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ON THE DECOMPOSITION OF PARTICLE SIZE DISTRIBUTION IN THE EXTRACTION REPLICA METHOD

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1. INTRODUCTION

In recent years we have been witnessing a very intensive development of quantitative methods for material structure evaluation. We meet this problem above all in metallography and petrography, but also in anatomy and haematology etc. Evaluating the spatial structure we start mostly from a model based on the assumption that in the space of material specimen spherical particles with diameter being random variable with specified probability density function (p.d.f.) are randomly and mutually independently distributed. Parameters of the corresponding distribution function (d.f.), the relevant moments and the mean number of these particles per volume unit represent the basic spatial structure parameters of the material.

Due to the opacity of the majority of the material specimens investigated we cannot observe the spatial structure directly and therefore we must use indirect information. Among other we can consider:

a) planar structure observable in the plane of section and formed by circular sections of spherical particles;

b) projected structure obtained from a thin foil by using parallel projection (the projection direction is perpendicular to the specimen surface). In the projection plane we can observe both actual diameters of spherical particles (their centres are located in the foil) and actual diameters of the base of particle cap (these particles are intercepted by the surface plane of the projected foil and their centres are located outside the foil);

c) extraction replica; this method is applied first of all for metalic specimens. After etching the surface of the specimen the fine particles embedded in the prepared surface are extracted using replica method.

For the method a), we gather our information on the p.d.f. $f(x)$ of the particle diameter from p.d.f. $g(y)$ of the planar section diameters of these particles and from the p.d.f. $l(p)$ of the chord length, intercepted on these sections by a line applied randomly to the microstructure in the polished plane. These problems as well as estimations of spatial structure parameters were studied from various points of view e.g. by Wicksell [14], Fullman [3], Saltykov [12], Horálek [4] and Bach [2]. The excelent paper of Nicholson [11] deals with generalizations of different models concerning the estimation of linear properties of particle size distributions. His theory unifies a large body of current literature and also provides the basis for including the effect of specimen preparation and various anomalies encountered in the measurement process. The influence of the resolution power of an instrument on the estimation of the basic material spatial structure parameters is in detail analyzed by Horálek [7]. For the areal and the lineal method new analytical forms of the p.d.f.'s $g(y)$ and $l(p)$ respecting the presence of a positive resolution point are derived. It is also shown that in practice, disregard of the reduced resolution power always leads to overestimating the actual mean value α_1 of the distribution of the sphere diameter and, at the same time, to underestimating the actual mean number $\varkappa$ of the sphere centres per volume unit; these two features become more evident if the lineal method is applied, as compared with the areal method.

Method b) was studied by Horálek [6] and Šidák [13] independently of one another. Their models assume that all particles and their caps embedded in the foil are projected. In [6] it is shown among other that the direct estimate of spatial structure parameters is possible only in the case of two prepared foils of different thickness fulfilling the requirement of minimal difference of these thicknesses. In [13] the author applied the general results to some special types of the particle diameter distribution and also introduced an estimate of the probability density function of particle diameters.

In the present paper we pay special attention to the method c). In this field the study of Ashby and Ebeling [1] must be considered a basic paper. Their model (in the following text referred to as the AE-model) enables us to determine two basic spatial structure parameters of the material (α_1 and $\varkappa$). It is constructed under the assumption that all particles of the investigated phase, embedded in the prepared surface on which

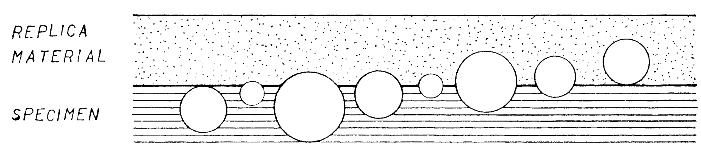

Fig. 1. Physical model of the pure extraction replica.

the replica material was deposited (i.e. all particles touching or intersecting the prepared plane V_a), are captured by the replica and on the exposure we can observe actual diameters of every captured particle (see Fig. 1). Horálek and Vobořil [10] have shown that under the above assumptions the AE-model gives exact values α_1 and $\varkappa$ only for the lognormal particle size distribution $f(x)$. An exact and also general theoretical solution of this problem under the same physical assumptions as in the AE-model has been published by Horálek [8]. Horálek and Vobořil ([9], [10]) summarized practical experience with the AE-model and came to the conclusion that its application in evaluating real cases of precipitation processes in special type of metalic material leads to a strong bias of estimations of the parameters α_1 and $\varkappa$, caused first of all by non-realistic physical assumptions of the originalAE-model. Therefore they have suggested in [9] a modification of these assumptions on the decomposition of the relevant conditional particle size distribution into four distributions different from the point of view of physical properties of particles sectioned by the plane V_a. These four groups of particles form disjoint sets. The modification under consideration is in detail described in Chap. 2 of this paper. The fractions of these four groups of particles are derived in Chap. 3 and the analytical forms of relevant probability density functions are determined in Chap. 4. The application of this model to the extraction replica evaluation is the subject of Chap. 5.

2. FORMULATION OF PROBLEMS

Let us consider the three-dimensional euclidean space E_3 with coordinate axes a, b and c. Let $\mathbf{S}_\varkappa$ be a random spatial structure represented by spheres in E_3, the centres S_j $(j = 1, 2, \ldots)$ of the spheres being located at the points of a homogeneous Poisson process with a constant density $\varkappa$, while the sphere diameters ξ_j are random variables mutually independent and identically distributed with a continuous probability density function $f(x)$, $0 < x < \infty$, and with a finite second moment α_2 [5].

