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MATHEMATICAL STUDY OF ROTATIONAL INCOMPRESSIBLE NON-VISCOUS FLOWS THROUGH MULTIPLY CONNECTED DOMAINS

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1. INTRODUCTION

In this paper we shall deal with the solvability of boundary value problems describing steady, generally rotational, plane or three-dimensional axially symmetric stream fields of an ideal (i.e. non-viscous) incompressible fluid. The rotational incompressible flows in simply connected domains were studied e.g. in [1, 4, 5, 6, 8, 9, 14]. The paper [6] was devoted to the existence and uniqueness of stream fields even in multiply connected domains, under the assumption that the mass flows per second of the fluid between the individual components of the boundary were given.

However, in many cases these mass flows are not known a priori. As an example we can use the plane flow round a group of profiles inserted into a bounded domain. On the basis of experiments and physical considerations we can conclude that of all mathematically possible stream fields, which differ from one another in the mass flows as mentioned above, only those are physically admissible that fulfil that co-called trailing conditions. It means that on every (plane, smooth, fixed and impermeable) profile a trailing point is given, at which the velocity of the fluid is zero.

This problem was solved in [7] for the class of models describing stream fields by means of a linear equation for the stream function. Among others, irrotational plane and axially symmetric flows belong to this class.

In this paper the results of [7] will be generalized to the case of rotational flows where the equation for the stream function is not linear any more.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

By the symbol E_k let us denote the Euclidean k-dimensional space. The distance of two points $\sigma, \sigma' \in E_k$ will be denoted by $|\sigma - \sigma'|$. As a rule, we shall use the notation $x = (x_1, x_2)$ for points of the space E_2.

Let $\Omega \subset E_2$ be an $(r + 1)$-multiply connected $(r \geqq 1)$, bounded domain. We assume that the boundary $\partial\Omega$ of the domain Ω has $r + 1$ components $C_0, C_1 \ldots, C_r$ which are geometric images of Jordan curves. Let $C_i \subset \operatorname{Int} C_0$ for $i = 1, \ldots, r$ (See Fig. 1.) The closure of the domain Ω will be denoted by $\overline{\Omega}$.

We shall consider the following boundary value problem:

$$Lu = f(x, u, \nabla u) \quad \text{in} \quad \Omega\,, \tag{2.1}$$

$$u \mid C_0 = \psi_0\,, \tag{2.2}$$

$$u \mid C_i - \psi_i = q_i\,, \quad i = 1, \ldots, r\,. \tag{2.3}$$

Here L denotes a uniformly elliptic second-order partial differential operator, $\nabla u =$

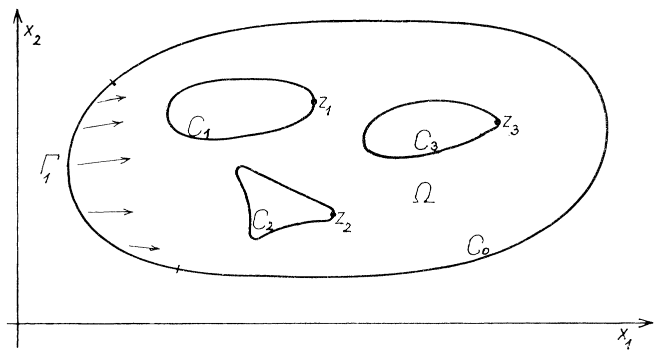

Fig. 1.

$= (u_{x_1}, u_{x_2}) = (\partial u/\partial x_1, \partial u/\partial x_2)$; $f = f(x, \xi_0, \xi_1, \xi_2) : \overline{\Omega} \times E_3 \to E_1$ and $\psi_i : C_i \to E_1$ $(i = 0, \ldots, r)$ are given functions, $q_1, \ldots, q_r$ are constants.

In studying the flow round a group of profiles given by the curves $C_1, \ldots, C_r$ we face the problem connected with the determination of the constants $q_1, \ldots, q_r$, which are not known in advance. They must be determined so that the solution u of the problem (2.1)–(2.3) satisfies the so-called trailing conditions (cf. [7])

$$\frac{\partial u}{\partial n}(z_i) = v_i\,, \quad i = 1, \ldots, r\,. \tag{2.4}$$

The given points $z_i \in C_i$ $(i = 1, \ldots, r)$ are called the trailing points, v_i are given real constants. $\partial/\partial n$ denotes the derivative in the direction of the outer normal to $\partial\Omega$.

