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RANKING AND SELECTION PROCEDURES FOR LOCATION PARAMETER CASE BASED ON L-ESTIMATES

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1. GENERAL BACKGROUND

There are many kinds of ranking and selection problems. Generally speaking, the goal is to select some "good" populations from given k populations $\pi_1, \ldots, \pi_k$. In this paper we will consider two problems: (a) selecting t "best " populations regardless of their order and (b) selecting t "best" populations with regard to their order $(1 \leqq t \leqq k - 1)$. (The complete ranking of all k populations is of course included in (b), namely for $t = k - 1$.) We will deal only with the special case of the location parameter and continuous distribution functions. This means, we shall assume that the population π_i has a distribution function $F_i(x) = F(x - \theta_i)$, $i = 1, \ldots, k$, $F(x)$ is continuous and the "bestness" of a population π_i is characterized by its location parameter θ_i, the best population being the one with the largest location parameter, etc.

Given k random samples, $X_{i1}, \ldots, X_{in}$ from π_i $(i = 1, \ldots, k)$, all of the same sample size n, a selection procedure is usually based on some statistics $Y_i = Y_i(X_{i1}, \ldots, X_{in})$ with a distribution depending on θ_i. We call a *selection correct* (CS), if the selected populations are really the best ones, respectively with the right order. We seek for such a procedure that the probability of correct selection exceeds a preassigned value $P^* < 1$ when $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$ lies in a certain subset D of the parameter space $\Omega = \{\boldsymbol{\theta}\}$, called the preference zone. Or, more specially, we demand

$$\inf_{\boldsymbol{\theta} \in D} \mathsf{P}\{\mathrm{CS}\} = P^* \tag{1}$$

and from this equation we determine the sample size n. The parameter vector $\boldsymbol{\theta} \in D$ for which $\mathsf{P}\{\mathrm{CS}\}$ attains its infimum on D is called the *least favorable configuration* (LFC) (if it exists). Then (1) can be written in the form

$$\mathsf{P}_{\mathrm{LFC}}\{\mathrm{CS}\} = P^* . \tag{2}$$

Let us denote by $\theta_{[1]} \leqq \theta_{[2]} \leqq \ldots \leqq \theta_{[k]}$ the ordered parameter values $\theta_1, \ldots, \theta_k$, by $\pi_{[1]}, \pi_{[2]}, \ldots, \pi_{[k]}$ the respective populations, by $Y_{(1)}, Y_{(2)}, \ldots, Y_{(k)}$ the respective statistics $Y_1, \ldots, Y_k$ and by $Y_{[1]} \leqq Y_{[2]} \leqq \ldots \leqq Y_{[k]}$ the ordered values of these statistics. (Of course, we do not know the correspondence between $\pi_1, \ldots, \pi_k$ and $\pi_{[1]}, \ldots, \pi_{[k]}$.) For problem (a), it is usual to put

$$D = \{\boldsymbol{\theta} \in \Omega : \theta_{[k-t]} \leqq \theta_{[k-t+1]} - \delta^*\}, \tag{3}$$

where $\delta^* > 0$ is to be chosen in advance, and for problem (b),

$$D = \{\boldsymbol{\theta} \in \Omega : \theta_{[i-1]} \leqq \theta_{[i]} - \delta^*, \ i = k - t + 1, \ldots, k\}. \tag{4}$$

Various nonsequential techniques for ranking and selection of populations were developed during the last 25 years. For F normal, there are Bechhofer procedures based on sample means $Y_i = n^{-1} \sum_{j=1}^{n} X_{ij}$ (see [2] or [10]). The LFC's for them are the so called slippage configurations:

$$\theta_{[1]} = \ldots = \theta_{[k-t]} = \theta_{[k-t+1]} - \delta^* = \ldots = \theta_{[k]} - \delta^* \tag{5}$$

for problem (a) and

$$\begin{gathered} \theta_{[1]} = \ldots = \theta_{[k-t]}, \\ \theta_{[i-1]} = \theta_{[i]} - \delta^*, \quad i = k - t + 1, \ldots, k \end{gathered} \tag{6}$$

for problem (b). For F continuous, procedures based on ranks of X_{ij}'s among $X_{11}, \ldots, X_{1n}, X_{21}, \ldots, X_{kn}$ (i.e., procedures based on the so called R-estimates) were also studied (among others). The authors of the paper [10] came to the conclusion that these "nonparametric" procedures were more robust in terms of the asymptotic relative efficiency (ARE) than the corresponding "parametric" procedures, the ARE of two proceduras being defined as the limiting ratio of the sample sizes required to ensure (1). However, the ARE's (both for (a) and (b)) were derived under the restrictive assumption $\theta_{[k]} - \theta_{[1]} = O(n^{-1/2})$; in this case slippage configurations are LFC's at least asymptotically. Later it was shown (see [11]) that generally slippage configurations need not be LFC's for procedures based on ranks, even asymptotically. It follows that for these procedures the ARE's are generally not known and the infimum of $\mathsf{P}\{\mathrm{CS}\}$ over D is not controlled even asymptotically.