Let us consider a plane V_a defined by $a = 0$ and let $R_x \subset E_3$ be the set of all spheres $(S_j; \xi_j) \in \mathbf{S}_\varkappa$. Then $R_y = R_x \cap V_a$ is the set of all circular planar sections of spheres in the plane V_a that form a random planar structure. The diameters η_j of the sections are random variables mutually independent and identically distributed with a continuous probability density function $g(y)$.

Let A be an event occuring when the sphere $(S_j; \xi_j) \in \mathbf{S}_\varkappa$ is intercepted by the plane V_a, and let $\xi_{j,A}$ be the diameter of this sphere. This diameter can be expressed as a sum of two random variables

$$\xi_{j,A} = \psi_j + \omega_j ,$$

where ψ_j is the height of the sphere cap and ω_j that of the sphere remainder (in what follows, the indexes j will be omitted). We shall denote the corresponding probability density functions by $f_A(x)$, $k(s)$ and $t(w)$ respectively.

Let us consider another plane V_0 parallel with the plane V_a and defined by $a = -v_0$. Every sphere intercepted by V_a belongs into one of these four disjoint sets (see Fig. 2):

a) sphere of type "1": its centre lies above or on V_a and $\psi < v_0$;

b) the sphere of type "2": its centre lies under V_a and $\omega < v_0$;

c) the sphere of type "3a": its centre lies above or on V_a and $\psi \geqq v_0$;

d) the sphere of type "3b": its centre lies under V_a and $\omega \geqq v_0$.

Compare now the AE-model with ours from the physical point of view. In agreement with the experimental data ([9], [10]) we assume that the extraction concerns

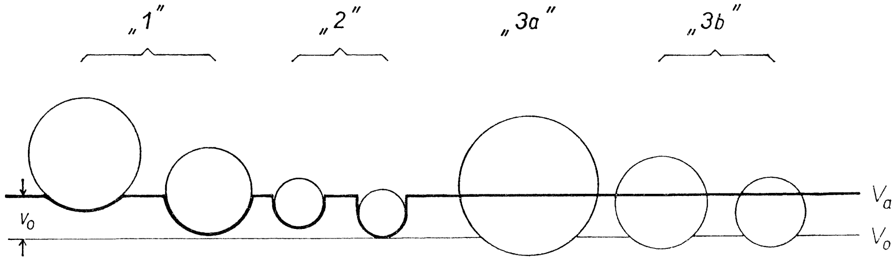

Fig. 2. Physical model of the modified extraction replica.

only those particles which left the surface after etching a small surface layer v_0 of the material or after underetching fine particles and remainders of them. In the exposures we can measure: for the particles of type "1", the actual diameter of the sphere cap embedded under the plane V_a, and for the particles of type "2", the actual diameter of the extracted particle; the particles of type "3" remain embedded in the specimen and the actual diameters of their planar sections in the plane V_a are observed for them. Therefore the relevant probability density functions will be denoted by $g_1^*(y)$, $f_2^*(x)$, $g_{3a}^*(y)$ and $g_{3b}^*(y)$, respectively.

Now we can formulate two basic problems solved in this paper:

– to determine the fractions r_1, r_2, r_{3a} and r_{3b} of spheres of the corresponding types into which the total distribution of particles intercepted by the plane V_a and having the p.d.f. $f_A(x)$ can be decomposed,

– to derive the p.d.f.'s $f_2^*(x)$, $g_1^*(y)$, $g_{3a}^*(y)$, $g_{3b}^*(y)$ of diameters of particles or their sections measured in the exposure.

3. DETERMINATION OF FRACTIONS r_1 TO r_{3b}

Theorem 1. *Let ξ_A be the diameter of a sphere belonging to a structure $\mathbf{S}_\varkappa$ and intercepted by the plane V_a. Then the p.d.f. $k(s)$ of the sphere cap height ψ $(0 < \psi < \xi_A/2)$ and the p.d.f. $t(w)$ of the sphere remainder height ω $(\xi_A/2 \leqq \omega < \xi_A)$ are equal to*

$$k(s) = \frac{2}{\alpha_1} \int_{2s}^{\infty} f(x)\,\mathrm{d}x \tag{3.1}$$

and

$$(3.2) \qquad t(w) = \frac{2}{\alpha_1} \int_w^{2w} f(x) \, dx \,,$$

where $\alpha_\nu = E(\xi^\nu)$, $\nu = 1, 2, \ldots$.

Proof. Let ζ_A be a random variable taking the value of the distance of the centre S of the sphere on which the event A occured, from V_a (see Fig. 3). Obviously, the relations

$$(3.3) \qquad \psi = 0{\cdot}5\xi_A - \zeta_A = 0{\cdot}5\xi_A(1 - \tau) = \xi_A \tau_0$$

hold, and the random variable

$$(3.4) \qquad \tau = \frac{2\zeta_A}{\xi_A}$$

has a uniform distribution on the interval $(0, 1)$ independent of ξ_A.

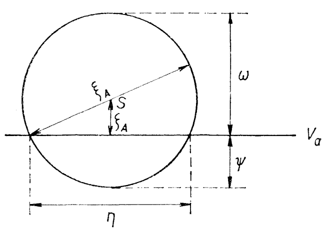

Fig. 3. Sphere $(S; \xi_A)$ intercepted by the plane V_a.

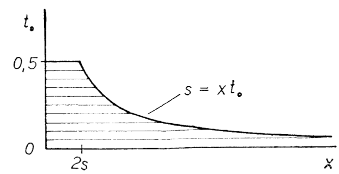

Fig. 4. Domain of integration used in (3.7).

