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CONTACT BETWEEN ELASTIC PERFECTLY PLASTIC BODIES

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INTRODUCTION

Unilateral contact problems of two bounded bodies within the framework of linear two-dimensional elasticity have been studied in the paper [1]. If the material of the bodies is elastic perfectly plastic, obeying the Hencky's law, the formulation in terms of stresses is more suitable than that in displacements. Thus we first extend the well-known Haar-Kármán principle to the case of a unilateral contact on the boundary. This is carried out in Section 1 for the problems with a bounded contact zone and with an enlarging contact zone.

Approximations to both types of contact problems are proposed in Section 2, based on piecewise constant triangular finite elements. Convergence of the method is proven for any regular family of triangulations. In Section 3, we present a simplification of the approximate problem with a bounded contact zone, which enables us to employ methods of nonlinear programming.

1. EXTENDED HAAR-KÁRMÁN PRINCIPLE

We assume:

- plane problems,
- bounded bodies,
- small deformations,
- zero friction,
- zero initial strain and stress fields,
- constant temperature field,
- Hencky's law.

Let two elastic perfectly plastic bodies occupy bounded domains $\Omega', \Omega'' \subset R^2$ with Lipschitz boundaries. Henceforth one or two primes denotes that the quantity is referred to the body Ω' and Ω'', respectively. We shall use the Cartesian coordinate system $\mathbf{x} = (x_1, x_2)$. The summation is implied over the range 1, 2 if an index is repeated. $H^j(\Omega)$ will denote the Sobolev space $W^{j,2}(\Omega)$ of functions with square-integrable derivatives up to the order j in the sense of distributions.

Let $\mathbb{R}_\sigma$ be the space of symmetric 2×2 matrices (strain or stress tensors). Assume that a yield function $f : \mathbb{R}_\sigma \to R$ is given, which is convex and continuous in $\mathbb{R}_\sigma$. We introduce the following notations:

$$S = \{\tau : \Omega \to \mathbb{R}_\sigma \mid \tau_{ij} \in L^2(\Omega) \ \forall i, j\}, \quad \Omega = \Omega' \cup \Omega''.$$

$$\langle \sigma, \mathbf{e} \rangle = \int_\Omega \sigma_{ij} e_{ij} \, \mathrm{d}x, \quad \|\sigma\|_0 = \langle \sigma, \sigma \rangle^{1/2}.$$

In the space S we introduce also the energy scalar product $(\sigma, \tau) = \langle c^{-1}\sigma, \tau \rangle$, $\|\sigma\| = (\sigma, \sigma)^{1/2}$, where $c : S \to S$ is the isomorphism defined by the generalized Hooke's law:

$$\sigma = c\mathbf{e} \Leftrightarrow \sigma_{ij} = c_{ijkm} e_{km}.$$

Here $c_{ijkm} \in L^\infty(\Omega^M)$, $M = ', ''$; σ and $\mathbf{e}$ are the stress and strain tensors, respectively;

$$\exists \alpha > 0, \quad \langle c\mathbf{e}, \mathbf{e} \rangle \geqq \alpha \|\mathbf{e}\|_0^2 \quad \forall \mathbf{e} \in S.$$

We introduce the set of plastically admissible stresses

$$B = \{\tau \in \mathbb{R}_\sigma \mid f(\tau) \leqq 1\}.$$

It is easy to see that B is convex and closed in $\mathbb{R}_\sigma$.

We define the set of plastically admissible stress fields

$$P = \{\tau \in S \mid \tau(\mathbf{x}) \in B \text{ a.e. in } \Omega' \cup \Omega''\}.$$

The set P is convex and closed in S.

The Hencky's law can be stated in the following way (cf. [2], [3]). Introducing the the projection $\Pi_B(\mathbf{x}) : \mathbb{R}_\sigma \to B$ onto the set B with respect to the scalar product $(c^{-1}(\mathbf{x})\,\sigma)_{ij}\,\tau_{ij}$, then

$$\sigma = \Pi_B(\mathbf{x})\, c\mathbf{e}.$$

Let us consider the actual strain tensor field $\mathbf{e}(\mathbf{u}) \in S$,

$$e_{ij}(\mathbf{u}) = \tfrac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i), \quad \text{where} \quad \mathbf{u}^M \in [H^1(\Omega^M)]^2, \quad M = ', ''$$

and the actual stress tensor field $\sigma \in S$. (Suppose the existence of all these fields for the time being).

Moreover, let $\Pi : S \to P$ be the projection onto the set P with respect to the energy scalar product (σ, τ). Then

$$(\Pi\tau)(\mathbf{x}) = \Pi_B(\mathbf{x})\, \tau(\mathbf{x})$$

holds almost everywhere in $\Omega' \cup \Omega''$ (see [3]). Hence we may write

$$\sigma = \Pi c\, \mathbf{e}(\boldsymbol{u})\,. \tag{0.1}$$

1.1 Bounded contact zone

In [1] – I we have distinguished two kinds of unilateral contact problems according to the geometrical shape of the bodies: (i) problems with a bounded contact zone, (ii) problems with an enlarging contact zone. Next let us consider the first class of problems and recall their variational formulations (see [1] – I and III).

Defining the contact zone

$$\Gamma_K = \partial\Omega' \cap \partial\Omega''\,,$$

we have the following decompositions

$$\partial\Omega' = \bar{\Gamma}_u \cup \bar{\Gamma}'_\tau \cup \Gamma_K\,, \quad \partial\Omega'' = \bar{\Gamma}_0 \cup \bar{\Gamma}''_\tau \cup \Gamma_K\,,$$

where Γ_u, Γ_0, Γ'_τ and Γ''_τ are mutually disjoint open parts of the boundaries; assume that Γ_u and Γ_K have positive measure. The remaining parts may be either of positive measure or empty.

We say that a unilateral bounded contact occurs on Γ_K if

$$u'_n + u''_n \leqq 0 \tag{1.1}$$

holds a.e. on Γ_K, where

$$u^M_n = u^M_i n^M_i\,, \quad M = {}',{}''$$

and $\boldsymbol{n}^M$ denotes the unit outward normal with respect to $\partial\Omega^M$. (See [1] – I for the derivation of the condition (1.1) of non-penetrating).

On Γ_u we consider the displacement condition

$$\boldsymbol{u}' = \boldsymbol{u}'_0\,, \tag{1.2}$$

where $\boldsymbol{u}'_0 \in [H^1(\Omega')]^2$ is given such that $u'_{0n} = 0$ on Γ_K.

On Γ_0 the bilateral contact conditions are prescribed:

$$u_n = 0\,, \quad T_t(\sigma) = 0\,, \tag{1.3}$$

where $T_t(\sigma) = \sigma_{ij} n_j t_i$, $\mathbf{t} = (-n_2, n_1)$, denotes the tangential stress vector component.

On $\Gamma_\tau = \Gamma'_\tau \cup \Gamma''_\tau$ the tractions are prescribed, i.e.,

$$T_i(\sigma) = \sigma_{ij} n_j = P_i\,, \quad i = 1, 2\,, \tag{1.4}$$

where $P_i \in L^2(\Gamma_\tau)$ are given.

On Γ_K we have the condition (1.1) and

$$T'_n(\sigma) = T''_n(\sigma) \leqq 0\,, \quad T'_n(\sigma)\,(u'_n + u''_n) = 0\,, \quad T'_t = T''_t = 0\,. \tag{1.5}$$

The stress field σ satisfies the equilibrium equations

$$\frac{\partial \sigma_{ij}}{\partial x_j} + F_i = 0 \quad \text{in} \quad \Omega^M , \quad M = ', '' , \tag{1.6}$$

where $F_i \in L^2(\Omega)$ are the body force components.