In the following, we shall introduce assumptions under which the solvability of our problem will be investigated.

(i) $$\alpha \in (0, 1)\,, \quad \partial\Omega \in C^{2,\alpha}\,.$$

(ii) $$a_{ij} \in C^{1,\alpha}(\overline{\Omega})\,, \quad a_i\,, \quad a \in C^{\alpha}(\overline{\Omega})\,, \quad a \geqq 0 \quad \text{in} \quad \overline{\Omega}\,,$$

$$Lu = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(a_{ij} u_{x_j}) + \sum_{i=1}^{2} a_i u_{x_i} - au\,. \tag{2.5}$$

(iii) There exist constants $M, \mu, \nu > 0$ such that

(2.6) $$|a_{ij}|, |a_i|, |a|, \left|\frac{\partial a_{ij}}{\partial x_k}\right| \leqq M \quad \text{in} \quad \overline{\Omega}, \quad i, j, k = 1, 2,$$

(2.7) $$\mu\tau^2 \leqq \sum_{i,j=1}^{2} a_{ij}(x)\, \tau_i \tau_j \leqq \nu\tau^2 \quad \forall x \in \overline{\Omega},$$
$$\forall \tau = (\tau_1, \tau_2) \in E_2 \quad (\tau^2 = \tau_1^2 + \tau_2^2).$$

(iv) $\psi_i \in C^{2,\alpha}(C_i)$, $i = 0, \ldots, r$.

(v) $f \in C^\alpha(\overline{\Omega} \times E_3)$, which means that the function f is continuous in $\overline{\Omega} \times E_3$ and

(2.8) $$\|f\|_{C^\alpha(\overline{\Omega} \times E_3)} = \|f\|_{C(\overline{\Omega} \times E_3)} + \langle f \rangle^{(\alpha)}_{\overline{\Omega} \times E_3} :=$$
$$:= \sup_{\overline{\Omega} \times E_3} |f| + \sup_{\substack{\sigma, \sigma' \in \Omega \times E_3 \\ \sigma \neq \sigma'}} |f(\sigma) - f(\sigma')|\, |\sigma - \sigma'|^{-\alpha} < +\infty .$$

Let $|f| \leqq M$.

The definitions of classes and spaces C^k, C^α, $C^{k,\alpha}$, etc., can be found e.g. in [2] or [11].

If $u \in C^2(\overline{\Omega})$, then the expression Lu can be written in the form

(2.9) $$Lu = \sum_{i,j=1}^{2} a_{ij} u_{x_i x_j} + \sum_{j=1}^{2} b_j u_{x_j} - au ,$$

where

$$u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}$$

and

(2.10) $$b_j = a_j + \sum_{i=1}^{2} \frac{\partial a_{ij}}{\partial x_i} .$$

If the assumption (ii) is satisfied, then $b_j \in C^\alpha(\overline{\Omega})$.

In view of [11], there exist functions $\varphi_i \in C^{2,\alpha}(\overline{\Omega})$ such that

(2.11) $$\varphi_0 \mid C_i = \psi_i , \quad i = 0, \ldots, r ,$$
$$\varphi_i \mid C_j = \delta_{ij} , \quad i = 1, \ldots, r , \quad j = 0, \ldots, r .$$

($\delta_{ii} = 1$, $\delta_{ij} = 0$, if $i \neq j$.)

At the end of this section, we introduce the definition of the classical solution of our problem.

Problem (P). *Let the assumptions* (i)–(v) *be satisfied and let a vector* $v = (v_1, \ldots, v_r) \in E_r$ *be given. Then a function* $u \in C^{2,\alpha}(\overline{\Omega})$ *and a vector* $q = (q_1, \ldots, q_r) \in E_r$ *will be called a solution of the problem* (P) *if they satisfy the equation* (2.1) *and the conditions* (2.2)–(2.4).