The p.d.f. of the random variable ξ_A is equal to

$$(3.5) \qquad f_A(x) = \frac{x f(x)}{\alpha_1} \,,$$

so that the joint p.d.f. $f(x, t_0)$ of random variables ξ_A and τ_0 defined in (3.3) has the form

$$(3.6) \qquad f(x, t_0) = \frac{2x f(x)}{\alpha_1} \,, \quad 0 < x < \infty; \quad 0 < t_0 < 0{\cdot}5 \,.$$

Therefore the d.f. $K(s)$ of the random variable ψ can be expressed as

$$(3.7) \qquad K(s) = \mathsf{P}(\xi_A \tau_0 < s) = \int_0^{1/2} \left[\int_0^{s/t_0} f(x, t_0) \, dx \right] dt_0 \,.$$

The domain of integration, for which $x t_0 < s$ holds, is indicated in Fig. 4.

In a similar way we determine the d.f. $T(w)$ of the random variable ω,

$$\omega = \xi_A - \psi = \xi_A \frac{1 + \tau}{2} = \xi_A \tau_1 . \tag{3.8}$$

In this case we obtain

$$T(w) = \mathsf{P}(\xi_A \tau_1 < w) = \int_{1/2}^{1} \left[\int_0^{w/t_1} f(x, t_1)\, \mathrm{d}x \right] \mathrm{d}t_1 , \tag{3.9}$$

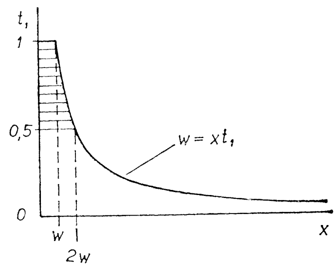

Fig. 5. Domain of integration used in (3.9).

where $f(x, t_1)$ is the joint p.d.f. of random variables ξ_A and τ_1, defined in (3.8). In Fig. 5 the corresponding domain of integration, for which $xt_1 < w$ holds, is hatched. By differentiating (3.7) and (3.9) with respect to s and w, respectively, we find (3.1) and (3.2).

Corollary 1.1. *The fractions r_1 to r_{3b} of the spheres of types* "1" *to* "3b" *are given by*

$$r_1 = 0{\cdot}5\, K(v_0) , \tag{3.10}$$

$$r_2 = 0{\cdot}5\, T(v_0) , \tag{3.11}$$

$$r_{3a} = 0{\cdot}5\, [1 - K(v_0)] , \tag{3.12}$$

$$r_{3b} = 0{\cdot}5\, [1 - T(v_0)] . \tag{3.13}$$

Proof. The subsets of the spheres of types "1", "2", "3a" and "3b" have no common element and their union coincides with the total system of events; therefore we have

$$r_1 + r_2 + r_{3a} + r_{3b} = 1 .$$

The fraction r_1 represents the probability that a sphere intercepted by the plane V_a and having its centre above or on this plane, has the height of sphere cap embedded under this plane smaller than v_0. In our model the sphere centres of type "1" lie only on one side from the plane V_a and not on both sides; this justifies the factor 0·5 in (3.10).

The equations (3.11) to (3.13) can be derived in an analogous way.

4. DETERMINATION OF PROBABILITY DENSITY FUNCTIONS

From the definitions of spheres of types "1" to "3b" it is clear that in order to determine the p.d.f.'s $g_1^*(y)$ and $g_{3a}^*(y)$ we need first to derive the joint probability density function $h(y, s)$ of random variables η and ψ; analogously, to determine the p.d.f.'s $f_2^*(x)$ and $g_{3b}^*(y)$ we must first find the joint p.d.f. $h(x, w)$ of the random variables ξ_A and ω and the joint p.d.f. $h(y, w)$ on the random variables η and ω, respectively.

Theorem 2. *Let $(S, \xi_A) \in \mathbf{S}_\varkappa$ be a sphere intercepted by the plane V_a, the diameter following the p.d.f. $f_A(x)$. Let ψ be the sphere cap height, ω the sphere remainder height and η the diameter of the planar section. Then the corresponding joint p.d.f.'s $h(y, s)$, $h(y, w)$ and $h(x, w)$ are given by*

$$h(y, s) = \frac{y}{\alpha_1 s} f\left(\frac{4s^2 + y^2}{4s}\right), \quad 0 < y < \infty, \quad 0 < s \leqq 0{\cdot}5y, \tag{4.1}$$

$$h(y, w) = \frac{y}{\alpha_1 w} f\left(\frac{4w^2 + y^2}{4w}\right), \quad 0 < y < \infty, \quad 0{\cdot}5y < w < \infty \tag{4.2}$$

and

$$h(x, w) = \frac{2}{\alpha_1} f(x), \quad 0 < x < \infty; \quad 0{\cdot}5x < w < x. \tag{4.3}$$

Proof. Besides (3.3) we also have the identity

$$\psi^2 - \psi\xi_A + 0{\cdot}25\eta^2 = 0.$$

The joint p.d.f. $h(x, t)$ of random variables ξ_A and τ, defined in (3.4), is equal to

$$h(x, t) = \frac{x f(x)}{\alpha_1}, \quad 0 < x < \infty; \quad 0 < t < 1. \tag{4.4}$$

Applying to (4.4) the transformation

$$x = \frac{4s^2 + y^2}{4s} \quad \text{and} \quad t = 1 - \frac{2s}{x} = \frac{y^2 - 4s^2}{y^2 + 4s^2}, \tag{4.5}$$

whose Jacobian is

$$|J| = \frac{4y}{4s^2 + y^2},$$

we get (4.1).