In the variational formulations we need the space

$$V = \{\mathbf{v} \in [H^1(\Omega')]^2 \times [H^1(\Omega'')]^2 \mid \mathbf{v} = 0 \text{ on } \Gamma_u , \quad v_n = 0 \text{ on } \Gamma_0\} ,$$

the cone of admissible virtual displacements

$$K = \{\mathbf{v} \in V \mid v'_n + v''_n \leqq 0 \text{ on } \Gamma_K\}$$

and the set of statically admissible stress fields

$$K^+ = \{\tau \in S \mid \langle \mathbf{e}(\mathbf{v}), \tau \rangle \geqq L(\mathbf{v}) \ \forall \mathbf{v} \in K\} ,$$

where

$$L(\mathbf{v}) = \int_\Omega F_i v_i \, \mathrm{d}\mathbf{x} + \int_{\Gamma_\tau} P_i v_i \, \mathrm{d}s .$$

Theorem 1.1. *Assume that fields of displacements* $\mathbf{u}$ *and stresses* σ *(sufficiently smooth) exist such that the conditions* (0.1), (1.1), ... (1.6) *are satisfied. Then the stress field* σ *solves the following problem*

$$(\mathscr{P}_1) \qquad \mathscr{S}(\tau) = \tfrac{1}{2}\|\tau\|^2 - \langle \mathbf{e}(\mathbf{u}_0), \tau \rangle = \min \quad \text{over} \quad K^+ \cap P .$$

Remark 1.1. This minimization problem represents an extension of the Haar-Kármán principle (cf. [2], [3]).

Proof. Instead of (0.1) we may write for any $\tau \in P$

$$0 \geqq (c\,\mathbf{e}(\mathbf{u}) - \sigma, \tau - \sigma) = \langle \mathbf{e}(\mathbf{u}), \tau - \sigma \rangle - (\sigma, \tau - \sigma) . \tag{1.7}$$

If we set $\mathbf{u} = \mathbf{u}_0 + \mathbf{w}$, where $\mathbf{u}''_0 \equiv 0$, then $\mathbf{w} \in K$ and

$$\langle \mathbf{e}(\mathbf{w}), \sigma \rangle = L(\mathbf{w}) + \int_{\Gamma_K} T'_n(\sigma)\,(w'_n + w''_n)\, \mathrm{d}s \tag{1.8}$$

can be deduced by integration by parts. Since $w'_n + w''_n = u'_n + u''_n$ holds on Γ_K, it follows from (1.5), that the integral over Γ_K vanishes. Moreover, $\sigma \in K^+$ can be verified on the basis of (1.3), (1.4), (1.5), (1.6) and Lemma 1.6 of [1] – III.

If $\tau \in K^+$, then

$$\langle \mathbf{e}(\mathbf{w}), \tau \rangle - L(\mathbf{w}) \geqq 0 . \tag{1.9}$$

Thus inserting into (1.7) from (1.8), (1.9), we may write for $\tau \in K^+ \cap P$

$$(\sigma, \tau - \sigma) \geqq \langle \mathbf{e}(\mathbf{u}_0), \tau - \sigma \rangle + \langle \mathbf{e}(\mathbf{w}), \tau \rangle - \langle \mathbf{e}(\mathbf{w}), \sigma \rangle \geqq \langle \mathbf{e}(\mathbf{u}_0), \tau - \sigma \rangle .$$

Since K^+ and P are convex sets and $\mathscr{S}$ is a convex functional, the variational inequality is equivalent to the problem $(\mathscr{P}_1)$.

Theorem 1.2. *Let the set $K^+ \cap P$ be non-empty. Then the problem $(\mathscr{P}_1)$ has a unique solution.*

Proof. The sets K^+ and P are convex and closed in S, the functional $\mathscr{S}$ is quadratic and strictly convex. Hence the existence and uniqueness follow.

Remark 1.2. The formulation in terms of displacements is more difficult to handle, as far as the existence and uniqueness is concerned – see [2], [3] and [5] for the classical boundary value problems.

1.2 Enlarging contact zone

If the bodies Ω' and Ω'' have smooth boundaries in a neighbourhood of $\partial\Omega' \cap \partial\Omega''$, the contact zone can enlarge during the deformation process. We introduce (see [1] – I) a local Cartesian coordinate system (ξ, η) at a point of $\partial\Omega' \cap \partial\Omega''$ such that the ξ-axis coincides with the direction of $\boldsymbol{n}''$ and define the function

$$\varepsilon(\eta) = f'(\eta) - f''(\eta)\,, \quad \eta \in \langle a, b\rangle\,,$$

which describes the distance between the bodies before the deformation.

The condition (1.1) of non-penetrating is now replaced by

$$u''_\xi - u'_\xi \leqq \varepsilon \quad \forall \eta \in \langle a, b\rangle \tag{1.10}$$

where $\langle a, b\rangle$ is an a priori estimate of the projection of the enlarged contact zone.

On Γ_u we have the condition (1.2), where let (u'_0, u''_0) be such that $u''_{0n} = 0$ on Γ_0 and $u''_{0\xi} - u'_{0\xi} = \varepsilon$ on $\langle a, b\rangle$. The conditions (1.3) on Γ_0, (1.4) on Γ_τ and (1.6) in $\Omega' \cup \Omega''$ remain unchanged, whereas (1.5) on Γ_K is to be replaced by the following three conditions

$$\begin{gathered} -T'_\xi(\sigma)(\cos\alpha')^{-1} = T''_\xi(\sigma)(\cos\alpha'')^{-1} \leqq 0\,, \\ T'_\eta(\sigma) = T''_\eta(\sigma) = 0\,, \\ T''_\xi(\sigma)(u''_\xi - u'_\xi - \varepsilon) = 0\,, \end{gathered} \tag{1.11}$$

which hold at almost all points of $\Gamma'_K \cup \Gamma''_K$ with the same coordinates $\eta \in \langle a, b\rangle$. Here α^M, $M = {}'$, $''$, denotes the angle between η-axis and the tangent to Γ^M_K.

We introduce the set of admissible virtual displacements

$$K_\varepsilon = \{\mathbf{v} \in V \mid v''_\xi - v'_\xi \leqq \varepsilon \ \forall \eta \in \langle a, b\rangle\}$$

and the set of statically admissible stress fields

$$\begin{gathered} K_0^+ = \{\tau \in S \mid \langle \mathbf{e}(\mathbf{v}), \tau\rangle \geqq L(v) \ \forall v \in K_0\}\,, \\ (K_0 = K_\varepsilon \text{ with } \varepsilon \equiv 0)\,. \end{gathered}$$

Theorem 1.3. *Assume that fields of displacements* $\boldsymbol{u}$ *and stresses* σ *exist (sufficiently smooth) such that the conditions* (0.1), (1.2), (1.3), (1.4), (1.6), (1.10), (1.11) *are satisfied.*

Then σ solves the following problem

$$(\mathscr{P}_2) \qquad \mathscr{S}(\tau) = \tfrac{1}{2}\|\tau\|^2 - \langle \mathbf{e}(\boldsymbol{u}_0), \tau \rangle = \min \quad \textit{over} \quad K_0^+ \cap P .$$

Proof. For any $\tau \in P$ we have (see (1.7))

$$(1.13) \qquad (\sigma, \tau - \sigma) \geqq \langle \mathbf{e}(\boldsymbol{u}), \tau - \sigma \rangle .$$

Considering any $\tau \in P \cap K_0^+$, we may write $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{w}$ with $\boldsymbol{w} \in K_0$ and

$$(1.14) \qquad \langle \mathbf{e}(\boldsymbol{u}), \tau - \sigma \rangle = \langle \mathbf{e}(\boldsymbol{u}_0), \tau - \sigma \rangle + \langle \mathbf{e}(\boldsymbol{w}), \tau - \sigma \rangle .$$

