\sigma^0 - \sigma^0_{\mathcal{D}}\|_0) \leqq$$
$$\leqq c_{\sigma^0} h^2_{\mathcal{D}} + c h^{-3/2}_{\mathcal{D}} (C' h^2_{\mathcal{D}} |\sigma^0|_2 + C h^2_{\mathcal{D}} |\sigma^0|_2) = O(h^{1/2}_{\mathcal{D}})\,. \quad ∎$$

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Souhrn

METODA ROVNOVÁŽNÝCH PRVKŮ V TROJROZMĚRNÉ PRUŽNOSTI

MICHAL KŘÍŽEK

Je vyšetřován čtyřstěnný rovnovážný prvek napětí, který vznikl zobecněním trojúhelníkového rovnovážného prvku zavedeného Watwoodem a Hartzem [6]. Na dané polyedrické oblasti jsou studovány dva různé typy po částech lineární aproximace duální úlohy lineární pružnosti metodou konečných prvků. Pro oba typy je dokázána konvergence v L_2-normě řádu $O(h^2)$ a v L_∞-normě řádu $O(h^{1/2})$ pro dostatečně hladké řešení. Za tím účelem je také dokázána existence silně regulárního systému rozkladů polyedru na čtyřstěny.

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