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MIXED FORMULATION OF ELLIPTIC VARIATIONAL INEQUALITIES AND ITS APPROXIMATION

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INTRODUCTION

A dualization technique is frequently employed to obtain an approximation of variational inequalities (see [4]). A properly chosen Lagrangian $\mathscr{L}$ enables us to transform the original minimization problem into a problem of finding its saddle-point on a certain convex set $K \times \Lambda$. This approach has some advantages:

– it avoides the complex construction of convex sets of admissible functions;

– it offers algorithms for numerical computations.

Last but not least, it makes it possible to approximate the Lagrange multipliers associated with the problem. Since these multipliers have usually a good physical meaning (for example outward fluxes, normal or friction forces), their knowledge is welcome.

In the present paper, conditions sufficient for the convergence of saddle-points $\{u_h, \lambda_H\}$ of $\mathscr{L}$ on $K_h \times \Lambda_H$ (approximation of $K \times \Lambda$) to the saddle-point $\{u, \lambda\}$ of $\mathscr{L}$ on $K \times \Lambda$ are studied. Applications to the unilateral problem and to problems with friction are presented.

1. MIXED FORMULATION OF VARIATIONAL INEQUALITIES

Let V, L be two real Hilbert spaces, with the norms $\| \ \|$, $| \ |$, respectively, and let V', L' be their dual spaces. On V, a quadratic functional $\mathscr{J}$ will be given

$$\mathscr{J}(v) = \tfrac{1}{2}\, a(v, v) - \langle f, v \rangle ,$$

where a is *a continuous, symmetric and V-elliptic bilinear form*, $f \in V'$ and $\langle \, , \rangle$ denotes the duality pairing between V' and V.

Let $\mathscr{L} : V \times L \to R_1$ be a functional of the form

$$\mathscr{L}(v, \mu) = \mathscr{J}(v) + b(v, \mu) - [g, \mu] , \qquad (1)$$

where $b : V \times L \to R_1$ is *a continuous bilinear form*, $g \in L'$ and $[g, \mu]$ denotes the value of g at μ. Finally, let $K \subseteq V$, $\Lambda \subseteq L$ be *non-empty, closed convex subsets.* We make the following assumptions, concerning K and Λ:

Λ is either

(CC) *a convex cone with its vertex at Θ (zero element of L) and $K = V$*

or

(BC) *a bounded convex subset of L.*

We shall consider the following problem:

$$(\mathscr{P}) \qquad \begin{cases} \text{to find an element } \{u, \lambda\} \in K \times \Lambda \text{ such that} \\ \mathscr{L}(u, \mu) \leqq \mathscr{L}(u, \lambda) \leqq \mathscr{L}(v, \lambda) \;\; \forall v \in K, \;\; \forall \mu \in \Lambda . \end{cases}$$

$\{u, \lambda\}$ will be called *a saddle-point* of $\mathscr{L}$ on $K \times \Lambda$.

Remark 1 $\{u, \lambda\} \in K \times \Lambda$ is a saddle-point of $\mathscr{L}$ on $K \times \Lambda$ if and only if

$$(2) \qquad \mathscr{L}(u, \lambda) = \min_K \sup_\Lambda \mathscr{L}(v, \mu) = \max_\Lambda \inf_K \mathscr{L}(v, \mu)$$

(see [2], [3]). Let us denote $j(v) = \sup_\Lambda \{b(v, \mu) - [g, \mu]\}$. It is easy to see that j is a lower semicontinuous convex function. With regard to (2) and $(\mathscr{P})$ we see that $u \in K$ solves the following problem:

$$(3) \qquad \mathscr{J}(u) + j(u) = \min_K \{\mathscr{J}(v) + j(v)\} .$$

An equivalent formulation of $(\mathscr{P})$ is the following ([3]):

$$(\mathscr{P})' \qquad \begin{cases} \text{to find } \{u, \lambda\} \in K \times \Lambda \text{ such that} \\ a(u, v - u) + b(v - u, \lambda) \geqq \langle f, v - u \rangle \;\; \forall v \in K \\ \qquad\qquad b(u, \mu - \lambda) \leqq [g, \mu - \lambda] \;\; \forall \mu \in \Lambda . \end{cases}$$

We present two typical examples, leading to the problem $(\mathscr{P})$.

Example 1 (*Dualization of constraints*). Let $u \in K$ be such that

$$(\mathscr{P}_1) \qquad \mathscr{J}(u) \leqq \mathscr{J}(v) \quad \forall v \in K .$$

We shall suppose that the following characterization of K holds:

$$K = \{v \in V \mid b(v, \mu) \leqq [g, \mu] \;\; \forall \mu \in \Lambda\} ,$$

where b, g have the above mentioned properties and Λ is a convex cone with the vertex at Θ. Then $(\mathscr{P}_1)$ leads to the search of a saddle-point $\{u, \lambda\}$ of $\mathscr{L}$ on $V \times \Lambda$ (see [2]) and j is the indicator function of K. In this case (CC) is satisfied.

Example 2. Let K be a closed, convex subset of the Sobolev space $H^1(\Omega)$ and

$\Omega \subset R_2$ a bounded domain with a continuous boundary $\partial\Omega$. We look for $u \in K$, satisfying

$$(\mathscr{P}_2) \qquad \mathscr{S}(u) \leqq \mathscr{S}(v) \quad \forall v \in K\,,$$

where $\mathscr{S}(v) = \mathscr{J}(v) + \int_{\partial\Omega} |v| \, \mathrm{d}s$ is *a non-differentiable functional.* Then $(\mathscr{P}_2)$ leads to the search of a saddle-point $\{u, \lambda\}$ of $\mathscr{L}(v, \mu) = \mathscr{J}(v) + \int_{\partial\Omega} \mu v \, \mathrm{d}s$ on $K \times \Lambda$, where

$$\Lambda = \{\lambda \in L^2(\partial\Omega) | \; |\lambda| \leqq 1 \text{ a.e. on } \partial\Omega\}\,.$$

In this case, $j(v) = \int_{\partial\Omega} |v| \, \mathrm{d}s$ and (BC) is satisfied. We see, that introducing the new variable $\mu \in \Lambda$,we obtain *a differentiable* functional $\mathscr{L}(v, \mu)$, which is more suitable for numerical calculations in many cases.