2. L-ESTIMATES

For the above-mentioned reasons, attempts to find robust procedures of other types have been made. One of many possibilities is to take linear functions of order statistics (the so called L-estimates) for Y_i's, i.e. to put

$$Y_i = S_i = \sum_{j=1}^{n} \lambda_j X_{i[j]}, \quad i = 1, \ldots, k,$$

where $X_{i[1]} \leqq X_{i[2]} \leqq \ldots \leqq X_{i[n]}$ are the ordered values $X_{i1}, \ldots, X_{in}$ and $\lambda_1, \ldots, \lambda_n$ are some suitably chosen coefficients. We shall suppose that these coefficients are generated by a weight function $J(u) \geqq 0$, $u \in (0, 1)$, such that $\int_0^1 J(u)\,\mathrm{d}u = 1$; namely,

$$\lambda_j = \frac{1}{n} J\left(\frac{j}{n+1}\right), \quad j = 1, \ldots, n\,.$$

We put $\lambda = \lambda^{(n)} = \sum_{j=1}^{n} \lambda_j$ and denote by $G(y; \theta_i)$ the distribution function of S_i. Then we have the following result:

Theorem 1. *The system of distribution functions $G(y; \theta)$ is stochastically increasing, i.e.*

$$\theta < \theta' \Rightarrow G(y; \theta) \geqq G(y; \theta') \quad \text{for all} \quad y\,.$$

Further, $G(y; \theta)$ is continuous both in y and in θ.

Proof. Using the notation $x_{[1]} \leqq x_{[2]} \leqq \ldots \leqq x_{[n]}$ for ordered values $x_1, \ldots, x_n$ and $\boldsymbol{r} = (r_1, \ldots, r_n)$ for any permutation of numbers $1, \ldots, n$, we get

$$G(y; \theta_i) = \int_{\sum_{j=1}^{n} \lambda_j x_{[j]} \leqq y} \ldots \int \mathrm{d}F(x_1 - \theta_i) \ldots \mathrm{d}F(x_n - \theta_i) = \sum_{\boldsymbol{r}} \int_{\substack{x_{r_1} \leqq \ldots \leqq x_{r_n} \\ \Sigma \lambda_j x_{r_j} \leqq y - \lambda\theta_i}} \ldots \int \mathrm{d}F(x_1) \ldots \mathrm{d}F(x_n)$$

and after the transformations $x_{r_j} = x'_j$, $j = 1, \ldots, n$ (for each $\boldsymbol{r}$),

$$G(y; \theta_i) = G(y - \lambda\theta_i)\,,$$

where

$$G(y) = n! \int_{\substack{x_1 \leqq \ldots \leqq x_n \\ \Sigma \lambda_j x_j \leqq y}} \ldots \int \mathrm{d}F(x_1) \ldots \mathrm{d}F(x_n)\,. \tag{7}$$

The first statement of the theorem follows from the fact that $\lambda > 0$ and the rest immediately from the identity

$$n! \int_{\substack{x_1 \leqq \ldots \leqq x_n \\ \Sigma \lambda_j x_j = y}} \ldots \int \mathrm{d}F(x_1) \ldots \mathrm{d}F(x_n) = 0\,,$$

which holds for every y.

Remark 1. As to the choice of the weight function $J(u)$, the condition

$$\mu(J, F) = \int_0^1 J(u)\, F^{-1}(u)\,\mathrm{d}u = 0 \tag{8}$$

ensures the asymptotical unbiasedness of the estimates, since according to Theorem 3 of [12] (under some conditions)

$$\lim_{n \to \infty} \mathsf{E} S_i = \mu(J, F_i) = \theta_i + \mu(J, F)\,.$$

But (8) is not necessary for our purposes, because the bias $\mu(J, F)$ is the same for all $S_1, \ldots, S_k$. If $F(x)$ has the density $f(x)$, then the choice