The solvability of the problem (P) will be proved on the basis of appropriate a priori estimates of its solutions. For this purpose the following known results will be used:

Theorem 3.1. (*the Schauder a priori estimate of a solution of a linear elliptic equation*). *Let the assumptions* (i)–(iii) *be satisfied and let* $g \in C^{\alpha}(\overline{\Omega})$. *Then there exists a constant* k_1 *that depends on the domain* Ω, *the constants* μ, ν, α *and the norms of the coefficients* a_{ij}, b_i, a *in the space* $C^{\alpha}(\overline{\Omega})$ (*i.e.* $k_1 = k_1(\Omega, \mu, \nu, \alpha, \|a_{ij}, b_i, a\|_{C^{\alpha}(\overline{\Omega})})$) *such that for an arbitrary solution* $u \in C^{2,\alpha}(\overline{\Omega})$ *of the equation*

$$Lu = g \quad in \quad \Omega \tag{3.1}$$

the following estimate holds:

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leqq k_1[\|g\|_{C^{\alpha}(\overline{\Omega})} + \|u\|_{C^{2,\alpha}(\partial\Omega)}]\,. \tag{3.2}$$

Proof. See [2], § 5.6 or [11], § 2 from Ch. III.

Theorem 3.2. *Let the assumptions* (i)–(iii) *be satisfied*, $g \in C(\overline{\Omega})$, $|g| \leqq M$, $\tilde{\psi} \in C^2(\overline{\Omega})$. *Let* $u \in C^2(\overline{\Omega})$ *be a solution of the equation* (3.1) *with the boundary value condition*

$$u(x) = \tilde{\psi}(x) \quad \forall x \in \partial\Omega\,. \tag{3.3}$$

Then

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leqq k_2(\mu, \nu, \alpha, M, \|\tilde{\psi}\|_{C^2(\overline{\Omega})}, \Omega)\,, \tag{3.4}$$

Proof follows from Theorem 15.1 from Ch. III in the monograph [11].

Theorem 3.3. (*the strong maximum principle*). *Let* $a_{ij} \in C^1(\overline{\Omega})$, $b_j, a \in C(\overline{\Omega})$, $a \geqq 0$ *in* Ω *and let the assumptions* (i) *and* (iii) *be satisfied. Further, let* $u \in C^2(\overline{\Omega})$ *be a solution of the equation* $Lu = 0$ *in* $\overline{\Omega}$. *Then*:

1) *If* u *has its positive maximum or negative minimum in* Ω, *then* u *is constant in* $\overline{\Omega}$.

2) *Let us assume that* $\hat{x} \in \partial\overline{\Omega}$ *and that* u *is not constant in* $\overline{\Omega}$. *If* $u(\hat{x}) = \max_{x \in \overline{\Omega}} u(x) > 0$ or $u(\hat{x}) = \min_{x \in \overline{\Omega}} u(x) < 0$, *then*

$$\frac{\partial u}{\partial n}(\hat{x}) > 0 \quad or \quad \frac{\partial u}{\partial n}(\hat{x}) < 0\,, \quad respectively. \tag{3.5}$$

Proof. See [2], § 2.2.

Theorem 3.4. (*on the solvability of a linear elliptic equation*). *Let the assumptions* (i)–(iii) *be satisfied*, $\tilde{\psi} \in C^{2,\alpha}(\partial\Omega)$, $g \in C^{\alpha}(\overline{\Omega})$. *Then there exists a unique solution* $u \in C^{2,\alpha}(\overline{\Omega})$ *of the problem* (3.1), (3.3).

Proof. See [11], Theorem 1.3 from Ch. III or [2], § 5.7.

First, we shall deal with the case when the function f depends on $x \in \overline{\Omega}$ only, so that $f(x, \xi_0, \xi_1, \xi_2) = g(x)$. Hence, the equation (2.1) is a linear equation of the form (3.1):

$$Lu = g .$$

We shall thus speak about the linear problem (P).

In the following ,we shall assume that the assumptions (i)–(iv) are satisfied, $\gamma \in (0, \alpha\rangle$ and $g \in C^\gamma(\overline{\Omega})$. Let us denote by u_i, $i = 0, \ldots, r$, solutions of the following problems:

$$(4.1) \qquad \begin{aligned} Lu_0 &= g \quad \text{in} \quad \Omega , \quad u_0 \mid \partial\Omega = \varphi_0 \mid \partial\Omega , \\ Lu_i &= 0 \quad \text{in} \quad \Omega , \quad u_i \mid \partial\Omega = \varphi_i \mid \partial\Omega , \quad i = 1, \ldots, r . \end{aligned}$$

(φ_i are the functions satisfying the conditions (2.11).)