The formulation $(\mathscr{P})$ (or $(\mathscr{P})'$) will be called a *mixed formulation of* (3).

Remark 2 $(\mathscr{P})'$ is meaningful for a general continuous, V-elliptic bilinear form a (not necessarily symmetric). In such a case, $(\mathscr{P})'$ is a mixed formulation of the following problem:

$$(4) \qquad \begin{cases} \textit{to find } u \in K \textit{ such that} \\ a(u, v - u) + j(v) - j(u) \geqq \langle f, v - u \rangle \quad \forall v \in K\,. \end{cases}$$

Let us mention briefly some well-known results on the existence and uniqueness of solutions of $(\mathscr{P})$.

Let $b : V \times L \to R_1$ satisfy *Babuška-Brezzi's condition*

$$(5) \qquad \exists \beta = \text{const.} > 0 : \sup_V \frac{b(v, \mu)}{\|v\|} \geqq \beta |\mu| \quad \forall \mu \in L\,.$$

Theorem 1. *Let* (5) *and* (CC) *be satisfied. Then there exists a unique solution of* $(\mathscr{P})$.

For the proof, see [1].

If (BC) is satisfied, the situation is much simpler.

Theorem 2. *Let* (BC) *be satisfied. Then* $(\mathscr{P})$ *has a solution, the first component of which is uniquely determined.*

Proof. The existence of a solution follows from the V-elipticity of a and the boundedness of Λ, the uniqueness of the first component from the V-elipticity of a (see [3]).

Approximation of $(\mathscr{P})$

Let $h, H \in (0, 1)$ be two parameters, tending to $0+$. To every couple h, H we associate *finite dimensional subspaces* $V_h \subset V$ and $L_H \subset L$, respectively. Let K_h and Λ_H be *closed, convex subsets* of V_h and L_H, respectively.

Similarly as in the continuous case we make the following assumptions:

Λ_H is either

(CC_H) *a convex cone with vertex at* Θ *and* $K_h = V_h$

or

(BC_H) *a convex subset of* L_H, *bounded uniformly in* L, *i.e. there exists a positive number* $c > 0$ *such that*

$$|\mu_H| \leqq c \quad \forall \mu_H \in \Lambda_H \quad \forall H \in (0, 1)\,.$$

By *the approximation* of $(\mathscr{P})$ we mean the problem of finding a saddle-point $\{u_h, \lambda_H\} \in K_h \times \Lambda_H$ of $\mathscr{L}$ on $K_h \times \Lambda_H$:

$$(\mathscr{P}_{hH}) \qquad \mathscr{L}(u_h, \mu_H) \leqq \mathscr{L}(u_h, \lambda_H) \leqq \mathscr{L}(v_h, \lambda_H) \quad \forall v_h \in K_h\,, \quad \forall \mu_H \in \Lambda_H\,,$$

or equivalently

$$(\mathscr{P}_{hH})' \qquad \begin{cases} \text{to find } \{u_h, \lambda_H\} \in K_h \times \Lambda_H \text{ such that} \\ a(u_h, v_h - u_h) + b(v_h - u_h, \lambda_H) \geqq \langle f, v_h - u_h \rangle \ \forall v_h \in K_h \\ b(u_h, \mu_H - \lambda_H) \leqq [g, \mu_H - \lambda_H] \ \forall \mu_H \in \Lambda_H. \end{cases}$$

Let us not that $K_h \not\subset K$ and $\Lambda_H \not\subset \Lambda$, in general.

Interpretation of $(\mathscr{P}_{hH})$

If we set $j_H(v_h) = \sup_{\Lambda_H} \{b(v_h, \mu_H) - [g, \mu_H]\}$, the first component $u_h \in K_h$ minimizes the functional $\mathscr{J}(v_h) + j_H(v_h)$ over K_h.

As far as the existence and uniqueness of $(\mathscr{P}_{hH})$ is concerned, results similar to those from Theorems 1, 2 hold. To this end let us suppose that there exists a positive number $\hat{\beta}$, independent of h, H and such that

$$(6) \qquad \sup_{V_h} \frac{b(v_h, \mu_H)}{\|v_h\|} \geqq \hat{\beta}|\mu_H| \quad \forall \mu_H \in L_H\,.$$

Theorem 3. *Let* (CC_H) *and* (6) *be satisfied. Then there exists a unique solution of* $(\mathscr{P}_{hH})$.

Theorem 4. *Let* (BC_H) *be satisfied. Then there exists a solution of* $(\mathscr{P}_{hH})$, *the first component of which is uniquely determined.*

The most difficult task is the verification of (6) in particular examples.

Example 3. Let us consider the problem $(\mathscr{P}_1)$ with

$$\mathscr{J}(v) = \tfrac{1}{2}\|v\|^2_{H^1(\Omega)} - (f, v)_0\,, \quad f \in L^2(\Omega)\,,$$

and

$$K = \{v \in H^1(\Omega) |\ v \geqq 0 \text{ on } \partial\Omega\}\,,$$

where $(\, , \,)_0$ denotes the $L^2(\Omega)$-scalar product. The corresponding mixed formulation is

$$\begin{cases} \textit{to find } \{u, \lambda\} \in H^1(\Omega) \times H_-^{-1/2}(\partial\Omega) \textit{ such that} \\ (\operatorname{grad} u, \operatorname{grad} v)_0 + (u, v)_0 + \langle v, \lambda \rangle = (f, v)_0 \quad \forall v \in H^1(\Omega) \\ \langle u, \mu - \lambda \rangle \leqq 0 \quad \forall \mu \in H_-^{-1/2}(\partial\Omega), \end{cases}$$

where $H_-^{-1/2}(\partial\Omega)$ denotes the convex cone of non-positive linear functionals over the space $H^{1/2}(\partial\Omega)$ and $\langle \, , \, \rangle$ is the corresponding duality pairing. It is easy to see ([2]) that $\lambda = -\partial u/\partial n$. One can prove ([6]) that Babuška-Brezzi's condition (5) holds with $\beta = 1$.