We can show that $\sigma \in K_0^+$. In fact, for any $\boldsymbol{v} \in K_0$

$$\langle \mathbf{e}(\boldsymbol{v}), \sigma \rangle = L(\boldsymbol{v}) + \int_a^b \left[v'_\xi T'_\xi(\sigma) (\cos \alpha')^{-1} + v''_\xi T''_\xi(\sigma) (\cos \alpha'')^{-1} \right] \mathrm{d}\eta$$

can be deduced by integration by parts and using (1.6) and the boundary conditions. By virtue of $(1.11)_1$, the last integral equals to

$$\int_a^b T''_\xi(\sigma) (\cos \alpha'')^{-1} (v''_\xi - v'_\xi) \, \mathrm{d}\eta \geqq 0 ,$$

which proves that $\sigma \in K_0^+$. On the other hand, we have in particular:

$$T''_\xi(\sigma) (w''_\xi - w'_\xi) = (u''_\xi - u'_\xi - \varepsilon) T''_\xi(\sigma) = 0 \quad \forall \eta \in \langle a, b \rangle ,$$

so that

$$\langle \mathbf{e}(\boldsymbol{w}), \sigma \rangle = L(\boldsymbol{w}) .$$

Thus we obtain for $\tau \in K_0^+$

$$\langle \mathbf{e}(\boldsymbol{w}), \tau - \sigma \rangle = \langle \mathbf{e}(\boldsymbol{w}), \tau \rangle - L(\boldsymbol{w}) \geqq 0$$

and inserting into (1.13), we arrive at

$$(1.15) \qquad (\sigma, \tau - \sigma) \geqq \langle \mathbf{e}(\boldsymbol{u}_0), \tau - \sigma \rangle \quad \forall \tau \in K_0^+ \cap P .$$

The equivalence of (1.15) and $(\mathscr{P}_2)$ is obvious, the set K_0^+ being convex.

Theorem 1.4. *Assume that the set $K_0^+ \cap P$ is non-empty. Then the problem $(\mathscr{P}_2)$ has a unique solution.*

Proof. The sets P and K_0^+ are convex and closed in S, the functional $\mathscr{S}$ is quadratic and strictly convex.

2. APPROXIMATIONS BY PIECEWISE CONSTANT STRESS FIELDS

We shall apply the finite element method to the approximate solution of the problems $(\mathscr{P}_1)$ and $(\mathscr{P}_2)$, using the simplest possible model, i.e. piecewise constant external approximations of the set of statically admissible stress fields.

2.1 Approximations to the problem with a bounded contact zone

Assume that Γ_K, Γ_u and Γ_0 consist of straight segments only. Let us consider triangulations $\mathcal{T}_h^M$ of Ω^M, $(M = ', '')$ such that their nodes on Γ_K coincide and the triangles $T \in \mathcal{T}_h^M$ adjacent to the boundaries $\partial\Omega^M$ may have curved sides along the boundary Γ_τ. We introduce the space of piecewise linear spline functions

$$V_h = \{v \in V | v|_T \in [P_1(T)]^2 \ \forall T \in \mathcal{T}_h\},$$

where $\mathcal{T}_h = \mathcal{T}_h' \cup \mathcal{T}_h''$ and $P_k(T)$ denotes the space of polynomials of the degree k, defined on T. Let h denote the maximal diameter of all triangles in $\mathcal{T}_h$. The minimal interior angle of all triangles in $\mathcal{T}_h$ will be denoted by ϑ_h. (If the triangle T has a curved side, then the interior angles are defined by the angles of the straight triangle with the same vertices.) We say that a family $\{\mathcal{T}_h\}$, $0 < h \leqq h_0$, of triangulations is regular, if ϑ_h is bounded away from zero by a number ϑ, independent of h.

Introducing the space of piecewise constant stress fields

$$S_h = \{\tau \in S | \tau_{ij}|_T \in P_0(T) \ \forall i, j, \ \forall T \in \mathcal{T}_h\},$$

we may define external approximations of the set K^+ as follows:

$$K_h^+ = \{\tau \in S_h | \ \langle \mathbf{e}(\mathbf{v}_h), \tau \rangle \geqq L(\mathbf{v}_h) \ \forall \mathbf{v}_h \in K_h\}$$

where $K_h = K \cap V_h = \{\mathbf{v} \in V_h \mid v_n' + v_n'' \leqq 0 \text{ on } \Gamma_K\}$.

Note that if the condition (1.1) of non-penetrating is fulfilled at the nodes of Γ_K, it holds on the whole Γ_K, by virtue of the assumed shape of Γ_K and the linearity of $\mathbf{v} \in V_h$ in any triangle.

Instead of the problem $(\mathcal{P}_1)$ we shall solve the approximate problem

$$\mathcal{S}(\sigma_h) = \min \quad \text{over} \quad K_h^+ \cap P. \tag{2.1}$$

To analyze the solvability of the problem (2.1), we introduce a projection $r_h : S \to \to S_h$ as follows:

$$\langle \tau - r_h\tau, \chi_h \rangle = 0 \quad \forall \chi_h \in S_h.$$

Lemma 2.1. *If $\tau \in K^+ \cap P$, then $r_h\tau \in K_h^+ \cap P$.*

Proof. It is obvious that

$$r_h\tau|_T = \frac{1}{\text{mes } T} \int_T \tau \, d\mathbf{x} \tag{2.2}$$

and it is well-known that

$$\|r_h\tau - \tau\|_0 \to 0 \quad \text{for} \quad h \to 0. \tag{2.3}$$

Then for any $\mathbf{v}_h \in K_h$ we obtain $\mathbf{e}(\mathbf{v}_h) \in S_h$ and therefore

$$\langle \mathbf{e}(\mathbf{v}_h), r_h\tau \rangle = \langle \mathbf{e}(\mathbf{v}_h), \tau \rangle \geqq L(\mathbf{v}_h),$$

making use of the fact, that $\mathbf{v}_h \in K_h \subset K$ and $\tau \in K^+$. As a consequence, $r_h\tau \in K_h^+$.

Since $\tau(\mathbf{x}) \in B$ a.e. and B is closed and convex in $\mathbb{R}_\sigma$, from (2.2) it follows that

$$r_h\tau\big|_T \in B \quad \forall T \in \mathscr{T}_h$$

and therefore $r_h\tau \in P$. Q.E.D.

Theorem 2.1. *Assume that $K^+ \cap P \neq \emptyset$ and there exists only a finite number of points $\Gamma_K \cap \bar{\Gamma}_\tau$, $\bar{\Gamma}_\tau \cap \bar{\Gamma}_0$, $\bar{\Gamma}_\tau \cap \bar{\Gamma}_u$. Then the approximate problem (2.1) has a unique solution σ_h and*

$$\|\sigma_h - \sigma\|_0 \to 0\,, \quad h \to 0\,, \tag{2.4}$$

provided that the family of triangulations is regular.

Proof. The set $K_h^+ \cap P$ is closed and convex. By virtue of Lemma 2.1 it is also non-empty. Hence the existence and uniqueness of σ_h follows.

To prove the convergence (2.4), we employ an abstract theorem (see [6] – chpt. 4 or [1] – II – Theorem 1.1) on the convergence of Ritz-Galerkin approximations. Thus it suffices to verify the following two conditions

(i) $\exists\{\tau_h\}$, $\tau_h \in K_h^+ \cap P$, $\tau_h \to \sigma$ in S for $h \to 0$;

(ii) $\tau_h \in K_h^+ \cap P$, $\tau_h \rightharpoonup \tau$ (weakly) in S implies that $\tau \in K^+ \cap P$.

The first condition is satisfied with $\tau_h = r_h\sigma$, by virtue of Lemma 2.1 and (2.3).