$$J(u) = \frac{\mathrm{d}\varphi(u, f)}{\mathrm{d}u} f(F^{-1}(u)) \left[\int_0^1 \frac{\mathrm{d}\varphi(v, f)}{\mathrm{d}v} f(F^{-1}(v))\, \mathrm{d}v \right]^{-1},$$

where

$$\varphi(u, f) = -\frac{f'(F^{-1}(u))}{f(F^{-1}(u))},$$

yields an asymptotically efficient estimate (in the sense of Cramér) (see [8]). But this weight function is nonnegative only if the density $f(x)$ is strongly unimodal. Of special interest from the point of view of robustness is the α-trimmed mean corresponding to the weight function

$$\begin{aligned} J(u) &= \frac{1}{1 - 2\alpha} \quad \text{for} \quad u \in \langle \alpha, 1 - \alpha \rangle \\ &= 0 \quad \text{otherwise}\,, \end{aligned} \tag{9}$$

since it limits the influence of outlying observations. In the concluding section, we give some numerical examples of ARE of procedure based on α-trimmed means relative to the Bechhofer procedure based on sample means.

3. PROBLEM (a)

We suggest a selection procedure for problem (a) based on L-estimates consisting in selecting the t populations associated with the t largest values $S_{[k-t+1]}, \ldots, S_{[k]}$ of $S_1, \ldots, S_k$. Then

$$\mathsf{P}\{\mathrm{CS}\} = \mathsf{P}\{\max(S_{(1)}, \ldots, S_{(k-t)}) < \min(S_{(k-t+1)}, \ldots, S_{(k)})\}\,. \tag{10}$$

Searching for infimum $\mathsf{P}\{\mathrm{CS}\}$ on D (and for LFC), we can apply the theorem of [1] saying that (10) is a nonincreasing function of $\theta_{[1]}, \ldots, \theta_{[k-t]}$ and a nondecreasing function of $\theta_{[k-t+1]}, \ldots, \theta_{[k]}$. (The assumptions of the theorem are fulfilled by Theorem 1.) So we have (see [1])

$$\inf_{\theta \in D} \mathsf{P}\{\mathrm{CS}\} = \inf_{\theta} \mathsf{Q}(\theta)\,,$$

where

$$\begin{aligned} \mathsf{Q}(\theta) &= \mathsf{P}\{\mathrm{CS}/\theta_{[1]} = \ldots = \theta_{[k-t]} = \theta,\ \theta_{[k-t+1]} = \ldots = \theta_{[k]} = \theta + \delta^*\} = \\ &= t \int_{-\infty}^{\infty} G^{k-t}(y + \lambda\delta^*)\,[1 - G(y)]^{t-1} \mathrm{d}G(y) \end{aligned}$$

in fact does not depend on θ. Hence LFC is the slippage configuration (5) and, given $\delta^* > 0$, the sample size is determined from the equation

$$\mathsf{P}_{\text{LFC}}\{\text{CS}\} = t\int_{-\infty}^{\infty} G^{k-t}(y + \lambda^{(n)}\delta^*)\,[1 - G(y)]^{t-1}\,\mathrm{d}G(y) = P^* . \tag{11}$$

(One must keep in mind the dependence of G on n.)

Further, we shall prove a theorem analogous to Lemmas 4A.1 and 4B.1 in [10], which gives a large sample solution of the sample size problem and enables us to find ARE of the investigated procedure relative to the Bechhofer procedure. To this end, we shall impose some of the following conditions on the functions $J(u)$ and $F(x)$, which guarantee the applicability of the results of [12].

(A) $J(u)$ is bounded on $(0,1)$.

(B) $J(u)$ is continuous a.e. F^{-1}.

(C) $J(u)$ satisfies the Hölder condition with $\beta > \frac{1}{2}$ except possibly at a finite number of points of F^{-1} measure zero.

(D) $J(u) = 0$ for $u \in (0, \alpha)$ and $u \in (1 - \alpha, 1)$, where $0 < \alpha < \frac{1}{2}$.

(E) $\displaystyle\int_{-\infty}^{\infty} x^2\,\mathrm{d}F(x) < \infty.$

(F) $\displaystyle\lim_{x\to\infty} x^{\gamma}[1 - F(x) + F(-x)] = 0$ for some $\gamma > 0$.

Let us now consider the k-sample problem described in Section 1 with an increasing value of the sample size n. P^* being fixed, write $\delta^{(n)}$ for δ^* and

$$\theta^{(n)}_{[1]} = \ldots = \theta^{(n)}_{[k-t]} = \theta^{(n)}_{[k-t+1]} - \delta^{(n)} = \ldots = \theta^{(n)}_{[k]} - \delta^{(n)} \tag{12}$$

for the respective LFC. We first prove a lemma.