Let $\{\mathcal{T}_h\}$ be a *regular family of triangulations* of $\bar{\Omega}$, whose nodes lying on $\partial\Omega$, form an equidistant partition of $\partial\Omega$. Let us denote them by $a_1, \ldots, a_m, a_{m+1} = a_1$. Now we set

$$V_h = \{v_h \in C(\bar{\Omega}) | \; v|_{T_i} \in P_1(T_i) \; \forall T_i \in \mathcal{T}_h\}$$

$$L_H \equiv L_h = \{\mu_h \in L^2(\partial\Omega) | \; \mu_h|_{a_i a_{i+1}} \in P_0(a_i a_{i+1}), \; i = 1, \ldots, m\}$$

$$\Lambda_H \equiv \Lambda_h = \{\mu_h \in L_h | \; \mu_h \leqq 0 \text{ on } \partial\Omega\} ,$$

where $P_1(T_i)$ and $P_0(a_i a_{i+1})$ are the spaces of *linear polynomials* on T_i and of *constant functions* on $a_i a_{i+1}$, respectively. Then the problem $(\mathcal{P}_{hH}) = (\mathcal{P}_h)$ has a solution $\{u_h, \lambda_h\}$ with a uniquely determined u_h (see [2]). Next we analyze the condition (6). Let $\mu_h \in L_h$ be such that

$$(6)' \qquad \int_{\partial\Omega} v_h \mu_h \, ds = 0 \quad \forall v_h \in V_h \Leftrightarrow \int_{\partial\Omega} \varphi_j \mu_h \, ds = 0 \quad j = 1, \ldots, m ,$$

where $\varphi_j \in V_h$, $\varphi_j(a_i) = \delta_{ij}$ and $\varphi_j = 0$ at the internal nodes of $\mathcal{T}_h$. (6)′ is equivalent to the following system of linear algebraic equations:

$$\begin{aligned} \mu_1 + \mu_2 &= 0 \\ \mu_2 + \mu_3 &= 0 \\ \vdots \quad & \quad \vdots \\ \mu_1 + \mu_m &= 0 \quad \mu_i = \mu|_{a_i a_{i+1}} . \end{aligned}$$

If the number m of $a_i a_{i+1}$ is *even*, the system has also a *non-trivial* solution. Consequently, the condition (6) cannot be satisfied and the second component λ_h is not uniquely determined, in general. In order to obtain (6), we use *two systems* of partitions $\{\mathcal{T}_h\}$, $\{\mathcal{T}_H\}$ of $\bar{\Omega}$ and $\partial\Omega$, respectively. Let $h = \max \operatorname{diam} T_i$, $H = \max$ *length* $a_i a_{i+1}$, a_i nodes of $\mathcal{T}_H$. We define V_h in the same way as above and

$$L_H = \{\mu_H \in L^2(\partial\Omega) | \; \mu_{H|a_i a_{i+1}} \in P_0(a_i a_{i+1}), \; i = 1, \ldots, m\}$$

$$\Lambda_H = \{\mu_H \in L_H | \; \mu_H \leqq 0 \text{ on } \partial\Omega\} .$$

If the ratio h/H is *sufficiently small*, then

$$\sup_{V_h} \frac{\langle v_h, \mu_H \rangle}{\|v_h\|_{H^1(\Omega)}} \geqq \hat{\beta} |\mu_H|_{H^{-1/2}(\partial\Omega)} ,$$

with $\hat{\beta}$ independent of h, H (see [6]).

2. ERROR ESTIMATES

Our aim is to establish relations between u_h, u and λ_H, λ. To this end we give another, equivalent form of $(\mathscr{P})'$.

Let $\mathscr{H} = V \times L$ be a Hilbert space, equipped with the norm:

$$\|V\|_{\mathscr{H}} = \{\|v\|^2 + |\mu|^2\}^{1/2}, \quad V = (v, \mu) \in \mathscr{H} ,$$

$\mathscr{A} : \mathscr{H} \times \mathscr{H} \to R_1$ a bilinear form

$$\mathscr{A}(U, V) = a(u, v) + b(v, \lambda) - b(u, \mu), \quad U = (u, \lambda) \in \mathscr{H}$$
$$V = (v, \mu) \in \mathscr{H}$$

and $\mathscr{F} : \mathscr{H} \to R_1$ a linear functional

$$\langle \mathscr{F}, V \rangle = \langle f, v \rangle - [g, \mu], \quad V = (v, \mu) \in \mathscr{H} .$$

The definition of $\mathscr{A}$ immediately implies

(7) $$\mathscr{A}(V, V) = a(v, v) \quad \forall V = (v, \mu) \in \mathscr{H} ;$$

(8) $$\exists M = \text{const.} > 0 : |\mathscr{A}(U, V)| \leqq M \|U\|_{\mathscr{H}} \|V\|_{\mathscr{H}} \quad \forall U, V \in \mathscr{H} .$$

It is readily seen that $(\mathscr{P})'$ is equivalent to

(P) $$\begin{cases} \text{to find } U = \{u, \lambda\} \in \mathscr{K} = K \times \Lambda \text{ such that} \\ \mathscr{A}(U, V - U) \geqq \langle \mathscr{F}, V - U \rangle \ \forall V \in \mathscr{K}. \end{cases}$$