To prove (ii), we consider an arbitrary element $\mathbf{v} \in K$. There exists a sequence $\{\mathbf{v}_h\}$, $\mathbf{v}_h \in K_h$ such that

$$\mathbf{v}_h \to \mathbf{v} \quad \text{in} \quad [H^1(\Omega')]^2 \times [H^1(\Omega'')]^2\,, \quad h \to 0\,. \tag{2.5}$$

In fact, we utilize first the fact that the set

$$K^\infty \equiv K \cap [C^\infty(\bar{\Omega}')]^2 \times [C^\infty(\bar{\Omega}'')]^2$$

is dense in K. The density follows from Lemma 3.1 of [1] – II., where the cases $\Gamma_K \cap \bar{\Gamma}_u \neq \emptyset$ and $\Gamma_K \cap \bar{\Gamma}_0 \neq \emptyset$ have been excluded. The proof, however, can be completed by consideration of the cases mentioned above, as follows.

Appendix to the proof of Lemma 3.1 of [1] – II. Let the domain B contain a point $\Gamma_K \cap \bar{\Gamma}_u \equiv P$, which is at the same time a vertex of $\partial\Omega'$. Using the skew coordinate basis (see Fig. 1), we have

$$u^{M(p)} = \mathbf{u}^M \,.\, \mathbf{n}^p\,, \quad M = {}'\,,{}''\,, \quad p = 1, 2\,,$$

$$u'_n + u''_n = u'^{(p)} - u''^{(p)} \quad \text{on} \quad \Gamma^{(p)}\,.$$

As a consequence,

$$\Gamma^{(1)} \subset \Gamma_K \Rightarrow u'^{(1)} - u''^{(1)} \leqq 0 \quad \text{on} \quad \Gamma^{(1)}\,,$$

$$\Gamma^{(2)} \subset \Gamma_u \Rightarrow u'^{(1)} = u'^{(2)} = 0 \quad \text{on} \quad \Gamma^{(2)}\,.$$

Let $Eu'^{(1)}$ be an extension of $u'^{(1)}$ into $B - \Omega'$ such that $Eu'^{(1)} \in H^1(B)$, supp $Eu'^{(1)} \subset \subset B$, $Eu'^{(1)} \equiv 0$ in an angular domain between Γ_τ and Γ_u (outside $\Omega' \cup \Omega''$).

Let $Eu''^{(1)} \in H^1(B)$ be an extension of $u''^{(1)}$ into $B - \Omega''$ such that supp $Eu''^{(1)} \subset B$.

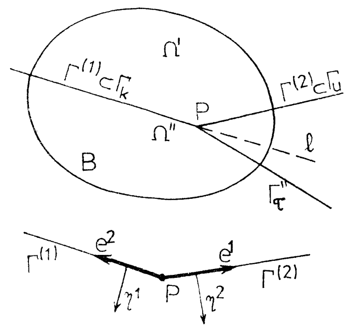

Fig. 1

Denote l the straight line containing $\Gamma^{(1)}$ and

$$\mathscr{U} = (u'^{(1)} - u''^{(1)}) .$$

Let $E_1\mathscr{U}$ be an extension of $\mathscr{U}$ onto the whole line l, such that $E_1\mathscr{U}$ is symmetric with respect to the point P. Consequently, we have

$$E_1\mathscr{U} \leqq 0 \text{ on } l , \quad \operatorname{supp} E_1\mathscr{U} \subset B \cap l .$$

There exists a function $v \in H^1(B)$, supp $v \subset B$ such that

$$v = E_1\mathscr{U} \text{ on } l , \quad v \leqq 0 \text{ in } B .$$

Then we may write

$$Eu'^{(1)} - Eu''^{(1)} = v + z \text{ on } B , \tag{2.6}$$

where $z \in H^1(B)$, supp $z \subset B$, $z = 0$ on $\Gamma^{(1)}$.

There exists a function $w \in H^1(B)$ such that

$$w = z \text{ on } \Gamma^{(1)} \cup \Gamma_\tau , \quad w = 0$$

in an angular neighbourhood of $\Gamma^{(1)}$.

Let us define a shifted function

$$w_\lambda(\mathbf{x}) = w(\mathbf{x} + \lambda \mathbf{e}^2) , \quad \lambda \in R , \quad \lambda > 0 .$$

A positive C exists such that for a regularizing operator $R_\varkappa$ we have

$$R_\varkappa w_\lambda = 0 \text{ on } \Gamma^{(1)} , \quad \text{if} \quad \varkappa < C\lambda ,$$

$$R_\varkappa w_\lambda \to w , \quad \lambda \to 0 , \quad \varkappa < C\lambda , \quad \text{in} \quad H^1(B) . \tag{2.7}$$

We may write
$$z = w + z_0\,, \quad z_0'' \equiv z_0|_{B\cap\Omega''} \in H_0^1(B \cap \Omega'')$$
and find

(2.8) $$z_{0\varkappa}'' \in C_0^\infty(B \cap \Omega'')\,, \quad z_{0\varkappa}'' \to z_0''\,, \quad \varkappa \to 0\,, \quad \text{in} \quad H^1(B \cap \Omega'')\,.$$

Let us introduce
$$(Eu'^{(1)})_\mu = Eu'^{(1)}(\mathbf{x} + \mu \mathbf{e})\,,$$
where $\mu \in R$, $\mu > 0$ and $\mathbf{e}$ is a unit vector in the direction of the axis between $\Gamma^{(2)}$ and Γ_τ. Then
$$R_\varkappa(Eu'^{(1)})_\mu = 0 \quad \text{on} \quad \Gamma^{(2)} \quad \text{for} \quad \varkappa < C_1\mu\,.$$
If we define
$$u_\varkappa'^{(1)} = R_\varkappa(Eu'^{(1)})_\mu|_{\Omega'}\,,$$
$$u_\varkappa''^{(1)} = [R_\varkappa(Eu'^{(1)})_\mu - (R_\varkappa v + R_\varkappa w_\lambda + z_{0\varkappa})]_{\Omega''}\,,$$
then it holds
$$u_\varkappa'^{(1)} - u_\varkappa''^{(1)} = [R_\varkappa v + R_\varkappa w_\lambda + z_{0\varkappa}]_{\Gamma^{(1)}} \leqq 0 \quad \text{on} \quad \Gamma^{(1)} \quad \text{for} \quad \varkappa < C\lambda\,,$$
$$u_\varkappa'^{(1)} = 0 \quad \text{on} \quad \Gamma^{(2)} \quad \text{for} \quad \varkappa < C_1\mu\,.$$
Moreover,
$$\|u_\varkappa'^{(1)} - u'^{(1)}\|_{1,\Omega'} \leqq \|R_\varkappa(Eu'^{(1)})_\mu - Eu'^{(1)}\|_{1,B} \to 0\,, \quad \mu \to 0\,, \quad \varkappa < C\mu\,,$$
$$\|u_\varkappa''^{(1)} - u''^{(1)}\|_{1,\Omega''} \leqq \|R_\varkappa(Eu'^{(1)})_\mu - Eu'^{(1)}\|_{1,B\cap\Omega''} +$$
$$+ \|Eu'^{(1)} - Eu''^{(1)} - (R_\varkappa v + R_\varkappa w_\lambda + z_{0\varkappa})\|_{1,B\cap\Omega''}\,,$$
where the last two terms tend to zero for $\mu \to 0$, $\lambda \to 0$, $\varkappa < \min(C\lambda, C_1\mu)$, as a consequence of (2.6), (2.7) and (2.8).

The component $u'^{(2)}$ will be approximated like z, changing only $\Gamma^{(1)}$ with $\Gamma^{(2)}$. The component $u''^{(2)}$ can be extended arbitrarily and regularized.