On the other hand, the random variables ξ_A, η and τ are related by the equation

$$\eta = \xi_A \sqrt{(1 - \tau^2)}; \tag{4.6}$$

this yields the transformation

$$x = \frac{4w^2 + y^2}{4w} \quad \text{and} \quad t = \frac{4w^2 - y^2}{4w^2 + y^2}\,.$$

Applying it to (4.4), we find the joint p.d.f. $h(y, w)$.

Finally, the application of the transformation

$$x = x \quad \text{and} \quad t = \frac{2w}{x} - 1\,,$$

the latter equation of which follows from (3.8), leads to the joint p.d.f. $h(x, w)$.

Corollary 2.1. *The p.d.f. $g_1^*(y)$ of the random variable η_1^* taking the value y of the planar section diameter of the sphere of type "1" is equal to*

$$g_1^*(y) = \frac{y}{\alpha_1\, K(v_0)} \int_y^\infty \frac{f(x)}{\sqrt{(x^2 - y^2)}}\, dx\,, \quad 0 < y < 2v_0\,, \tag{4.7a}$$

$$= \frac{y}{\alpha_1\, K(v_0)} \int_{c(y,v_0)}^\infty \frac{f(x)}{\sqrt{(x^2 - y^2)}}\, dx\,, \quad 2v_0 \leqq y < \infty\,, \tag{4.7b}$$

where

$$c(y, v_0) = \frac{4v_0^2 + y^2}{4v_0}\,. \tag{4.7c}$$

Proof. In view of (4.1), the p.d.f. $g(y)$ of the random variable η equals

$$g(y) = \int_0^{y/2} h(y, s)\, ds = \frac{y}{\alpha_1} \int_y^\infty \frac{f(x)}{\sqrt{(x^2 - y^2)}}\, dx\,, \quad y > 0\,, \tag{4.8}$$

where we have used the substitution for x following from (4.5) and taken into account the relation

$$\frac{4s}{y^2 - 4s^2} = \frac{1}{\sqrt{(x^2 - y^2)}}\,.$$

Our result (4.8) is in full agreement with the p.d.f. of planar section diameters derived previously by Wicksell [14], Fullman [3] and Horálek ([4], [5]) by other methods.

From the distributions of all planar section diameters we must now take into

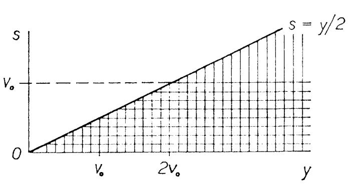

Fig. 6. Definition regions of the joint p.d.f. $h(y, s)$ from the point of view of the derivation of the p.d.f. $g_1^*(y)$.

account only those sections which belong to the spheres of type "1". In Fig. 6 the area corresponding to the regions of definition of the random variables η and ψ, connected by the relation (4.1), is hatched vertically. Its part hatched horizontally represents the spheres whose cap heights under or above the plane V_a are smaller than v_0.

Now from (4.8) and (3.10) we find the p.d.f. $g_1^*(y)$ of the random variable η_1^* in the form

$$g_1^*(y) = \frac{y}{\alpha_1 K(v_0)} \int_0^{y/2} f\left(\frac{4s^2 + y^2}{4s}\right) \frac{1}{s}\, ds\,, \quad 0 < y < 2v_0\,,$$

$$= \frac{y}{\alpha_1 K(v_0)} \int_0^{v_0} f\left(\frac{4s^2 + y^2}{4s}\right) \frac{1}{s}\, ds\,, \quad 2v_0 \leqq y < \infty\,.$$

The use of the same procedure as in (4.8) completes the proof of Corollary 2.1.

Corollary 2.2. *The p.d.f.* $f_2^*(x)$ *of the random variable* ξ_A^* *taking the value* x *of the sphere diameter of type "2" is equal to*

$$f_2^*(x) = \frac{x}{\alpha_1\, T(v_0)} f(x)\,, \quad 0 < x < v_0\,, \tag{4.9a}$$

$$= \frac{2v_0 - x}{\alpha_1 T(v_0)} f(x)\,, \quad v_0 \leqq x < 2v_0\,. \tag{4.9b}$$

Proof. From (4.3) we obtain

$$f_A(x) = \frac{x}{\alpha_1} f(x)\,, \quad 0 < x < \infty\,, \tag{4.10}$$

which agrees with (3.5).

From the total distribution of sphere diameters, intercepted by the plane V_a, we must consider only those whose properties are identical with those of the spheres

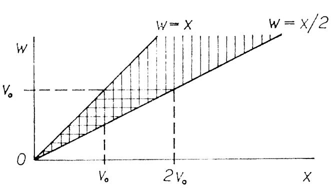

Fig. 7. Definition regions of the joint p.d.f. $h(x, w)$ from the point of view of the derivation of the p.d.f. $f_2^*(x)$.

of type "2". In Fig. 7 the area corresponding to the regions of definition of the random variables ξ_A and ω, connected by the equation (4.3), is hatched vertically. Its part hatched horizontally represents the spheres whose spherical remainder under or above the plane V_a is smaller than v_0.

Using (4.10) and (3.11) we can easily prove both the equations (4.9).

Corollary 2.3. *The p.d.f. $g^*_{3a}(y)$ of the random variable η^*_{3a} taking the value y of the planar section diameter of the sphere of type "3a" is equal to*

(4.11a) $$g^*_{3a}(y) = 0\,, \quad y \leqq 2v_0$$

(4.11b) $$= \frac{y}{\alpha_1[1 - K(v_0)]} \int_y^{c(y,v_0)} \frac{f(x)}{\sqrt{(x^2 - y^2)}}\, \mathrm{d}x\,, \quad 2v_0 < y < \infty\,.$$

where $c(y, v_0)$ has the form given in (4.7c).