Next, let $\mathscr{K}_{hH} = K_h \times \Lambda_H$ be a closed, convex subset of $\mathscr{H}$; $\mathscr{K}_{hH} \not\subset \mathscr{K}$, in general. The problem

$(\mathbf{P}_{hH})$ $$\begin{cases} \text{to find } \mathfrak{U} = \{u_h, \lambda_H\} \in \mathscr{K} \text{ such that} \\ \mathscr{A}(\mathfrak{U}, \mathfrak{V} - \mathfrak{U}) \geqq \langle \mathscr{F}, \mathfrak{V} - \mathfrak{U} \rangle \ \forall \mathfrak{V} \in \mathscr{K}_{hH} \end{cases}$$

represents an approximation of (**P**), equivalent to $(\mathscr{P}_{hH})'$ (or $(\mathscr{P}_{hH})$).

First we prove an auxiliary lemma.

Lemma 1. *Let $\{u, \lambda\}$ and $\{u_h, \lambda_H\}$ be solutions of $(\mathscr{P})'$ and $(\mathscr{P}_{hH})'$, respectively. Then*

(9) $$c\|u - u_h\|^2 \leqq c_1\{\|u - v_h\|^2 + |\lambda - \mu_H|^2\} + A_1(v_h) + $$
$$+ A_2(v) + \{b(u, \lambda_H - \mu) - [g, \lambda_H - \mu]\} +$$
$$+ \{b(u, \lambda - \mu_H) - [g, \lambda - \mu_H]\} + c_2|\lambda - \lambda_H|^2$$

holds for every $v_h \in K_h$, $v \in K$, $\mu_H \in \Lambda_H$, $\mu \in \Lambda$, *where*

$$A_1(v_h) = a(u, v_h - u) + b(v_h - u, \lambda) + \langle f, u - v_h \rangle$$
$$A_2(v) = a(u, v - u_h) + b(v - u_h, \lambda) + \langle f, u_h - v \rangle$$

and c, c_1, c_2 *are positive constants independent of* h, H.

Proof. By virtue of (7) and the definitions of $(\mathbf{P})$ and $(\mathbf{P}_{hH})$, we get – using the definitions of $\mathscr{A}$ and $\mathscr{F}$

$$(10) \qquad \alpha\|u - u_h\|^2 \leqq \mathscr{A}(U - \mathfrak{U},\ U - \mathfrak{U}) = \mathscr{A}(U, U) - \mathscr{A}(\mathfrak{U}, U) - - \mathscr{A}(U, \mathfrak{U}) + \mathscr{A}(\mathfrak{U}, \mathfrak{U}) \leqq \langle \mathscr{F}, U - V \rangle + \mathscr{A}(U, V) + + \langle \mathscr{F}, \mathfrak{U} - \mathfrak{V} \rangle + \mathscr{A}(\mathfrak{U}, \mathfrak{V}) - \mathscr{A}(\mathfrak{U}, U) - \mathscr{A}(U, \mathfrak{U}) = = \langle \mathscr{F}, U - \mathfrak{V} \rangle + \langle \mathscr{F}, \mathfrak{U} - V \rangle + \mathscr{A}(U, V - \mathfrak{U}) + + \mathscr{A}(\mathfrak{U} - U,\ \mathfrak{V} - U) + \mathscr{A}(U, \mathfrak{V} - U) = A_1(v_h) + A_2(v) + + \{b(u, \lambda_H - \mu) - [g, \lambda_H - \mu]\} + \{b(u, \lambda - \mu_H) - - [g, \lambda - \mu_H]\} + a(u_h - u, v_h - u) + b(v_h - u,\ \lambda_H - \lambda) - - b(u_h - u,\ \mu_H - \lambda)\,.$$

The boundedness of a, b together with the inequality $2hf \leqq 1/\varepsilon\, h^2 + \varepsilon f^2$ yields

$$(11) \qquad \alpha\|u - u_h\|^2 \leqq A_1(v_h) + A_2(v) + \{b(u, \lambda_H - \mu) - [g, \lambda_H - \mu]\} + + \{b(u, \lambda - \mu_H) - [g, \lambda - \mu_H]\} + M_1\varepsilon\|u - u_h\|^2 + + M_1/\varepsilon\|u - v_h\|^2 + M_2/\varepsilon\|v_h - u\|^2 + M_2\varepsilon|\lambda_H - \lambda|^2 + + M_2\varepsilon\|u - u_h\|^2 + M_2/\varepsilon|\lambda - \mu_H|^2\,.$$

For $\varepsilon > 0$ sufficiently small, we arrive at (9).

As a direct consequence of Lemma 1, we obtain

Theorem 5. *Let* (CC), (CC_H) *and* (6) *be satisfied. Let there exist a solution* $\{u, \lambda\}$ *of* $(\mathscr{P})'$. *Then*

$$(12) \qquad c\|u - u_h\|^2 \leqq c_1\{\|u - v_h\|^2 + |\lambda - \mu_H|^2\} + + \{b(u, \lambda_H - \mu) - [g, \lambda_H - \mu]\} + \{b(u, \lambda - \mu_H) - [g, \lambda - \mu_H]\}\,,$$

$$(13) \qquad |\lambda - \lambda_H| \leqq c\{\|u - u_h\| + |\lambda - \mu_H|\}$$

hold for any $v_h \in V_h$, $\mu \in \Lambda$, $\mu_H \in \Lambda_H$ *with positive constants* c, c_1.