Finally, let us consider the case $\Gamma_K \cap \bar{\Gamma}_0 \neq \emptyset$. For instance, let the situation be as in Fig. 1, where Γ_u is replaced by Γ_0 and Ω' by Ω''. Then the condition $u_n'' = u''^{(2)} = 0$ holds on $\Gamma_0 \supset \Gamma^{(2)}$.

The components $u^{M(1)}$ can be approximated like in the 2. group of the proof of Lemma 3.1, the components $u''^{(2)}$ like the function z there. The components $u'^{(2)}$ can be regularized arbitrarily.

To prove (2.5), it suffices to consider
$$\mathbf{v}_\varkappa \in K^\infty\,, \quad \|\mathbf{v}_\varkappa - \mathbf{v}\|_1 < \varepsilon/2$$
and Lagrange interpolations $\mathbf{v}_{\varkappa I} \in K_h$ of $\mathbf{v}_\varkappa$. Since it holds
$$\|\mathbf{v}_{\varkappa I} - \mathbf{v}_\varkappa\|_1 \leqq C\,h|\mathbf{v}_\varkappa|_2\,,$$
(2.5) follows with $\mathbf{v}_h = \mathbf{v}_{\varkappa I}$.

Then $\mathbf{e}(\mathbf{v}_h) \to \mathbf{e}(\mathbf{v})$ in S and

$$\langle \tau_h, \mathbf{e}(\mathbf{v}_h) \rangle \geqq L(\mathbf{v}_h) .$$

By passing to the limit in h, we obtain $\tau \in K^+$. Since P is weakly closed, $\tau \in P$.

2.2 Approximations to the problem with enlarging contact zone

To approximate the problem $(\mathscr{P}_2)$, we employ again piecewise constant external approximations of the set of statically admissible stress fields.

Let us consider triangulations $\mathscr{T}_h^M$ of Ω^M, $M = {}', {}''$, such that the triangles adjacent to the boundaries may have curved sides along the boundary and the vertices on Γ_K^M lie on straight-lines parallel with the ξ-axis. If Γ_K^M contains a point of inflexion, than there is always a vertex.

If a curved triangle $T \in \mathscr{T}_h$ adjacent to Γ_K^M is convex, it will be divided by the chord into a "straight" triangle T_0 and a segment T_s, so that $T = T_0 \cup T_s$. If $T_c \in \mathscr{T}_h$ adjacent to Γ_K^M is non-convex, then one of its sides is parallel with the ξ-axis.

We define

$$V_h = \Big\{ \mathbf{v} \in V \big| \mathbf{v}|_{T_0} \in [P_1(T_0)]^2 \quad \forall T_0 \subset T \in \mathscr{T}_h \text{ adjacent to } \Gamma_K^M ,$$

$$\left(\frac{\partial}{\partial \xi} \mathbf{v} \right)_{T_s} = 0 \quad \forall T_s \subset T \in \mathscr{T}_h \text{ adjacent to } \Gamma_K^M ,$$

$\left(\frac{\partial}{\partial \xi} \mathbf{v} \right)_{T_c} = 0$ for all non-convex triangles T_c adjacent to Γ_K^M, $\mathbf{v}|_T \in [P_1(T)]^2$ for all remaining triangles$\Big\}$.

In other words, V_h consists of piecewise linear vector-functions, which are extended onto the segments and onto the non-convex triangles adjacent to Γ_K^M continuously by constants in ξ-direction.

Moreover, let us introduce

$$S_h = \{ \tau \in S \mid \tau_{ij} \in P_0(T^*) \ \forall T^* = T, T_0, T_s, T_c \in \mathscr{T}_h, \ i, j = 1, 2 \}$$

$$K_{0h} = \{ \mathbf{v} \in V_h \mid v''_\xi - v'_\xi \leqq 0 \ \forall \eta \in \langle a, b \rangle \text{ on } \Gamma'_K \cup \Gamma''_K \} ,$$

$$K_{0h}^+ = \{ \tau \in S_h \mid \langle \tau, \mathbf{e}(\mathbf{v}_h) \rangle \geqq L(\mathbf{v}_h) \ \forall \mathbf{v}_h \in K_{0h} \} .$$

It is easy to see that if the condition $v''_\xi - v'_\xi \leqq 0$ holds at the nodes of Γ_K^M, it holds on the whole interval $\langle a, b \rangle$ by virtue of the definition of V_h.

Instead of $(\mathscr{P}_2)$ we define the approximate problem

$$\mathscr{S}(\tau_h) = \min \quad \text{over} \quad K_{0h}^+ \cap P . \tag{2.9}$$

We again introduce a projection mapping $r_h : S \to S_h$ as follows

$$\langle \tau - r_h \tau, \chi_h > 0 \quad \forall \chi_h \in S_h .$$

Lemma 2.2. *If* $\tau \in K_0^+ \cap P$, *then* $r_h\tau \in K_{0h}^+ \cap P$.

Proof. It holds (2.2) for any $T^* \in \mathscr{T}_h$ and (2.3). Let $\mathbf{v}_h \in K_{0h}$. Then $\mathbf{e}(\mathbf{v}_h) \in S_h$ and we may write

$$\langle r_h\tau, \mathbf{e}(\mathbf{v}_h)\rangle = \langle \tau, \mathbf{e}(\mathbf{v}_h)\rangle \geqq L(\mathbf{v}_h),$$

since $K_{0h} \subset K_0$. Thus $r_h\tau \in K_{0h}^+$ follows.

Since $\tau(\mathbf{x}) \in B$ a.e. and B is convex in R_σ, the mean values of τ in T, T_0, T_s, T_c belong to B, as well. Therefore $r_h\tau \in P$.

Theorem 2.2. *Assume that* $f^M \in C^2$, $M = {}', {}''$, *in a neighbourhood of the interval* $\langle a, b\rangle$, $\Gamma'_K \cap \bar{\Gamma}_u = \emptyset$, $\Gamma''_K \cap \Gamma_0 = \emptyset$, *there exists only a finite number of points* $\bar{\Gamma}_\tau \cap \bar{\Gamma}_u$, $\bar{\Gamma}_\tau \cap \bar{\Gamma}_0$ *and* $K_0^+ \cap P \neq \emptyset$.

Then the approximate problem (2.9) *has a unique solution* σ_h *and*

$$\|\sigma_h - \sigma\|_0 \to 0\,, \quad h \to 0 \tag{2.10}$$

holds for any regular family of triangulations.

Proof. The set $K_{0h}^+ \cap P$ is convex and closed in S_h. Lemma 2.2 implies that it is also non-empty. Hence the existence and uniqueness of σ_h follows.

To prove the convergence (2.10), we employ the abstract theorem like in the proof of Theorem 2.1. The condition (i) follows from Lemma 2.2 and (2.3) with $\tau_h = r_h\sigma$.

Thus it remains to verify the condition (ii) (where K_h^+ and K^+ is replaced by K_{0h}^+ and K_0^+, respectively). We shall need two auxiliary lemmas.

Lemma 2.3. *Let the assumptions of Theorem* 2.2 *on* Γ_K^M, Γ_u, Γ_0, Γ_τ *be satisfied, except that* $f^M \in C^m(a - \delta, b + \delta)$, $\delta > 0$, $m \geqq 1$.

Then the set

$$\mathscr{K} \equiv K_0 \cap [C^m(\bar{\Omega}')]^2 \times [C^m(\bar{\Omega}'')]^2$$

is dense in K_0.