Proof. The proof proceeds in the same way as that of Corollary 2.1. In Fig. 8 the area horizontally hatched represents the region covered by the spheres with the cap height under or above the plane V_a equal at least to v_0. Using (4.8) and (3.12) we obtain

$$g^*_{3a}(y) = \frac{y}{\alpha_1[1 - K(v_0)]} \int_{v_0}^{y/2} f\left(\frac{4s^2 + y^2}{4s}\right) \frac{1}{s}\, \mathrm{d}s\,, \quad 2v_0 < y < \infty\,.$$

The rest of the proof is clear.

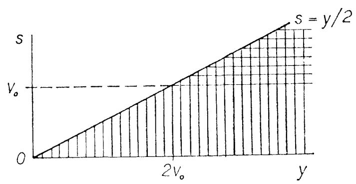

Fig. 8. Definition regions of the joint p.d.f. $h(y,s)$ from the point of view of the derivation of the p.d.f. $g^*_{3a}(y)$.

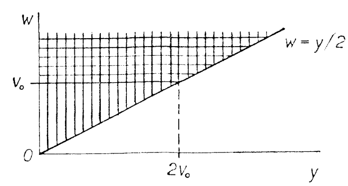

Fig. 9. Definition regions of the joint p.d.f. $h(y, w)$ from the point of view of the derivation of the p.d.f. $g^*_{3b}(y)$.

Corollary 2.4. *The p.d.f. $g^*_{3b}(y)$ of the random variable η^*_{3b} taking the value y of the planar section diameter of the sphere of type "3b" is equal to*

(4.12a) $$g^*_{3b}(y) = \frac{y}{\alpha_1[1 - T(v_0)]} \int_{c(y,v_0)}^{\infty} \frac{f(x)}{\sqrt{(x^2 - y^2)}}\, \mathrm{d}x\,, \quad 0 < y < 2v_0\,,$$

(4.12b) $$= \frac{y}{\alpha_1[1 - T(v_0)]} \int_y^{\infty} \frac{f(x)}{\sqrt{(x^2 - y^2)}}\, \mathrm{d}x\,, \quad 2v_0 \leqq y < \infty\,.$$

where $c(y, v_0)$ has the form given in (4.7c).

Proof. In Fig. 9 the area corresponding to the regions of definition of the random variables η and ω, connected by the relation (4.2), is hatched vertically. Its part hatched horizontally represents the spheres with the remainder height under or above the plane V_a equal at least to v_0, in agreement with the properties of the spheres of type "3b".

By means of (4.2) and (3.13) the p.d.f. $g^*_{3b}(y)$ may be found in the form

$$g^*_{3b}(y) = \frac{y}{\alpha_1[1 - T(v_0)]} \int_{v_0}^{\infty} f\left(\frac{4w^2 + y^2}{4w}\right) \frac{1}{w} \, \mathrm{d}w \,, \quad 0 < y < 2v_0 \,,$$

$$= \frac{y}{\alpha_1[1 - T(v_0)]} \int_{y/2}^{\infty} f\left(\frac{4w^2 + y^2}{4w}\right) \frac{1}{w} \, \mathrm{d}w \,, \quad 2v_0 \leqq y < \infty \,.$$

Using the same procedure as in (4.8) we can reduce these equations to the form (4.12).

5. APPLICATION

5.1 Problem formulation

From the negative of a microstructure photograph we can estimate the values of fractions

(5.1a) $$r_{12} = r_1 + r_2$$

of extracted particles and

(5.1b) $$r_3 = r_{3a} + r_{3b}$$

of sectioned (and remaining embedded) particles per unit test area. Further, we can measure the diameters of the white features (particles of type "1" and "2") and black features (particles of types "3") investigated. So we can calculate estimates of the first moments α^*_ν of the mixed population formed by white and black features and, using them, we can find estimates of the parameters of the p.d.f. $f(x)$ and especially of the basic spatial structure parameter, the mean value α_1. The value v_0 may be known, but usually it is unknown. In the solution given below we shall assume that $f(x)$ follows the Rayleigh distribution with a parameter μ.

5.2 Problem solution

Wicksell [14] proved that for the spherical particle diameter ξ following the Rayleigh distribution, the diameter η of a circular planar section of the sphere has the same distribution

$$g(y) = \frac{y}{\alpha_1} \int_y^{\infty} \frac{f(x)}{\sqrt{(x^2 - y^2)}} \, \mathrm{d}x = \frac{y}{\mu} \, \mathrm{e}^{-y^2/2\mu} \,, \quad \mu > 0 \,,$$

with $\mathbf{E}(\eta^\nu) = \beta_\nu$, $\nu = 1, 2, \ldots$. Hence the relevant moments α_ν and β_ν coincide.

The first moment of the mixed population investigated, described in section 5.1, can be written in the form

(5.2) $$\alpha^*_1 = r_1 \int_{Q_1} y \, g^*_1(y) \, \mathrm{d}y + r_2 \int_{Q_2} x \, f^*_2(x) \, \mathrm{d}x + r_{3a} \int_{Q_{3a}} y \, g^*_{3a}(y) \, \mathrm{d}y + r_{3b} \int_{Q_{3b}} y \, g^*_{3b}(y) \, \mathrm{d}y \,,$$

where Q_i $(i = 1, 2, 3a, 3b)$ are the definition regions of the corresponding probability density functions.