Proof. Since (CC) and (CC_H) are satisfied, $K = V$, $K_h = V_h$, i.e. K and K_h are linear sets. Therefore, in $(\mathscr{P})'_2$ and $(\mathscr{P}_{hH})'_2$ the sign of equality can be written, so that

$$(14) \qquad A_1(v_h) = 0 \quad \forall v_h \in V_h\,.$$

As $K = V$ and $V_h \subset V\ \forall h \in (0, 1)$, we can choose $v = u_h$ in (9). Hence

(15) $$A_2(v) = 0 .$$

Let $\mu_H \in \Lambda_H$ be arbitrary. From (6) we obtain

(16) $$\hat{\beta}|\lambda_H - \mu_H| \leqq \sup_{V_h} \frac{b(v_h, \mu_H - \lambda_H)}{\|v_h\|} .$$

Using $(\mathscr{P}_{hH})'_2$ and $(\mathscr{P})'_2$, we may write

$$b(v_h, \mu_H - \lambda_H) = b(v_h, \mu_H) - b(v_h, \lambda_H) = b(v_h, \mu_H) +$$
$$+ a(u_h, v_h) - \langle f, v_h \rangle = b(v_h, \mu_H) + a(u_h, v_h) - a(u, v_h) -$$
$$- b(v_h, \lambda) = b(v_h, \mu_H - \lambda) + a(u_h - u, v_h) \leqq c\{|\mu_H - \lambda| + \|u_h - u\|\} \|v_h\| .$$

This identity together with (16) implies

$$|\mu_H - \lambda_H| \leqq c\{\|u - u_h\| + |\lambda - \mu_H|\} \quad \forall \mu_H \in \Lambda_H .$$

Using the triangle inequality

$$|\lambda - \lambda_H| \leqq |\lambda - \mu_H| + |\mu_H - \lambda_H| \quad \forall \mu_H \in \Lambda_H ,$$

we obtain (13). Finally, replacing the term $M_2 \varepsilon |\lambda_H - \lambda|$ on the right hand side of (11) by (13) and making use of (14) and (15), we obtain (12) for $\varepsilon > 0$ sufficiently small.

Remark 3. If $\Lambda_H \subset \Lambda$ for $\forall H \in (0, 1)$, we can insert $\mu = \lambda_H$ into (12). Therefore, (12) takes the following simpler form:

(12') $$c\|u - u_h\|^2 \leqq c_1\{\|u - v_h\|^2 + |\lambda - \mu_H|^2\} +$$
$$+ \{b(u, \lambda - \mu_H) - [g, \lambda - \mu_H]\} \quad \forall v_h \in V_h , \quad \mu_H \in \Lambda_H .$$

Theorem 6. *Let* (BC) *and* (BC_H) *be satisfied. Then*

(17) $$c\|u - u_h\|^2 \leqq A_1(v_h) + A_2(v) + c_1\{\|u - v_h\|^2 + |\lambda - \mu_H|^2\} +$$
$$+ c_2\|u - v_h\| + \{b(u, \lambda_H - \mu) - [g, \lambda_H - \mu]\} +$$
$$+ \{b(u, \lambda - \mu_H) - [g, \lambda - \mu_H]\}$$

holds for any $v_h \in K_h$, $v \in K$, $\mu \in \Lambda$, $\mu_H \in \Lambda_H$.

(18) *Moreover if* $K = V$, $K_h = V_h$ *and* (6) *is satisfied, then* (12) *and* (13) *hold.*

Proof. We have to prove (17) only. As Λ, Λ_H are bounded in L,

$$|b(v_h - u, \lambda_H - \lambda)| \leqq c\|v_h - u\| \quad \forall v_h \in K_h .$$

Hence (17) follows by virtue of (10).

Remark 4. If $K_h \subset K$, $\Lambda_H \subset \Lambda$ $\forall h, H \in (0, 1)$ then setting $v = u_h$, $\mu = \lambda_H$, we obtain $A_2(v) = 0$, $b(u, \lambda_H - \mu) - [g, \lambda_H - \mu] = 0$.

Next, let us suppose that the pair of real parameters h, H satisfies

$$h \to 0+ \Leftrightarrow H \to 0+ .$$

Relations (12), (13) and (17) can be used to estimate the rate of convergence of u_h to u and λ_H to λ, provided the exact solution is smooth enough. Other application are given by the following convergence theorems.

Theorem 7. *Let* (BC), (BC_H) *be satisfied and, moreover let*

(19) $$\forall v \in K \quad \exists v_h \in K_h : v_h \to v \ \ in \ \ V ;$$

(20) $$\forall \mu \in \Lambda \quad \exists \mu_H \in \Lambda_H : \mu_H \to \mu \ \ in \ \ L ;$$

(21) $$v_h \in K_h , \quad v_h \rightharpoonup v \quad (weakly) \ in \ V implies \quad v \in K ;$$

(22) $$\mu_H \in \Lambda_H , \quad \mu_H \rightharpoonup \mu \quad in \ L implies \quad \mu \in \Lambda ;$$

(23) $$\exists r > 0 \quad \exists \{v_h\} , \quad v_h \in K_h \quad such \ that \quad \|v_h\| \leqq r \quad \forall h \in (0, 1) .$$

Let the solution $\{u, \lambda\} \in K x \Lambda$ *of* $(\mathscr{P})'$ *be unique. Then*

$$u_h \to u \text{ in } V, \quad \lambda_H \rightharpoonup \lambda \text{ in } L.$$

Proof. First, $\{u_h\}$, $\{\lambda_H\}$ are bounded. For $\{\lambda_H\}$ this follows from (BC_H), for $\{u_h\}$ from (23) and $(\mathscr{P}_{hH})'_2$. Hence, there exists a subsequence $\{u_{h'}, \lambda_{H'}\} \subset \{u_h, \lambda_H\}$ and $\{u^*, \lambda^*\} \in V \times L$ such that

(24) $$u_{h'} \rightharpoonup u^* \text{ in } V, \quad \lambda_{H'} \rightharpoonup \lambda^* \text{ in } L.$$