Lemma 1. *For P^* fixed, let $\delta^{(n)}$ be such that* (11) *holds with $\delta^* = \delta^{(n)}$; $n = 1, 2, \ldots$. Let the conditions* (A), (B), (E) *or the conditions* (A), (B), (D), (F) *be fulfilled. Then* $\lim_{n\to\infty} \delta^{(n)} = 0$.

Proof. Suppose the assertion of the lemma is not true. Then there exist a number $\varepsilon > 0$ and a subsequence $\{m_n\}$, $m_n \geqq n$, so that $\delta^{(m_n)} \geqq \varepsilon$ for all n. Let us put $\tilde{S}_i = \sum_{j=1}^{n} \lambda_j \tilde{X}_{ij}$, where $\tilde{X}_{ij} = X_{ij} - \theta^{(n)}_i$. Random variables $\tilde{X}_{ij}$ $(i = 1, \ldots, k;\, j = 1, \ldots, n)$ are i.i.d. with the distribution function $F(x)$ and also $S_1, \ldots, S_k$ are i.i.d. From (11), (10) and the fact that LFC is the slippage configuration (5) we get

$$\begin{aligned} P^* &= \mathsf{P}\{\max_{1\leqq i\leqq k-t} (\tilde{S}_{(i)} - \mathsf{E}\tilde{S}_{(i)}) \leqq \min_{k-t+1\leqq l\leqq k} (\tilde{S}_{(l)} - \mathsf{E}\tilde{S}_{(l)}) + \delta^{(n)}\lambda^{(n)}\} \geqq \\ &\geqq \mathsf{P}\{|\tilde{S}_i - \mathsf{E}\tilde{S}_i| \leqq \tfrac{1}{2}\delta^{(n)}\lambda^{(n)},\ i = 1, \ldots, k\} \geqq \\ &\geqq \left[1 - 4\frac{\operatorname{Var}\tilde{S}_1}{(\delta^{(n)}\lambda^{(n)})^2}\right]^k \end{aligned}$$

and passing to the subsequence $\{m_n\}$,

$$P^* \geqq \left[1 - \frac{4}{m_n} \cdot \frac{m_n \operatorname{Var} \tilde{S}_1}{(\varepsilon \lambda^{(m_n)})^2}\right]^k \tag{13}$$

for all n. But $\lim_{n \to \infty} \lambda^{(m_n)} = \int_0^1 J(u)\, du = 1$ and

$$\lim_{m_n \to \infty} m_n \operatorname{Var} \tilde{S}_1 = K(J, F),$$

where

$$K(J, F) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J[F(x)]\, J[F(y)]\, [F(\min(x, y)) - F(x) F(y)]\, dx\, dy \tag{14}$$

is finite: first, if the second moment of F is finite, then according to Lemma 2 of [9]

$$\sigma^2(F) = \operatorname{Var} \tilde{X}_{ij} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [F(\min(x, y)) - F(x) F(y)]\, dx\, dy < \infty$$

and it remains to use the boundedness of $J(u)$; second, if $J(u)$ trims the extremes, then in (14) we integrate only over the bounded area

$$\langle F^{-1}(\alpha), F^{-1}(1 - \alpha)\rangle \times \langle F^{-1}(\alpha), F^{-1}(1 - \alpha)\rangle .$$

Thus

$$\lim_{n \to \infty} \left[1 - \frac{4}{m_n} \cdot \frac{m_n \operatorname{Var} \tilde{S}_1}{(\varepsilon \lambda^{(m_n)})^2}\right]^k = 1$$

and (13) leads to a contradiction with the assumption $P^* < 1$.

Theorem 2. *For P^* fixed, let $\delta^{(n)}$ be such that (11) holds with $\delta^* = \delta^{(n)}$; $n = 1, 2, \ldots$. Let the conditions (A), (C), (E) or (A), (C), (D), (F) be fulfilled. Then as $n \to \infty$,*

$$\delta^{(n)} = \delta [K(J, F)]^{1/2}\, n^{-1/2} + o(n^{-1/2}), \tag{15}$$

where $K(J, F)$ is given by (14), δ is determined by the condition

$$P^* = t Q_{k-1}(\underbrace{\delta\, 2^{-1/2}, \ldots, \delta\, 2^{-1/2}}_{(k - t) \text{ times}}, \underbrace{0, \ldots, 0}_{(t - 1) \text{ times}}) \tag{16}$$