Inserting (3.10) into (3.13) and (4.7), (4.9), (4.11) and (4.12) into (5.2) we get

$$\alpha_1^* = \frac{1}{2\alpha_1}\left\{\int_0^{2v_0} y^2 \int_y^\infty \frac{f(x)}{\sqrt{(x^2 - y^2)}}\,dx\,dy + \int_{2v_0}^\infty y^2 \int_{c(y,v_0)}^\infty \frac{f(x)}{\sqrt{(x^2 - y^2)}}\,dx\,dy + \right.$$

$$+ \int_0^{v_0} x^2 f(x)\,dx + \int_{v_0}^{2v_0} (2v_0 - x)\,x f(x)\,dx +$$

$$+ \int_{2v_0}^\infty y^2 \int_y^{c(y,v_0)} \frac{f(x)}{\sqrt{(x^2 - y^2)}}\,dx\,dy + \int_0^{2v_0} y^2 \int_{c(y,v_0)}^\infty \frac{f(x)}{\sqrt{(x^2 - y^2)}}\,dx\,dy +$$

$$\left. + \int_{2v_0}^\infty y^2 \int_y^\infty \frac{f(x)}{\sqrt{(x^2 - y^2)}}\,dx\,dy\right\}.$$

The above introduced property of the Rayleigh distribution enables us to rewrite the last equation as follows:

$$\alpha_1^* = \frac{1}{2}\left\{\beta_1 + \int_{2v_0}^\infty y\,g(y)\,dy + \frac{2v_0}{\alpha_1}\int_{v_0}^{2v_0} x f(x)\,dx + \right. \tag{5.3}$$

$$+ \frac{1}{\alpha_1}\int_0^{v_0} x^2 f(x)\,dx - \frac{1}{\alpha_1}\int_{v_0}^{2v_0} x^2 f(x)\,dx +$$

$$\left. + \frac{1}{\alpha_1}\int_0^{2v_0} y^2 \int_{c(y,v_0)}^\infty \frac{f(x)}{\sqrt{(x^2 - y^2)}}\,dx\,dy\right\}.$$

Hence it is clear that $\alpha_1^* = \beta_1$ for $v_0 = 0$ and therefore the introduced equality between the moments holds. On the other hand, $K(0) = T(0) = 0$; therefore $r_1 = r_2 = 0$ and $r_{3a} = r_{3b} = 0{\cdot}5$, i.e. the sets of types "1" and "2" are empty. The particles intercepted by the plane V_a are only intersected.

For $v_0 \to \infty$ we get from (5.3)

$$\lim_{v_0 \to \infty} \alpha_1^* = \frac{1}{2}\left(\beta_1 + \frac{\alpha_2}{\alpha_1}\right). \tag{5.4}$$

In the author's paper [8] it is shown that the ν-th moment α_ν^{**} of the distribution of the extracted particle diameters obtained under the physical conditions (see Fig. 1) suggested by Ashby and Ebeling [1] is given by

$$\alpha_\nu^{**} = \frac{\alpha_{\nu+1}}{\alpha_1}. \tag{5.5}$$

Hence we have

$$\lim_{v_0 \to \infty} \alpha_1^* = \tfrac{1}{2}(\beta_1 + \alpha_1^{**}).$$

On the other hand, $K(\infty) = T(\infty) = 1$; therefore $r_1 = r_2 = 0{\cdot}5$ and $r_{3a} = r_{3b} = 0$, i.e. in this limit case, from all particles intercepted by the plane V_a exactly one half of them belongs to the set of type "1" while the other half belongs to the set of type "2"; the sets "3*a*" and "3*b*" are empty.

From these two investigated cases ($v_0 = 0$ and $v_0 \to \infty$) we can see that for every fixed v_0 ($0 < v_0 < \infty$) the first moment $\alpha_1^*(v_0)$ satisfies the inequalities

$$\beta_1 < \alpha_1^*(v_0) < \tfrac{1}{2}(\beta_1 + \alpha_1^{**}) .$$

Further, the integrals in (5.3) can be expressed in the form

$$\int_{2v_0}^{\infty} y\, g(y)\, \mathrm{d}y = \alpha_1 \left[1 - \mathsf{P}\left(\frac{4v_0^2}{\mu} \mid 3\right)\right],$$

$$\frac{2v_0}{\alpha_1} \int_{v_0}^{2v_0} x\, f(x)\, \mathrm{d}x = 2v_0 \left[\mathsf{P}\left(\frac{4v_0^2}{\mu} \mid 3\right) - \mathsf{P}\left(\frac{v_0^2}{\mu} \mid 3\right)\right],$$

$$\frac{1}{\alpha_1} \int_0^{v_0} x^2 f(x)\, \mathrm{d}x = \frac{4\alpha_1}{\pi}\, \mathsf{P}\left(\frac{v_0^2}{\mu} \mid 4\right),$$

$$\frac{1}{\alpha_1} \int_{v_0}^{2v_0} x^2 f(x)\, \mathrm{d}x = \frac{4\alpha_1}{\pi} \left[\mathsf{P}\left(\frac{4v_0^2}{\mu} \mid 4\right) - \mathsf{P}\left(\frac{v_0^2}{\mu} \mid 4\right)\right],$$

where $\mathsf{P}(\chi^2 \mid k)$ is the χ^2-distribution function with k degrees of freedom. The last integral in (5.3) can be written in the form

$$\frac{1}{\alpha_1} \int_0^{2v_0} y^2 \int_{c(y,v_0)}^{\infty} \frac{f(x)}{\sqrt{(x^2 - y^2)}}\, \mathrm{d}x\, \mathrm{d}y =$$

$$= \alpha_1 \mathsf{P}\left(\frac{4v_0^2}{\mu} \mid 3\right) - \sqrt{\left(\frac{2\pi}{\mu}\right)} \frac{1}{\alpha_1} \int_0^{2v_0} y^2 e^{-y^2/2\mu}\, \Phi\left(\frac{4v_0^2 - y^2}{4v_0 \sqrt{\mu}}\right) \mathrm{d}y$$

and the second term can be calculated by numerical integration.