By virtue of (21), (22), $u^* \in K$, $\lambda^* \in \Lambda$. Let us show that $\{u^*, \lambda^*\}$ is a solution of $(\mathscr{P})'$. Let $\{v, \mu\} \in K \times \Lambda$ be an arbitrarily chosen element. From (19), (20) we conclude that there exist $v_h \in K_h$, $\mu_H \in \Lambda_H$ such that

(25) $$v_h \to v \text{ in } V, \quad \mu_H \to \mu \text{ in } L.$$

Since $\{u_{h'}, \lambda_{H'}\}$ is a solution of $(\mathscr{P}_{h'H'})'$, it satisfies

(26) $$a(u_{h'}, u_{h'} - v_{h'}) + b(u_{h'} - v_{h'}, \lambda_{H'}) \leqq \langle f, u_{h'} - v_{h'}\rangle \quad \forall v_{h'} \in K_{h'}$$

(27) $$b(u_{h'}, \mu_{H'} - \lambda_{H'}) \leqq [g, \mu_{H'} - \lambda_{H'}] \quad \forall \mu_{H'} \in \Lambda_{H'} .$$

Passing to the limit for $h', H' \to 0+$ in (26), together with (24), (25) implies that

(28) $$a(u^*, u^* - v) + \liminf_{h',H'} b(u_{h'}, \lambda_{H'}) - b(v, \lambda^*) \leqq \langle f, u^* - v\rangle \quad \forall v \in K .$$

The same procedure is applicable to (27):

(29) $$b(u^*, \mu) - [g, \mu - \lambda^*] \leqq \liminf_{h',H'} b(u_{h'}, \lambda_{H'}) \quad \forall \mu \in \Lambda .$$

Setting $\mu = \lambda^*$ in (29), we obtain

$$(30) \qquad b(u^*, \lambda^*) \leqq \liminf_{h',H'} b(u_{h'}, \lambda_{H'}) .$$

Substitution of (30) into (28) yields:

$$a(u^*, u^* - v) + b(u^* - v, \lambda^*) \leqq \langle f, u^* - v\rangle \quad \forall v \in K .$$

The choice $v = u^*$ in (28) implies:

$$\liminf_{h',H'} b(u_{h'}, \lambda_{H'}) \leqq b(u^*, \lambda^*) .$$

From this and (29), we have

$$b(u^*, \mu - \lambda^*) \leqq [g, \mu - \lambda^*] \quad \forall \mu \in \Lambda .$$

Thus $\{u^*, \lambda^*\}$ is a solution of $(\mathscr{P})'$. By virtue of its uniqueness, the whole sequences $\{u_h\}$, $\{\lambda_H\}$ tend weakly to u, λ. Let us show that $u_h \to u$ strongly in V. Let $\{\bar{v}_h\}$, $\bar{v}_h \in K_h$, $\{\bar{\mu}_H\}$, $\bar{\mu}_H \in \Lambda_H$ be such that

$$\bar{v}_h \to u , \quad \bar{\mu}_H \to \lambda .$$

Applying (17) with $v = u$, $\mu = \lambda$, $v_h = \bar{v}_h$, $\mu_H = \bar{\mu}_H$ and using the weak convergence $u_h \rightharpoonup u$, $\lambda_H \rightharpoonup \lambda$, we obtain $u_h \to u$ in V.

Remark 5. If $K_h \subset K$ and $\Lambda_H \subset \Lambda$, the conditions (21) and (22) respectively, are satisfied.

Theorem 8. *Let* (CC), (CC$_H$) *and* (6) *be satisfied. Let* $\{u, \lambda\}$ *be the unique solution of* $(\mathscr{P})'$. *Moreover, let us suppose that*

$$(31) \qquad \forall v \in V \quad \exists v_h \in V_h : v_h \to v \ \text{in } V ;$$

$$(32) \qquad \forall \mu \in \Lambda \quad \exists \mu_H \in \Lambda_H : \mu_H \to \mu \ \text{in } L ;$$

$$(33) \qquad \mu_H \in \Lambda_H , \quad \mu_H \rightharpoonup \mu \quad \text{in } L \text{ implies} \quad \mu \in \Lambda ;$$

(34) *there exist a real number* d, *a positive number* c *and a bounded sequence* $\{\bar{v}_h\}$, $\bar{v}_h \in V_h$ *such that* $j_H(v_h) \geqq d$ $\forall v_h \in V_h$, $\forall h$, $H \in (0, 1)$, $j_H(\bar{v}_h) \leqq c$ $\forall h$, $H \in \in (0, 1)$.

Then $u_h \to u$, $\lambda_H \to \lambda$.

Proof. We shall prove the boundedness of $\{u_h\}$ and $\{\lambda_H\}$ only. The rest of the proof is analogous to that of Theorem 7. The convergence of λ_H to λ follows from (13).

According to the interpretation of $(\mathscr{P}_{hH})'$, $u_h \in V_h$ satisfies

$$a(u_h, v_h - u_h) + j_H(v_h) - j_H(u_h) \geqq \langle f, v_h - u_h\rangle \quad \forall v_h \in V_h .$$

Hence

$$a(u_h, u_h) + j_H(u_h) \leqq a(u_h, \bar{v}_h) + j_H(\bar{v}_h) - \langle f, \bar{v}_h - u_h\rangle .$$

This and (34) implies the boundedness of $\{u_h\}$ and by virtue of (13) we deduce the boundedness of $\{\lambda_H\}$.

Remark 6. If $\Lambda_H \subset \Lambda\ \forall H \in (0, 1)$, (33) is automatically satisfied.