and Q_{k-1} is the distribution function of a normally distributed vector $(U_1, \ldots, U_{k-t}, W_{k-t+1}, \ldots, W_{k-1})$ with

$$\mathsf{E} U_i = \mathsf{E} W_l = 0\,, \quad \operatorname{Cov}(U_i, U_{i'}) = \tfrac{1}{2}(\delta_{ii'} + 1)\,,$$

$$\operatorname{Cov}(W_l, W_{l'}) = \tfrac{1}{2}(\delta_{ll'} + 1),\ \operatorname{Cov}(U_i, W_l) = -\tfrac{1}{2}\,,$$

$$i, i' = 1, \ldots, k - t\,, \quad l, l' = k - t + 1, \ldots, k - 1$$

($\delta_{ii'}$ is the Kronecker symbol).

Proof. Similarly as in the proof of Lemma 1, we introduce the variables $\tilde{X}_{ij}$ and $\tilde{S}_i$ and we start from (11) and (10):

$$P^* = \sum_{r=k-t+1}^{k} \mathsf{P}_{\mathrm{LFC}}\{S_{(i)} - S_{(r)} \leqq 0\,, \quad i = 1, \ldots, k - t\,;$$

$$S_{(r)} - S_{(l)} \leqq 0\,, \quad l = k - t + 1, \ldots, k;\ l \neq r\} =$$

$$= \sum_{r=k-t+1}^{k} \mathsf{P}\{n^{1/2}[2K(J, F)]^{-1/2} (\tilde{S}_{(i)} - \mathsf{E}\tilde{S}_{(i)} - \tilde{S}_{(r)} + \mathsf{E}\tilde{S}_{(r)}) \leqq$$

$$\leqq \delta^{(n)}\lambda^{(n)}\, n^{1/2}[2K(J, F)]^{-1/2}\,, \quad i = 1, \ldots, k - t\,;$$

$$n^{1/2}[2K(J, F)]^{-1/2} (\tilde{S}_{(r)} - \mathsf{E}\tilde{S}_{(r)} - \tilde{S}_{(l)} + \mathsf{E}\tilde{S}_{(l)}) \leqq 0\,,$$

$$l = k - t + 1, \ldots, k\,;\ l \neq r\}\,.$$

This equality holds for every n. According to Theorems 1 and 2 (or Theorem 5) of [12], the random variables $n^{1/2}[2K(J, F)]^{-1/2} (\tilde{S}_i - \mathsf{E}\tilde{S}_i)$, $i = 1, \ldots, k$, are for $n \to \infty$ asymptotically normally distributed $N(0, \frac{1}{2})$. If we further use a well-known result concerning the limiting distribution if a linear transformation of random variables and the fact that convergence to a continuous distribution function is uniform in argument, we obtain

$$P^* = t \lim_{n\to\infty} Q_{k-1}(\underbrace{\delta^{(n)}\lambda^{(n)}\, n^{1/2}[2K(J, F)]^{-1/2}, \ldots, \delta^{(n)}\lambda^{(n)}\, n^{1/2}[2K(J, F)]^{-1/2}}_{(k-t)\ \text{times}}, \underbrace{0, \ldots, 0}_{(t-1)\ \text{times}})\,.$$

Consequently, (16) is equivalent to

$$\lim_{n\to\infty} \delta^{(n)}\lambda^{(n)}\, n^{1/2}\delta^{-1}[K(J, F)]^{-1/2} = 1\,. \tag{17}$$

Let us now examine the asymptotic behaviour of $\lambda^{(n)}$. Suppose first that $J(u)$ satisfies the Hölder condition on (0, 1), i.e.

$$|J(u) - J(u')| \leqq C|u - u'|^{\beta}\,, \quad \beta > \tfrac{1}{2}\,, \tag{18}$$

for all $u, u' \in (0, 1)$. Then

$$n^{1/2}|\lambda^{(n)} - 1| = n^{1/2}\left|\sum_{j=1}^{n} \int_{(j-1)/n}^{j/n} \left[J\left(\frac{j}{n+1}\right) - J(u)\right] \mathrm{d}u\right| \leqq$$

$$\leqq n^{1/2}\, C \sum_{j=1}^{n} \int_{(j-1)/n}^{j/n} n^{-\beta}\, \mathrm{d}u = Cn^{1/2-\beta}.$$