The relationship between α_1^* and α_1 is displayed in Fig. 10 for various levels of v_0 in the range from $2 \,.\, 10^{-4}$ to 10^{-3}. The straight line α_1^{**} has the analytical form

$$\alpha_1^* = \frac{4}{\pi}\, \alpha_1 ,$$

which follows from (5.5) and from

$$\alpha_\nu = (2\mu)^{\nu/2} \Gamma\left(\frac{\nu + 2}{2}\right), \quad \nu > -2 ,$$

which is valid for the Rayleigh distribution.

In practice it is very difficult to determine the depth of the etching v_0. The knowl-

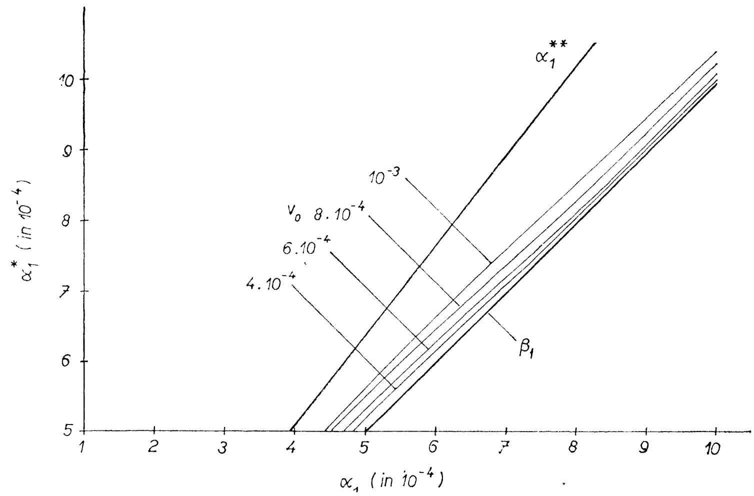

Fig. 10a. Graphical representation of the function $\alpha_1^*(\alpha_1; v_0)$ — part 1.

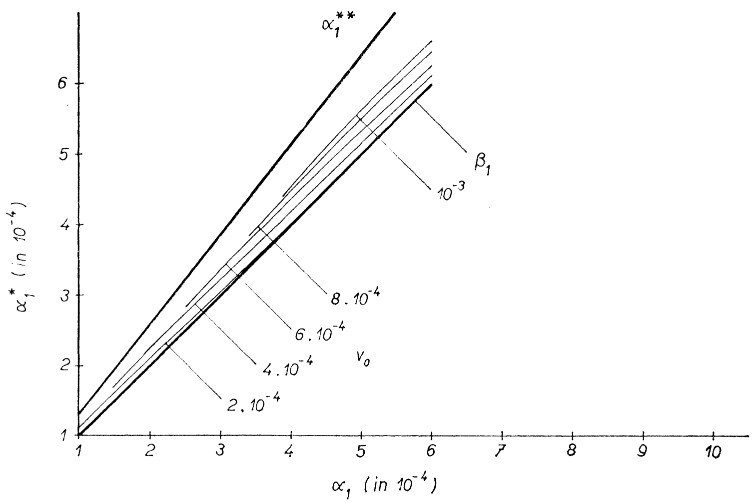

Fig. 10b. Graphical representation of the function $\alpha_1^*(\alpha_1; v_0)$ — part 2.

edge of fractions r_{12} and r_3, defined in (5.1), enables us to overcome this disadvantage. In Fig. 11 the fractions r_{12} and r_3 as functions of α_1 for various levels of v_0 are plotted. They were calculated by means of (3.10) to (3.13). When $f(x)$ follows the Rayleigh distribution, they become

$$r_{12} = 2\Phi\left(\frac{v_0}{\alpha_1}\sqrt{\frac{\pi}{2}}\right) - 1$$

and

$$r_3 = 2\left[1 - \Phi\left(\frac{v_0}{\alpha_1}\sqrt{\frac{\pi}{2}}\right)\right],$$

where $\Phi(z)$ is the normal distribution function. From Fig. 11 we can see a strong dependence of fractions r_{12} and r_3 on the actual mean value α_1 and on the depth of the etching v_0.

For v_0 unknown we can proceed in the following steps, using the knowledge of estimates of $\hat{\alpha}_1^*$ and $\hat{r}_{12}$ and $\hat{r}_3$, respectively, obtained as results of the measurement

a) we find the values $\alpha_1^* = \hat{\alpha}_1^*$ in Fig. 10 and the corresponding value $r_{12} = \hat{r}_{12}$ or $r_3 = \hat{r}_3$ in Fig. 11;