Condition (6), guaranteeing the convergence of λ_H to λ is very restrictive. That is why we shall be interested in the convergence u_h to u only if (CC) and (CC$_H$) hold. To this end let us suppose that the functions

$$j(v) = \sup_{\Lambda} \{b(v, \mu) - [g, \mu]\}$$

$$j_H(v_h) = \sup_{\Lambda_H} \{b(v_h, \mu_H) - [g, \mu_H]\}$$

take their values from the set $\{0, +\infty\}$. We shall denote by

$$\mathscr{K} = \{v \in V \mid j(v) = 0\}$$

$$\mathscr{K}_{hH} = \{v_h \in V_h \mid j_H(v_h) = 0\},$$

i.e. j and j_H are the *indicator functions* of the closed convex sets $\mathscr{K}$ and $\mathscr{K}_{hH}$, respectively. Let $\{u, \lambda\} \in V \times \Lambda$ and $\{u_h, \lambda_H\} \in V_h \times \Lambda_H$ be solutions of $(\mathscr{P})$ and $(\mathscr{P}_{hH})$, respectively. From the interpretation of these problems we see that $u \in \mathscr{K}$ and $u_h \in \mathscr{K}_{hH}$ are solutions of the minimizing problems:

$$\mathscr{J}(u) \leqq \mathscr{J}(v) \quad \forall v \in \mathscr{K}$$

and

$$\mathscr{J}(u_h) \leqq \mathscr{J}(v_h) \quad \forall v_h \in \mathscr{K}_{hH},$$

respectively.

As far as the convergence of u_h to u is concerned, we have

Theorem 9. *Let (CC), (CC$_H$) be satisfied and there exist solutions $\{u, \lambda\}$ and $\{u_h, \lambda_H\}$ of $(\mathscr{P})$ and $(\mathscr{P}_{hH})$, respectively, the first components of which are uniquely determined. Let*

(35) $$\forall v \in \mathscr{K} \quad \exists v_h \in \mathscr{K}_{hH} : v_h \to v \text{ in } V;$$

(36) $$v_h \in \mathscr{K}_{hH}, \quad v_h \rightharpoonup v \textit{ in } V \textit{ implies } v \in \mathscr{K}.$$

Then $u_h \to u$ in V.

Proof is a direct consequence of Th. 0.6 from [2].

3. APPLICATIONS

Example A. Let us consider the unilateral boundary value problem introduced in Example 3, with the same definitions of V_h, L_H and Λ_H. First, we consider the case, when $h = H$, i.e. the partition of $\partial\Omega$ is generated by the triangulation $\mathscr{T}_h$ of $\bar{\Omega}$. In that case

$$\mathscr{K}_{hH} \equiv \mathscr{K}_h = \{v_h \in V_h | \; v_h(a_{i+1/2}) \geqq 0, \; i = 1, \ldots, m\},$$

where $a_{i+1/2}$ is the midpoint of $a_i a_{i+1}$. It means that $\mathscr{K}_h$ contains all piecewise linear functions, the mean values of which are non-negative on $a_i a_{i+1}$. The function $j_h(v_h) = \sup_{\Lambda_h} \langle v_h, \mu_h \rangle$ is the indicator function of $\mathscr{K}_h$.

Now, let us suppose that h/H is sufficiently small. Then the condition (6) holds and one can use Theorem 5 for estimating the rate of convergence of u_h to u and λ_H to λ under some additional assumptions. We can prove the following result:

Theorem 10. *Let*

(i) $u \in K \cap H^2(\Omega)$;

(ii) $u \in H^{1,\infty}(a_i a_{i+1})$, $i = 1, \ldots, m$;

(iii) *the set of points where u changes from $u > 0$ to $u = 0$ is finite.*

Then

$$\|u - u_h\|_{H^1(\Omega)} \leqq c(u)(h + H)$$
$$\|\lambda - \lambda_H\|_{H^{-1/2}(\partial\Omega)} \leqq c(u, \lambda)(h + H)$$
$$\|\lambda - \lambda_H\|_{L^2(\partial\Omega)} \leqq c(u, \lambda)\, h^{-1/2}(h + H).$$

For the proof see [6].

Example B. Let us define the following problem:

$$\begin{cases} \text{to find } u \in H^1(\Omega) \text{ such that} \\ \mathscr{S}(u) \leqq \mathscr{S}(v) \quad \forall v \in H^1(\Omega), \end{cases}$$

where $\mathscr{S}(v) = \frac{1}{2}\|v\|^2_{H^1(\Omega)} + g \int_{\partial\Omega} |v| \, ds - (f, v)_0$ with $g \in R_1$, $g > 0$, $f \in L^2(\Omega)$. The corresponding Lagrangian of this problem is

$$\mathscr{L}(v, \mu) = \frac{1}{2}\|v\|^2_{H^1(\Omega)} + g \int_{\partial\Omega} \mu v \, ds - (f, v)_0,$$

$(v, \mu) \in H^1(\Omega) \times \Lambda$ and

$$\Lambda = \{\mu \in L^2(\partial\Omega) | \; |\mu| \leqq 1 \text{ a.e. on } \partial\Omega\}.$$

It is easy to see that there exists a unique saddle-point $\{u, \lambda\}$ of $\mathscr{L}$ on $H^1(\Omega) \times \Lambda$ and $\partial u/\partial n = -\lambda g$.

We define V_h as in the example A, $K_h = V_h$ and

$$\Lambda_H \equiv \Lambda_h = \{\mu_h \in L^2(\partial\Omega) | \; \mu_{h|a_i a_{i+1}} \in P_0(a_i a_{i+1}), \; |\mu_h| \leqq 1 \text{ on } \partial\Omega\}.$$

It is easy to verify that the conditions (19)–(23) are satisfied. Hence $u_h \to u$ in $H^1(\Omega)$, $\lambda_h \rightharpoonup \lambda$ in $L^2(\partial\Omega)$.

If the ratio h/H is sufficiently small, then Babuška-Brezzi's condition (6) is fulfilled and a result, similar to Theorem 10 can be obtained .