An analogous inequality may be proved in the general case: if (18) is satisfied on some m subintervals (u_{i-1}, u_i) of (0, 1) ($i = 1, \ldots, m$; $u_0 = 0$, $u_m = 1$), then we have just $n + m$ summands, each of them being bounded by $n^{-1/2} \sup_{u\in(0,1)} J(u)$, so that

$$n^{1/2}|\lambda^{(n)} - 1| \leqq Cn^{1/2-\beta} + mn^{-1/2} \sup_{u\in(0,1)} J(u)\,.$$

In any case we get $\lim_{n\to\infty} n^{1/2}|\lambda^{(n)} - 1| = 0$. As a consequence of Lemma 1, the sequence $\{\delta^{(n)}\}$ is bounded. It follows that (17) is equivalent to

$$\lim_{n\to\infty} \delta^{(n)} n^{1/2} \delta^{-1}[K(J, F)]^{-1/2} = 1$$

which gives (15).

Let us now assume that we are given a value δ^*. Then $\delta^{(n)}$ is set equal to δ^* and, from the above theorem, a large sample solution of (11) is given by

$$n_L = \left(\frac{\delta}{\delta^*}\right)^2 K(J, F) . \tag{19}$$

If we further equate (4A.7) of [10] to (15), we obtain the following theorem.

Theorem 3. *Let the conditions* (A), (C), (E) *be satisfied. Then the asymptotic efficiency of the procedure based on L-estimates relative to the Bechhofer procedure is*

$$e_{L(J),B}(F) = \frac{\sigma^2(F)}{K(J,F)}, \tag{20}$$

where $\sigma^2(F)$ is the variance of F.

Remark 2. Since sample means are special cases of L-estimates, procedures based on L-estimates are in fact generalizations of the Bechhofer procedure. So Theorem 2 also is a generalization of Lemma 4A.1 of [10].

Notice that (20) coincides with the ARE of the L-estimate of location with the weight function J with respect to the sample mean.

4. PROBLEM (b)

The procedure for selection of t best populations with regard to order (from the best one to the t-th best one) is to select populations associated with $S_{[k]}, \dots, S_{[k-t+1]}$ in this order. Then obviously

$$\mathsf{P}\{\mathrm{CS}\} = \mathsf{P}\{\max(S_{(1)}, \dots, S_{(k-t)}) < S_{(k-t+1)} < \dots < S_{(k)}\} . \tag{21}$$

As far as I know, there is no general analogue of the theorem of [1] for problem (b). So the search for infimum $\mathsf{P}\{\mathrm{CS}\}$ on D (given by (4)) is a little more complicated. Nevertheless, as in the previous case, $\mathsf{P}\{\mathrm{CS}\}$ is a nonincreasing function of $\theta_{[1]}, \dots, \theta_{[k-t]}$ and cannot be increased by setting $\theta_{[1]} = \theta_{[k-t]} = \theta_{[k-t+1]} - \delta^*$. So with help of (21),

$$\mathsf{P}\{\mathrm{CS}\} = \sum_{r=1}^{k-t} \int_{-\infty}^{\infty} \prod_{\substack{i=1 \\ i \neq r}}^{k-t} G(y - \lambda\theta_{[i]}) \int_y^{\infty} \int_y^{y_k} \dots \int_y^{y_{k-t+2}} \mathrm{d}G(y_{k-t+1} - \lambda\theta_{[k-t+1]}) \dots$$
$$\dots \mathrm{d}G(y_{k-1} - \lambda\theta_{[k-1]})\, \mathrm{d}G(y_k - \lambda\theta_{[k]})\, \mathrm{d}G(y - \lambda\theta_{[r]}) \geqq$$

$$\geqq (k-t)\int_{-\infty}^{\infty} G^{k-t-1}(y-\lambda\theta_{[k-t+1]}+\lambda\delta^*)\int_y^{\infty}\int_y^{y_k}\ldots\int_y^{y_{k-t+2}} \mathrm{d}G(y_{k-t+1}-$$

$$-\lambda\theta_{[k-t+1]})\ldots \mathrm{d}G(y_{k-1}-\lambda\theta_{[k-1]})\,\mathrm{d}G(y_k-\lambda\theta_{[k]})\,\mathrm{d}G(y-\lambda\theta_{[k-t+1]}+\lambda\delta^*)\,.$$