Proof. Let $\{B_i\}_{i=0}^r$ be a system of open domains, which covers $\bar{\Omega}' \cup \bar{\Omega}''$ and such that

$$\bar{B}_0 \subset \Omega'\,, \quad \bar{B}_1 \subset \Omega''\,, \quad \Gamma'_K \cap \Gamma''_K \subset \bigcup_{j=2}^{k} B_j\,,$$

$$\Gamma'_K \cup \Gamma''_K \cap B_i \neq \emptyset \Leftrightarrow 2 \leqq i \leqq k\,.$$

Let the union of arcs $\widehat{PQ'} \cup \widehat{PQ''}$ (see Fig. 2) be contained in one and only one domain B_j. Assume that the other domains contain at most one angular point of the boundaries or a point $\bar{\Gamma}_u \cap \bar{\Gamma}_\tau$, $\bar{\Gamma}_0 \cap \bar{\Gamma}_\tau$.

Let us consider the corresponding decomposition of unity and construct a smooth approximation to every function $\boldsymbol{u}^j = \boldsymbol{u}\varphi_j$, where $\boldsymbol{u} \in K_0$, $\varphi_j \in C_0^\infty(B_j)$. We can use the same approach as in the proof of Lemma 3.1 of [1] – II, except the configurations of the type depicted on Fig. 2. In such case we use the following technique.

By assumption, we have $\varphi_j = 1$ on $\Gamma'_K \cup \Gamma''_K \cap B_j$. Henceforth we omit the sub-

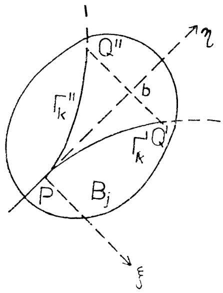

Fig. 2

scripts and superscripts j. Let us map $\Omega' \cap B$ into the halfplane $\hat{\xi} > 0$ and $\Omega'' \cap B$ into the halfplane $\hat{\xi} < 0$ by means of the following two mappings:

$$\hat{\mathbf{x}} = T^M \mathbf{x} = \{\hat{\xi}^M = \xi - f^M(\eta),\ \hat{\eta}^M = \eta\}\,, \quad M = {}',{}''\,,$$

where $\hat{\mathbf{x}} = (\hat{\xi}, \hat{\eta})$, $\mathbf{x} = (\xi, \eta)$.

Denote

$$\hat{B} = T'(\bar{\Omega}' \cap B) \cup T''(\bar{\Omega}'' \cap B)\,,$$

$$\hat{\mathbf{u}}^M(\hat{\mathbf{x}}) = \mathbf{u}^M((T^M)^{-1}\, \hat{\mathbf{x}})$$

and define

$$\hat{u}^M_{\xi\lambda}(\hat{\xi}, \hat{\eta}) = \hat{u}^M_{\xi}(\hat{\xi}, \hat{\eta} - \lambda)\,, \quad \lambda > 0\,,$$

$$\hat{u}^M_{\eta\lambda} = \hat{u}^M_{\eta}\,,$$

$$U_\lambda = (\hat{u}''_{\xi\lambda} - \hat{u}'_{\xi\lambda})|_{\hat{\xi}=0}\,.$$

It is easy to see that

$$U_\lambda \leqq 0\,, \quad \hat{\eta} < b + \lambda\,.$$

Let us define the extension $E\hat{u}^M_{\xi\lambda}$ across the axis $\hat{\xi} = 0$ as an even function of $\hat{\xi}$ and

$$\varepsilon_\lambda = (E\hat{u}''_{\xi\lambda} - E\hat{u}'_{\xi\lambda})^+|_{\hat{\xi}=0}\,.$$

Then it holds

$$U_\lambda - \varepsilon_\lambda \leqq 0\,, \quad \operatorname{supp}(U_\lambda - \varepsilon_\lambda) \subset \hat{B} \cap \{\hat{\xi} = 0\}$$

for sufficiently small λ. Therefore a function $\hat{v} \in H^1(\hat{B})$ exists such that $\hat{v} \leqq 0$ in $\hat{B}$, $\operatorname{supp} \hat{v} \subset \hat{B}$ and

$$\hat{v}|_{\hat{\xi}=0} = U_\lambda - \varepsilon_\lambda\,.$$

Since $\varepsilon_\lambda = 0$ for $\hat{\eta} < b + \lambda$ and $\varepsilon_\lambda \in H^{1/2}$ on the $\hat{\xi}$-axis, there exists a function $\hat{w} \in H^1(\hat{B})$ such that:

$$\hat{w}|_{\hat{\xi}=0} = \varepsilon_\lambda\,, \quad \operatorname{supp} \hat{w} \subset \hat{B}\,,$$

$\hat{w} = 0$ in a neighbourhood of the straight-line segment $\{\eta \leqq b + \frac{1}{2}\lambda,\ \hat{\xi} = 0\}$. It is readily seen that

$$E\hat{u}''_{\xi\lambda} - E\hat{u}'_{\xi\lambda} - \hat{w} = \hat{v} + \hat{z}\,,$$

where

$$\hat{z}|_{\xi=0} = U_\lambda - \varepsilon_\lambda - (U_\lambda - \varepsilon_\lambda) = 0\,, \quad \hat{z} \in H^1(\hat{B})\,.$$

Regularizing $\hat{v}$ and $\hat{z}$, we obtain for $\varkappa \to 0$ $R_\varkappa \hat{v} + \hat{z}_\varkappa \to \hat{v} + \hat{z}$ in $H^1(\hat{B})$, where $\hat{z}_\varkappa|_{\xi=0} = 0, (R_\varkappa \hat{v} + \hat{z}_\varkappa)|_{\xi=0} \leqq 0.$

We set

$$\hat{u}''_{\xi\lambda\varkappa} = R_\varkappa E\hat{u}''_{\xi\lambda}|_{T''\Omega''}\,,$$

$$\hat{u}'_{\xi\lambda\varkappa} = [R_\varkappa E\hat{u}''_{\xi\lambda} - R_\varkappa\hat{v} - \hat{z}_\varkappa - R_\varkappa\hat{w}]_{T'\Omega'}\,.$$

On the axis $\hat{\xi} = 0$ it holds

$$\hat{u}''_{\xi\lambda\varkappa} - \hat{u}'_{\xi\lambda\varkappa} = R_\varkappa\hat{v} + \hat{z}_\varkappa + R_\varkappa\hat{w} \leqq 0\,, \quad \hat{\eta} < b\,, \quad \varkappa < \lambda/2\,, \tag{2.11}$$

since $R_\varkappa\hat{w} = 0$ for $\hat{\eta} < b$, $\varkappa < \frac{1}{2}\lambda$.

Furthermore, we have for $\varkappa \to 0$, $\lambda \to 0$, $\varkappa < \frac{1}{2}\lambda$:

$$\hat{u}^M_{\xi\lambda\varkappa} \to \hat{u}^M_\xi \text{ in } H^1(\hat{B} \cap T^M\Omega^M)\,, \quad M = ', ''\,.$$

Finally, we set (after a suitable extension)

$$\hat{u}^M_{\eta\lambda\varkappa} = R_\varkappa \hat{u}^M_\eta$$

and define

$$u^M_{\xi\lambda\varkappa} = \hat{u}^M_{\xi\lambda\varkappa} \circ T^m\,, \quad u^M_{\eta\lambda\varkappa} = \hat{u}^M_{\eta\lambda\varkappa} \circ T^M\,.$$

Since both T^M and $(T^M)^{-1}$ are Lipschitz mappings, it holds

$$\|u^M_{\xi\lambda\varkappa} - u^M_\xi\|_{1,\Omega^M} \leqq C\|\hat{u}^M_{\xi\lambda\varkappa} - \hat{u}^M_\xi\|_{1,\hat{B}\cap T^M\Omega^M} \to 0$$

for $\varkappa < \frac{1}{2}\lambda$, $\lambda \to 0$ and a parallel assertion is true for $u^M_{\eta\lambda\varkappa}$.