Example C. (*Signorini problem with friction.*) Let $\Omega \subset R_2$ be a bounded, polygonal domain, the boundary of which is decomposed as follows: $\partial\Omega = \bar{\Gamma}_u \cup \bar{\Gamma}_K$, where Γ_u, Γ_K are non-empty and open subsets of $\partial\Omega$. Let

$$\boldsymbol{V} = \{\boldsymbol{v} \in (H^1(\Omega))^2 |\ \boldsymbol{v} = 0 \text{ on } \Gamma_u\},$$

$$\boldsymbol{K} = \{\boldsymbol{v} \in \boldsymbol{V} |\ v_n \leqq 0 \text{ on } \Gamma_K\},$$

where $v_n = \boldsymbol{v} \cdot \boldsymbol{n}$ is the normal component of $\boldsymbol{v}$. We shall consider the problem

$$\begin{cases} \text{to find } \boldsymbol{u} \in \boldsymbol{K} \text{ such that} \\ \mathscr{S}(\boldsymbol{u}) \leqq \mathscr{S}(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{K}, \end{cases}$$

where $\mathscr{S}(\boldsymbol{v}) = \frac{1}{2} \int_{\partial\Omega} \tau_{ij}(\boldsymbol{v})\, \varepsilon_{ij}(\boldsymbol{v})\, \mathrm{d}\boldsymbol{x} + g \int_{\partial\Omega} |v_t|\, \mathrm{d}s - \int_\Omega f_i v_i\, \mathrm{d}x$, $\varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$ and $\tau_{ij}(\boldsymbol{v})$ are components of the *strain* and *stress tensor*, respectively, corresponding to the displacement $\boldsymbol{v}$ and mutually coupled by the linear Hooke's law. Finally, let $f = (f_1, f_2) \in (L^2(\Omega))^2$, $g \in R_1$, $g > 0$ and $v_t = \boldsymbol{v} \cdot \boldsymbol{t}$ be the tangential component of $\boldsymbol{v}$. The corresponding Lagrangian is defined on $\boldsymbol{K} \times \Lambda$, where

$$\Lambda = \{\mu \in L^2(\Gamma_K) |\ |\mu| \leqq 1 \text{ a.e. on } \Gamma_K\},$$

as follows

$$\mathscr{L}(\boldsymbol{v}, \mu) = \frac{1}{2} \int_\Omega \tau_{ij}(\boldsymbol{v})\, \varepsilon_{ij}(\boldsymbol{v})\, \mathrm{d}x + g \int_{\Gamma_k} \mu v_t\, \mathrm{d}s - \int_\Omega f_i v_i\, \mathrm{d}x\,.$$

It is readily seen that there exists a unique saddle-point $\{\boldsymbol{u}, \lambda\}$ of $\mathscr{L}$ on $\boldsymbol{K} \times \Lambda$ and $T_t(\boldsymbol{u}) = -g\lambda$, where $T_t(\boldsymbol{u})$ denotes the tangential traction component on Γ_K. Application of this formulation will be discussed in [7].

Example D. (*Signorini problem with friction.*) We shall consider the problem from Example C. Let $\Lambda = \Lambda_1 \times \Lambda_2$ be a closed convex subset of $(H^{-1/2}(\Gamma_K))^2$ (dual space to $(H^{1/2}(\Gamma_K))^2$), where

$$\Lambda_1 = \{\mu_1 \in H^{-1/2}(\Gamma_K),\ \mu_1 \geqq 0\}$$

$$\Lambda_2 = \{\mu_2 \in L^2(\Gamma_K),\ |\mu_2| \leqq g \text{ a.e. on } \Gamma_K\}.$$

Moreover, we suppose that Γ_K is a straight segment. Let

$$\mathscr{L}(\boldsymbol{v}, \mu_1, \mu_2) = \frac{1}{2} \int_\Omega \tau_{ij}(\boldsymbol{v})\, \varepsilon_{ij}(\boldsymbol{v})\, \mathrm{d}x + \langle \mu_1, v_n \rangle + \langle \mu_2, v_t \rangle - \int_\Omega f_i v_i\, \mathrm{d}x$$

be the Lagrangian, defined on $\boldsymbol{V}x\ \Lambda_1 \times \Lambda_2$. It can be proved that $\mathscr{L}$ has a unique saddle-point $\{\boldsymbol{u}, \lambda_1, \lambda_2\}$ on $\boldsymbol{V}x\ \Lambda_1 \times \Lambda_2$ and $\lambda_1 = -T_n(\boldsymbol{u})$, $\lambda_2 = -T_t(\boldsymbol{u})$, where $T_n(\boldsymbol{u})$ denotes the normal traction component on Γ_K. Analysis of this above formulation will be discussed in [5]. Let us mention, that although the theory, presented here is not directly, applicable to this formulation, a slight modification will do.

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Souhrn

SMÍŠENÁ FORMULACE ELIPTICKÝCH VARIAČNÍCH NEROVNOSTÍ A JEJÍ APROXIMACE

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V této práci se studuje aproximace smíšené formulace eliptických variačních nerovnic. Smíšená formulace je definována jako problém nalezení sedlového bodu Lagrangeovy funkce $\mathscr{L}$ na kartézském součinu konvexních množin $K \times \Lambda$. Její aproximace je pak definována jako úloha nalezení sedlového bodu $\mathscr{L}$ na $K_h \times \Lambda_H$, kde K_h, Λ_H jsou konečně-dimensionální aproximace K, Λ. Jsou vysloveny postačující podmínky k tomu, aby takto nalezené aproximace na $K_h \times \Lambda_H$ konvergovaly k sedlovému bodu $\mathscr{L}$ na $K \times \Lambda$. Obecné výsledky jsou pak aplikovány na konkrétní příklady.

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