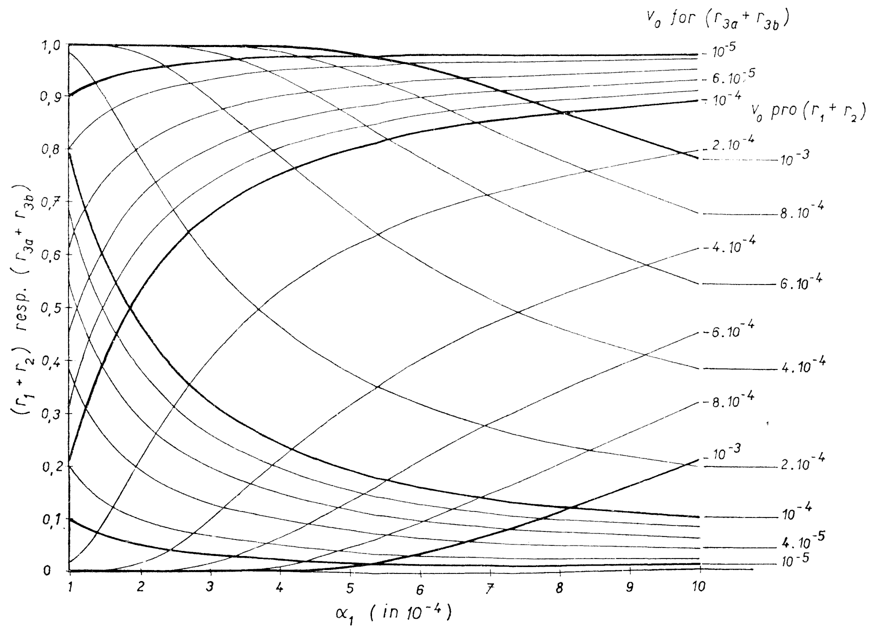

Fig. 11. Graphical representation of the functions $r_{12}(\alpha_1; v_0)$ or $r_3(\alpha_1; v_0)$.

b) in both figures we follow the relevant parallel straight line with α_1 until we reach the same level v_0 in both charts; for this value v_0 we read off the corresponding mean value of the particle diameter on the axis α_1.

In practice we take as boundary levels of v_0 those which limit an interval covering the observed value $\hat{\alpha}_1^*$ on the axis α_1 in Fig. 11 for the observed value $\hat{r}_{12}$ or $\hat{r}_3$, respectively. For these two levels, which we shall denote by v_{01} and v_{02}, we read off the corresponding values α_1 in Figs. 10 and 11 and we denote them by $({}_1\alpha_1^{(0)}$ and ${}_2\alpha_1^{(0)})$ and $({}_1\alpha_1^{(1)}$ and ${}_2\alpha_1^{(1)})$, respectively. Now in the plane $(\alpha_1^{(0)}, \alpha_1^{(1)})$ we construct a straight line passing through the points $({}_1\alpha_1^{(0)}, {}_1\alpha_1^{(1)})$ and $({}_2\alpha_1^{(0)}, {}_2\alpha_1^{(1)})$. The point of intersection of this straight line with the straight line $\alpha_1^{(1)} = \alpha_1^{(0)}$gives the chosen value $\hat{\alpha}_1$. Clearly, this value satisfies the identity

$$(5.6) \qquad \hat{\alpha}_1 = \frac{{}_2\alpha_1^{(1)} \, . \, {}_1\alpha_1^{(0)} - {}_1\alpha_1^{(1)} \, . \, {}_2\alpha_1^{(0)}}{{}_1\alpha_1^{(0)} + {}_2\alpha_1^{(1)} - {}_2\alpha_1^{(0)} - {}_1\alpha_1^{(1)}} \, .$$

5.3 Numerical example

For the material studied the values

$$\hat{\alpha}_1^* = 8 \, . \, 10^{-4} \, (\text{in mm}) \quad \text{and} \quad \hat{r}_3 = 0{\cdot}25$$

were calculated. From Fig. 11 we can see that the observed value $\hat{\alpha}_1^* = 8 \, . \, 10^{-4}$ is covered by the interval

$$\langle {}_1\alpha_1^{(1)} = 6{\cdot}5 \, . \, 10^{-4}; \; {}_2\alpha_1^{(1)} = 8{\cdot}73 \, . \, 10^{-4} \rangle$$

corresponding to the points of intersection of the straight line $\hat{r}_3 = 0{\cdot}25$ with isolines $v_{01} = 6 \, . \, 10^{-4}$ and $v_{02} = 8 \, . \, 10^{-4}$. From Fig. 10 we read off the values

$${}_1\alpha_1^{(0)} = 7{\cdot}70 \, . \, 10^{-4} \quad \text{and} \quad {}_2\alpha_1^{(0)} = 7{\cdot}90 \, . \, 10^{-4} \, .$$

Inserting these four values ${}_i\alpha_1^{(j)}$ $(i = 1, 2; \; j = 0, 1)$ into (5.6), we get $\hat{\alpha}_1 = 7{\cdot}8 \, . \, 10^{-4}$.

Using an earlier model constructed for pure extraction and independent of v_0 (see the straight line α_1^{**} in Fig. 10), we find for the same value of α_1^* that it is equal to $6{\cdot}30 \, . \, 10^{-4}$. In comparison with our result $\hat{\alpha}_1 = 7{\cdot}8 \, . \, 10^{-4}$ we can see a very substantial underestimating of the actual mean value.

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Souhrn

O ROZKLADU ROZDĚLENÍ VELIKOSTI ČÁSTIC PŘI METODĚ EXTRAKČNÍ REPLIKY

VRATISLAV HORÁLEK

Při hodnocení struktur niklových slitin se setkáváme se snímky, které obsahují jak extrahované, tak otištěné částice. Extrahovány jsou pouze jemné částice a vrchlíky částic, které leží při povrchu připravené roviny a po podleptání jsou uvolněny. V práci je předložen model, který umožňuje pro tento typ informace o prostorové struktuře stanovit neznámou střední hodnotu průměru kulových částic, náhodně rozmístěných v prostoru vzorku, a neznámou hloubku podleptání. Model je založen na rozkladu rozdělení velikosti částic zasažených připravenou rovinou. Pro případ Rayleighova rozdělení průměru částic je výsledné řešení úlohy podáno grafickou formou.

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