As a consequence of (2.11) $u''_{\xi\lambda\varkappa} - u'_{\xi\lambda\varkappa} \leqq 0$ for $\eta < b$ and therefore $\boldsymbol{u}_{\lambda\varkappa} \in K_0$. Since $f^M \in C^m$, $\boldsymbol{u}^M_{\lambda\varkappa} \in [C^m(\bar{\Omega}^M)]^2$.

Lemma 2.4. *Let* $\mathbf{v} \in [H^2(\Omega')]^2 \times [H^2(\Omega'')]^2$, $f^M \in C^1(\langle a, b\rangle)$. *If we define a Lagrange linear interpolate* $\mathbf{v}_I \in V_h$ *by an obvious way, then*

$$\|\mathbf{v}_I - \mathbf{v}\|_{1,\Omega'\cup\Omega''} \to 0\,, \quad h \to 0$$

holds for any regular family of triangulations.

Proof. Let V_h^0 be the space of piecewise linear functions, continuous in $\bar{\Omega}^M$ ($M = ', ''$), over the triangulation $\mathscr{T}_h$, where each convex curved triangle $T = T_0 \cup T_s$ remains undivided and the functions from V_h^0 are linear in the whole T. Let $v_h^0 \in V_h^0$ denote the Lagrange interpolate of v.

First we prove that

$$\|v_h^0 - v\|_{1,\Omega^M} \leqq C\, h \|v\|_{2,\Omega^M}\,, \quad M = {}', {}'' . \tag{2.12}$$

In fact, there exists an extension $Ev \in H^2(\mathbb{R}^2)$ such that

$$\|Ev\|_{2,R^2} \leqq C\|v\|_{2,\Omega^M} .$$

For the curved triangles, adjacent to Γ_K^M, we define (see Fig. 3):

$$\tilde{T} = \Delta \tilde{a}_i a_j \tilde{a}_k \quad (\text{i.e. twice enlarged } T_0)$$

if T is convex,

$$\tilde{T} = \Delta a_i a_j a_k \quad \text{if } T \text{ is non-convex} .$$

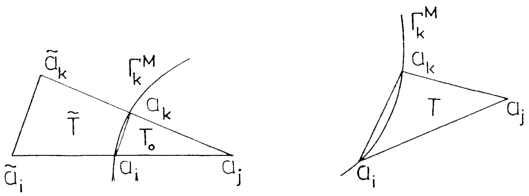

Fig. 3

Let π_2 denote the linear interpolation on $\tilde{T}$ with the nodes a_i, a_j, a_k. Making use of the affine equivalence and the regularity of $\{\mathcal{T}_h\}$, we derive the estimate

$$\|\pi_2 Ev - Ev\|_{1,\tilde{T}} \leqq C\, h |Ev|_{2,\tilde{T}}\,,$$

where C is independent of h and Ev.

Since $\pi_2 Ev = v_h^0$ holds on T, we may write

$$\|v - v_h^0\|_{1,\Omega^M}^2 = \sum_{T \in \mathcal{T}_h{}^M} \|v - v_h^0\|_{1,T}^2 \leqq \sum_{T \in \mathcal{T}_h{}^M} \|Ev - \pi_2 Ev\|_{1,\tilde{T}}^2 \leqq$$

$$\leqq Ch^2 \sum_{T \in \mathcal{T}_h{}^M} \|Ev\|_{2,\tilde{T}}^2 \leqq 2C\, h^2 \|Ev\|_{2,R^2}^2 \leqq C_1\, h^2 \|v\|_{2,\Omega^M}^2 .$$

Second, we shall prove that

$$\|v_I - v_h^0\|_{1,\Omega^M} \to 0\,, \quad h \to 0\,, \quad M = {}', {}'' . \tag{2.13}$$

In fact, it is readily seen that

$$\operatorname{supp}(v_I - v_h^0) \subset D_h = \bigcup T_s \cup T_c\,,$$

where D_h is the union of all segments and curved non-convex triangles adjacent to Γ_K^M.

In each T_s or T_c we have:

$$v_h^0(\xi, \eta) - v_I(\xi, \eta) = v_h^0(\xi, \eta) - v_I(\xi(s), \eta) = (\xi - \xi(s)) \frac{\partial v_h^0}{\partial \xi}\,,$$

(where the point $(\xi(s), \eta)$ lies on the chord or at the straight side of $\partial D_h \dot{-} \Gamma_K^M$),

$$|\xi - \xi(s)| \leqq h\,,$$

$$\int_{T_s} (v_h^0 - v_I)^2 \,\mathrm{d}\xi\,\mathrm{d}\eta \leqq h^2 \int_{T_s} \left(\frac{\partial v_h^0}{\partial \xi}\right)^2 \mathrm{d}\xi\,\mathrm{d}\eta$$

and the same estimate for integrals over T_c.

Furthermore, it holds

$$\frac{\partial}{\partial \xi}(v_h^0 - v_I) = \frac{\partial v_h^0}{\partial \xi}\,, \quad \frac{\partial}{\partial \eta}(v_h^0 - v_I) = -\frac{\partial v_h^0}{\partial \xi}\frac{\partial \xi(s)}{\partial \eta}\,, \quad \left|\frac{\partial \xi(s)}{\partial \eta}\right| = |\mathrm{tg}\,\alpha|\,,$$

where α is the angle between the η-axis and the chord or the straight side of $\partial D_h \dot{-} \Gamma_K^M$, respectively. Since $\Gamma_K^M \in C^1$ and the family $\{\mathscr{T}_h\}$ is regular,

$$|\mathrm{tg}\,\alpha| \leqq \max\left(|f^M|_{C^1(\langle a,b\rangle)}\,, (\sin \vartheta)^{-1}\right).$$

For sufficiently small h we obtain

$$\|v_h^0 - v_I\|_{1,T_s}^2 \leqq (C + h^2) \int_{T_s} \left(\frac{\partial v_h^0}{\partial \xi}\right)^2 \mathrm{d}\xi\,\mathrm{d}\eta \leqq C_1 \|v_h^0\|_{1,T_s}^2$$

and a parallel estimate for T_c.

Thus we have

$$\|v_h^0 - v_I\|_{1,\Omega^M}^2 \leqq \sum_{T_s \in D_h} \|v_h^0 - v_I\|_{1,T_s}^2 + \sum_{T_c \in D_h} \|v_h^0 - v_I\|_{1,T_c}^2 \leqq C \|v_h^0\|_{1,D_h}^2\,.$$

On the other hand,

$$\|v_h^0\|_{1,D_h} \leqq \|v\|_{1,D_h} + \|v_h^0 - v\|_{1,D_h} \to 0\,, \quad h \to 0$$

holds by virtue of (2.12) and the fact that mes $D_h \to 0$.

Finally, from (2.12) and (2.13) the assertion of Lemma 2.4 follows.

Proof of Theorem 2.2 – continuation. Making use of Lemma 2.3, to any $\mathbf{v} \in K_0$ we can find $\mathbf{v}_\varkappa \in \mathscr{K}_2$ such that

$$\|\mathbf{v}_\varkappa - \mathbf{v}\|_{1,\Omega' \cup \Omega''} < \varkappa\,.$$

From Lemma 2.4 it follows that

$$\|\mathbf{v}_{\varkappa I} - \mathbf{v}_\varkappa\|_{1,\Omega' \cup \Omega''} \to 0\,, \quad h \to 0\,.$$

Moreover, the interpolate $\mathbf{v}_{\varkappa I} \in K_{0h}$. Altogether, we have

$$\|\mathbf{v}_{\varkappa I} - \mathbf{v}\|_{1,\Omega' \cup \Omega''} \to 0\,, \quad h \to 0\,, \quad \varkappa \to 0\,.$$

If $\tau_h \in K_{0h}^+$, then

$$\langle \tau_h, \mathbf{e}(\mathbf{v}_{\varkappa I}) \rangle \geqq L(\mathbf{v}_{\varkappa I})\,.$$

Since $\tau_h \rightharpoonup \tau$ weakly in S and $\mathbf{e}(\mathbf{v}_{\varkappa I}) \to \mathbf{e}(\mathbf{v})$ in S, passing to the limit in h, we obtain

$$\langle \tau, \mathbf{e}(\mathbf{v}) \rangle \geqq L(\mathbf{v}),$$

i.e., $\tau \in K_0^+$.