Theorem 4.1. *The problems* (4.1) *have unique solutions* $u_0, \ldots, u_r$. *There exists a constant* c_1 *which depends on* μ, ν, γ, Ω *and on the norms of the coefficients* a_{ij}, b_j, a *in the space* $C^\gamma(\overline{\Omega})$ i.e. $c_1 = c_1(\mu, \nu, \gamma, \Omega, \|a_{ij}, b_j, a\|_{C^\gamma(\overline{\Omega})})$ *such that*

$$(4.2) \qquad \begin{aligned} \|u_0\|_{C^{2,\gamma}(\overline{\Omega})} &\leqq c_1[\|g\|_{C^\gamma(\overline{\Omega})} + \|\varphi_0\|_{C^{2,\gamma}(\partial\Omega)}] , \\ \|u_i\|_{C^{2,\gamma}(\overline{\Omega})} &\leqq c_1 , \quad i = 1, \ldots, r . \end{aligned}$$

Moreover, if $\beta \in (0, \alpha\rangle$ *and* $|g| \leqq M$, *then there exist constants* $c_2 = c_2(\mu, \nu, \beta, M, \|\varphi_0\|_{C^2(\overline{\Omega})}, \Omega)$ and $c_3 = c_3(\mu, \nu, \beta, M, \Omega)$ *such that*

$$(4.3) \qquad \begin{aligned} \|u_0\|_{C^{1,\beta}(\overline{\Omega})} &\leqq c_2 , \\ \|u_i\|_{C^{1,\beta}(\overline{\Omega})} &\leqq c_3 , \quad i = 1, \ldots, r . \end{aligned}$$

Proof follows immediately from Theorems 3.1, 3.2 and 3.4.

We shall seek a solution of the linear problem (P) in the form

$$(4.4) \qquad u = u_0 + \sum_{i=1}^{r} q_i u_i ,$$

where $q_1, \ldots, q_n$ are unknown constants. If we choose $q = (q_1, \ldots, q_r)$, then it is evident that the function (4.4) is a unique solution of the problem (3.1), (2.2), (2.3). We want to determine the vector q so that the function u satisfies the conditions (2.4). By substituting (4.4) into (2.4), we get a system of linear equations for the unknown values q_i of the form

$$(4.5) \qquad \mathbb{A}q = h ,$$

where

$$(4.6) \qquad \mathbb{A} = (\alpha_{ij})_{i,j=1}^r , \quad \alpha_{ij} = \frac{\partial u_j}{\partial n}(z_i) ,$$

$$h = (h_1, \ldots, h_r) , \quad h_i = v_i - \frac{\partial u_0}{\partial n}(z_i) .$$

It is evident that the solution of the linear problem (P) is equivalent to the solution of the system (4.5).

Theorem 4.2. *The matrix* $\mathbb{A}$, *defined in* (4.6), *is regular.*

Proof. It is sufficient to prove the implication "$\mathbb{A}q = 0 \Rightarrow q = 0$". The system (4.5) is homogeneous if $g = 0$, $\psi_i = 0$ for $i = 0, \ldots, r$ (so that $u_0 = 0$) and $v = 0$. Let the system $\mathbb{A}q = 0$ have a non-zero solution q^*. Then there exists a function $u^* \in C^{2,\alpha}(\overline{\Omega})$ which is not identically equal to zero in $\overline{\Omega}$, solves the equation $Lu = 0$ and fulfils the conditions (2.2)–(2.4) with $\psi_i = 0$, $i = 0, \ldots, r$, $q = q^*$ and $v = 0$. The function u^* is not constant in $\overline{\Omega}$ and has a positive maximum or a negative minimum on a certain curve C_i $(i = 1, \ldots, r)$. Since $u^* \mid C_i$ is constant, then, in view of Theorem 3.3, $(\partial u^*/\partial n)(z_i) > 0$ or < 0, which is a contradiction to (2.4).

Let us introduce the following notation: For $v = (v_1, \ldots, v_r) \in E_r$ the symbol $\|v\|_1$ denotes the norm of the vector v, defined by the relation $\|v\|_1 = \sum_{i=1}^{r} |v_i|$. If $\mathbb{B}$ is a square matrix of the type $r \times r$, then the symbol $\|\mathbb{B}\|_1$ denotes the norm of the matrix $\mathbb{B}$ induced by the norm $\|\ldots\|_1$, defined in E_r. The inequality $\|\mathbb{B}v\|_1 \leqq \|\mathbb{B}\|_1 \, . \, \|v\|_1$ holds.