After a transformation of variables given by $z = y - \lambda\theta_{[k-t+1]}$, $z_i = y_i - \lambda\theta_{[i]}$, $i = k - t + 1, \ldots, k$, it turns out that the last expression is a nondecreasing function of the differences $\theta_{[k-t+2]} - \theta_{[k-t+1]}, \ldots, \theta_{[k]} - \theta_{[k-1]}$. Thus that expression attains its infimum on D for $\theta_{[i]} - \theta_{[i-1]} = \delta^*$, $i = k - t + 2, \ldots, k$, and this is also the infimum of $\mathsf{P}\{\mathrm{CS}\}$ on D. It follows that the sample size n is obtained by solving the equation

$$\text{(22)}\qquad \mathsf{P}_{\mathrm{LFC}}\{\mathrm{CS}\} = (k-t)\int_{-\infty}^{\infty} G^{k-t-1}(y)\int_{y-t\lambda^{(n)}\delta^*}^{\infty}\int_{y-(t-1)\lambda^{(n)}\delta^*}^{y_k+\lambda^{(n)}\delta^*}\ldots$$

$$\ldots\int_{y-\lambda^{(n)}*\delta}^{y_{k-t+2}+\lambda^{(n)}\delta^*} \mathrm{d}G(y_{k-t+1})\ldots \mathrm{d}G(y_{k-1})\,\mathrm{d}G(y_k)\,\mathrm{d}G(y) = P^*$$

and LFC is clearly the slippage configuration (6).

Lemma 2. *For P^* fixed, let $\delta^{(n)}$ be such that (22) holds with $\delta^* = \delta^{(n)}$; $n = 1, 2, \ldots$. Let the conditions (A), (B), (E) or (A), (B), (D), (F) be fulfilled. Then* $\lim_{n\to\infty} \delta^{(n)} = 0$.

Proof. The proof of Lemma 1 can be repeated literally with the only difference that we have (21) instead of (10).

The proof of the following theorem, which is an analogue of the proof of Theorem 2, is also omitted.

Theorem 4. *For P^* fixed, let $\delta^{(n)}$ be such that (22) holds with $\delta^* = \delta^{(n)}$; $n = 1, 2, \ldots$. Let the conditions (A), (C), (E) or (A), (C), (D), (F) be satisfied. Then as $n \to \infty$,*

$$\text{(23)}\qquad \delta^{(n)} = \delta[K(J,F)]^{1/2}\, n^{-1/2} + o(n^{-1/2})\,,$$

where $K(J, F)$ is given by (14), δ is determined by the condition

$$\text{(24)}\qquad P^* = (k-t)\, Q_{k-1}(\underbrace{0,\ldots,0}_{(k-t-1)\text{ times}},\underbrace{\delta\, 2^{-1/2},\ldots,\delta\, 2^{-1/2}}_{t\text{ times}})$$

and Q_{k-1} is the distribution function of a normally distributed vector $(U_1, \ldots, U_{k-t-1}, W_{k-t}, \ldots, W_{k-1})$ such that

$$\mathsf{E}U_i = \mathsf{E}W_l = 0\,,\qquad \mathsf{Cov}(U_i, U_{i'}) = \tfrac{1}{2}(\delta_{ii'} + 1)\,,$$

$$\begin{aligned} \mathsf{Cov}(W_l, W_{l'}) &= 1 \quad \text{for} \quad l = l'\,, & \mathsf{Cov}(U_i, W_l) &= -\tfrac{1}{2} \quad \text{for} \quad l = k-t\,,\\ &= -\tfrac{1}{2} \quad \text{for} \quad |l - l'| = 1\,, & &= 0 \quad \text{for} \quad l > k-t\,,\\ &= 0 \quad \text{for} \quad |l - l'| > 1\,, \end{aligned}$$

$$i, i' = 1, \ldots, k-t-1\,,\quad l, l' = k-t, \ldots, k-1\,.$$

From (23) it is clear that the large sample solution of (22) is again given by (19) (where δ is of course determined by (24)). Finally, with help of Lemma 5A.1 of [10] (or Theorem 4 for $J(u) \equiv 1$) we obtain that the asymptotic efficiency of the procedure based on L-estimates relative to the Bechhofer procedure is equal to (20).

5. CONCLUDING REMARKS, EXAMPLES

From the practical point of view, the feasibility of the procedures suggested is of great importance. Let us at least have a look at the case of α-trimmed means and the large sample solution of the sample size problem given by (19). To be able to use this formula, we must know $K(J, F)$ and δ. Using the idea of the proof of Lemma 2 of [9], we can derive for (14), where $J(u)$ is given by (9) and the distribution F is symmetric about zero $(F(-x) = 1 - F(x))$,

$$K(J, F) = \frac{1}{(1-2\alpha)^2}\left[\int_{-F^{-1}(1-\alpha)}^{F^{-1}(1-\alpha)} x^2 \, \mathrm{d}F(x) + 2\alpha(F^{-1}(1-\alpha))^2\right]. \tag{25}$$