Since P is weakly closed in S, $\tau \in P$, as well. Q.E.D.

3. ON THE SOLUTION OF THE APPROXIMATE PROBLEMS

In the approximate problem (2.1), the set K_h^+ seems to cause difficulties, at a first glance. We can simplify the situation, however, by eliminating the auxiliary test functions $\mathbf{v}_h$, as follows.

Let us denote

$$\mathbf{v}_h(\mathbf{x}) = \sum_{i=1}^{N} q_i \, \varphi_i(\mathbf{x}),$$

where q_i are the values of displacement components at the nodes of the triangulation $\mathscr{T}_h$. If we write down the conditions (1.1) at the nodes of Γ_K, then precisely 4 components $\{q_{k_1}, q_{k_2}, q_{k_3}, q_{k_4}\}$ occur at each (double) node $A_k \in \bar{\Gamma}_K$.

In fact, assume for simplicity, that Γ_K is a single straight-line segment with $n_2'' \neq 0$. Then the condition (1.1) gives

$$\sum_{j=1}^{4} b_j q_{k_j} \leqq 0,$$

where $b_j = n_j'$ for $j = 1, 2$ and $b_j = n_{j-2}''$ for $j = 3, 4$. Introducing a linear transformation $\mathbf{q} = F_k \mathbf{y}$, $F_k : R^4 \to R^4$ by means of the relations

$$y_{k_j} = q_{k_j}, \quad j = 1, 2, 3,$$

$$y_{k_4} = \sum_{j=1}^{4} b_j q_{k_j},$$

we find out that F_k is regular. Let us consider the same transformation in each quadruplet $M_k = \{q_{k_1}, q_{k_2}, q_{k_3}, q_{k_4}\}$, $k = 1, \ldots, Q$, corresponding to each node $A_k \in \bar{\Gamma}_K$. Setting also $y_p = q_p$ for $q_p \notin \bigcup_{k=1}^{Q} M_k$, $1 \leqq p \leqq N$, altogether we have $\mathbf{q} = F\mathbf{y}$, $F : R^N \to R^N$ and

$$\mathbf{v}_h \in K_h \Leftrightarrow \mathbf{q} \in \mathscr{K}_q \Leftrightarrow \mathbf{y} \in \mathscr{K}_y = \{\mathbf{y} \in R^N \mid y_{k_4} \leqq 0, \ k = 1, \ldots, Q\}. \tag{3.1}$$

Let ψ_T denote the characteristic function of the triangle $T \in \mathscr{T}_h$. Then we have

$$\tau_h \in S_h \Leftrightarrow \tau_h(\mathbf{x}) = \sum_{T \in \mathscr{T}_h} \tau(T) \, \psi_T(\mathbf{x}) \tag{3.2}$$

and denoting

$$\mathbf{t}^{\mathsf{T}} = (\tau_{11}(T_1), \tau_{22}(T_1), \tau_{12}(T_1), \tau_{11}(T_2), \tau_{22}(T_2), \ldots) \tag{3.3}$$

the corresponding vector in R^M, we obtain

$$\langle \tau, \mathbf{e}(\mathbf{v}_h) \rangle = \sum_{T \in \mathscr{T}_h} \int_T \tau(T) \sum_{i=1}^{N} q_i \, \mathbf{e}(\varphi_i) \, \mathrm{d}\mathbf{x} = (E\mathbf{t}, \mathbf{q}) ,$$

where E is a $N \times M$ matrix, $(E\mathbf{t}, \mathbf{q}) = \mathbf{q}^{\mathsf{T}} E\mathbf{t}$. (Note that $N < M$).

Since

$$L(\mathbf{v}_h) = \sum_{i=1}^{N} q_i \, L(\varphi_i) = (\mathbf{l}, \mathbf{q}) \equiv \mathbf{q}^{\mathsf{T}} \mathbf{l} ,$$

where $\mathbf{l} \in R^N$ is a fixed vector, the condition $\tau \in K_h^+$ can finally be rewritten in the form

$$(\mathbf{l} - E\mathbf{t}, \mathbf{q}) \leqq 0 \quad \forall \mathbf{q} \in \mathscr{K}_q .$$

This means that the vectors $\mathbf{l} - E\mathbf{t}$ belong to the polar cone $\mathscr{K}_q^0$ of the cone $\mathscr{K}_q$. Employing the mapping F, we obtain an equivalent condition

(3.4) $$(\mathbf{l} - E\mathbf{t}, F\mathbf{y}) \leqq 0 \quad \forall \mathbf{y} \in \mathscr{K}_y .$$

Let I^- be the set of all indices k_4, $k = 1, 2, \ldots, Q$ and $I_0 = \{1, \ldots, N\} \dot{-} I^-$ the set of remaining indices. Since the cone $\mathscr{K}_y$ is generated by the vectors

$$\{\pm \mathbf{e}_j, \ j \in I_0, \ -\mathbf{e}_m, \ m \in I^-\}$$

where $\mathbf{e}_j$ and $\mathbf{e}_m$ form an orthonormal basis in R^N, (3.4) is equivalent to the following system

(3.5) $$g_j(\mathbf{t}) \equiv (\mathbf{l} - E\mathbf{t}, F\mathbf{e}_j) = 0 , \qquad j \in I_0 ,$$

(3.6) $$g_m(\mathbf{t}) \equiv (\mathbf{l} - E\mathbf{t}, -F\mathbf{e}_m) \leqq 0 , \quad m \in I^- .$$

Moreover, $\tau_h \in P$ if and only if

$$f(\tau_h(T)) \leqq 1 \quad \forall T \in \mathscr{T}_h ,$$

which may be written in the form

(3.7) $$f_T(\mathbf{t}) - 1 \leqq 0 \quad \forall T \in \mathscr{T}_h .$$

Finally, inserting (3.2) and (3.3) into the functional $\mathscr{S}(\tau_h)$, we are led to the following problem of nonlinear programming: $\mathscr{S}_0(\mathbf{t}) = \min$ over the set of $\mathbf{t} \in R^M$, satisfying (3.5), (3.6) and (3.7).

Remark 3.1. If Γ_K has a vertex, we define $y_{k_j} = q_{k_j}$, $j = 1, 2$,

$$y_{k_3} = \sum_{j=1}^{4} b_j^{(1)} q_{k_j} , \quad y_{k_4} = \sum_{j=1}^{4} b_j^{(2)} q_{k_j} ,$$

where $b_j^{(1)}$ and $b_j^{(2)}$ correspond to the normals $\mathbf{n}'$, $\mathbf{n}''$ on both sides of the vertex.

Remark 3.2. A similar approach can be applied to the approximate problem (2.9).

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Souhrn

KONTAKT MEZI PRUŽNĚ — DOKONALE PLASTICKÝMI TĚLESY

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Jednostranný kontakt dvou těles z materiálu, který se řídí zákonem Henckyho, je studován na základě formulace v napětích. Nejprve je rozšířen známý Haarův-Kármánův princip na úlohy s jednostranným kontaktem. Jsou navrženy aproximace metodou konečných prvků s funkcemi po částech konstantními na triangulacích. Pro každý regulární systém triangulací je dokázáno, že metoda konverguje.

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