(This formula was obtained directly for α-trimmed mean in [3].) With help of (25), we may easily evaluate $K(J, F)$ for various symmetric distributions; e.g. for the standard normal distribution $(F = \Phi)$

$$K(J, F) = \frac{1}{(1-2\alpha)^2}\left[2\alpha(\Phi^{-1}(1-\alpha))^2 - \left(\frac{2}{\pi}\right)^{1/2}\Phi^{-1}(1-\alpha)\, \mathrm{e}^{-\frac{(\Phi^{-1}(1-\alpha))^2}{2}} + \right.$$
$$\left. + 1 - 2\alpha\right],$$

for the uniform distribution $(F(x) = x + \frac{1}{2}, \quad -\frac{1}{2} \leqq x \leqq \frac{1}{2})$

$$K(J, F) = \tfrac{1}{12}(1 + 4\alpha),$$

for the double exponential distribution $(F(x) = \frac{1}{2}\mathrm{e}^x$ for $x < 0$, $F(x) = 1 - \frac{1}{2}\mathrm{e}^{-x}$ for $x \geqq 0)$

$$K(J, F) = \frac{1}{(1-2\alpha)^2}(2\alpha \log 2\alpha + 1 - 2\alpha),$$

for the logistic distribution $(F(x) = 1/(1 + e^{-x}))$

$$K(J, F) = \frac{4}{(1-2\alpha)^2}\left[\log\frac{1-\alpha}{\alpha}\log(1-\alpha) + \frac{\pi^2}{12} - \sum_{m=1}^{\infty}(-1)^m \frac{1}{m^2}\left(\frac{\alpha}{1-\alpha}\right)^m\right].$$

(With general known scale parameter σ, we make use of the identity $K(J, F(x/\sigma)) = \sigma^2 K(J, F(x))$.) In the table below, we present some numerical examples of these values for various α's. Interesting studies and discussions on α-trimmed mean may be found e.g. in [3], [5] and [7].

It remains to find the value of δ. For problem (a) we have to solve the equation (16). With regard to the form of the covariance matrix of Q_{k-1}, we can make use of formula (1.3) of [6]; after easy calculations (16) reduces to

$$P^* = t \int_{-\infty}^{\infty} \Phi^{k-t}(y + \delta) [1 - \Phi(y)]^{t-1} \, d\Phi(y) .$$

The value δ as a solution of this equation was tabulated in [2] for various values of k, t and P^*. When solving equation (24) (problem (b)) we should notice that the right-hand side of (24) coincides with the ritht-hand side of (21) where $S_{(i)}$ has the distribution function $\Phi(y - \theta_{[i]})$ and $\boldsymbol{\theta}$ has the form (6) with $\delta^* = \delta$. In [4] a quick algorithm is given for an approximate evaluation of the latter expression so that P^* can be easily tabulated (for given k and t) as a function of δ. For the case of complete ranking, there is a table of P^* in [4] for $\delta = 0{,}0\ (0{,}1)\ 4{,}2$ and $k = 2(1)\ 7$.

In the table below, for each distribution, in the left column we give the values of $K(J, F)$ and in the right column those of $e_{L(J),B}$.

Table

α \ F	Normal		Uniform		Double exponential		Logistic	
0·01	1·004	0·996	0·087	0·962	1·878	1·065	3·191	1·031
0·05	1·026	0·974	0·100	0·833	1·654	1·209	3·059	1·075
0·10	1·060	0·943	0·117	0·714	1·494	1·339	3·017	1·090
0·15	1·100	0·909	0·133	0·625	1·383	1·446	3·031	1·085
0·20	1·145	0·874	0·150	0·556	1·297	1·542	3·080	1·068
0·25	1·195	0·837	0·167	0·500	1·227	1·629	3·158	1·042

In addition, the algorithm described in [4] enables us to solve the sample size problem for small samples: we can evaluate (10) or (21) under LFC as a function of n and δ provided we know $G(y)$ (see (7)) or have it tabulated with sufficient accuracy.

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Souhrn

PROCEDURY USPOŘÁDÁVÁNÍ A SELEKCE PRO PŘÍPAD PARAMETRU POLOHY ZALOŽENÉ NA *L*-ODHADECH

V článku jsou studovány vlastnosti některých procedur uspořádávání a selekce populací založených na robustních *L*-odhadech parametru polohy. Je nalezena nejméně příznivá konfigurace parametrů a asymptotická relativní eficience vzhledem k procedurám založeným na výběrových průměrech.

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