As a consequence of the preceding theorems we get

Theorem 4.3. *For a given function* $g \in C^{\gamma}(\overline{\Omega})$ $(\gamma \in (0, \alpha\rangle)$ *and a given vector* $v \in E_r$, *the linear problem* (P) *has a unique solution* $u \in C^{2,\gamma}(\overline{\Omega})$, $q \in E_r$. *The function* u *satisfies the estimate*

$$(4.7) \qquad \|u\|_{C^{2,\gamma}(\overline{\Omega})} \leqq c_4[\|g\|_{C^{\gamma}(\overline{\Omega})} + \|\varphi_0\|_{C^{2,\gamma}(\partial\Omega)}] + c_5\|v\|_1$$

with $c_4 = c_1[1 + r\, c_1\|\mathbb{A}^{-1}\|_1]$ *and* $c_5 = c_1\|\mathbb{A}^{-1}\|_1$.

Further, if $\beta \in (0, \alpha\rangle$ *and* $|g| \leqq M$, *then*

$$(4.8) \qquad \|u\|_{C^{1,\beta}(\overline{\Omega})} \leqq c_6 + c_7\|v\|_1\,,$$

where $c_6 = c_2[1 + r\, c_3\|\mathbb{A}^{-1}\|_1]$ *and* $c_7 = c_3\|\mathbb{A}^{-1}\|_1$. *(*$c_1$, c_2, c_3 *are the constants from Theorem* 4.1.*)*

Proof. It is evident that the linear problem (P) has a unique solution. It is given by the formula (4.4), where $q = (q_1, \ldots, q_r)$ solves the system (4.5).

We denote either $\|u\| = \|u\|_{C^{2,\gamma}(\overline{\Omega})}$ or $\|u\| = \|u\|_{C^{1,\beta}(\overline{\Omega})}$. Then

$$(4.9) \qquad \|u\| = \|u_0 + \sum_{i=1}^{r} q_i u_i\| \leqq \|u_0\| + \sum_{i=1}^{r} |q_i|\, \|u_i\| \leqq \|u_0\| + \|q\|_1 \max_{i=1,\ldots,r} \|u_i\|\,.$$

We have $q = \mathbb{A}^{-1}h$ and

$$(4.10) \qquad \|q\|_1 \leqq \|\mathbb{A}^{-1}\|_1\, \|h\|_1\,.$$

Moreover,

$$\|h\|_1 \leqq \|v\|_1 + \sum_{i=1}^{r} \left| \frac{\partial u_0}{\partial n} (z_i) \right| .$$

If we use Theorem 4.1, we get the estimates

$$\left| \frac{\partial u_0}{\partial n} (z_i) \right| \leqq c_1 [\|g\|_{C^\gamma(\bar{\Omega})} + \|\varphi_0\|_{C^{2,\gamma}(\partial\Omega)}] ,$$

$$\left| \frac{\partial u_0}{\partial n} (z_i) \right| \leqq c_2$$

and hence,

$$\|h\|_1 \leqq \|v\|_1 + r\, c_1 [\|g\|_{C^\gamma(\bar{\Omega})} + \|\varphi_0\|_{C^{2,\gamma}(\partial\Omega)}] , \tag{4.11}$$

$$\|h\|_1 \leqq \|v\|_1 + r\, c_2 .$$

Now, let us substitute (4.2), (4.3), (4.10) and (4.11) into (4.9). We get the inequalities

$$\|u\|_{C^{2,\gamma}(\bar{\Omega})} \leqq c_1 [\|g\|_{C^\gamma(\bar{\Omega})} + \|\varphi_0\|_{C^{2,\gamma}(\partial\Omega)}] + c_1 \|\mathbb{A}^{-1}\|_1 \,.$$

$$. \{\|v\|_1 + r\, c_1 [\|g\|_{C^\gamma(\bar{\Omega})} + \|\varphi_0\|_{C^{2,\gamma}(\partial\Omega)}]\} ,$$

$$\|u\|_{C^{1,\beta}(\bar{\Omega})} \leqq c_2 + c_3 \|\mathbb{A}^{-1}\|_1 [\|v\|_1 + r\, c_2] ,$$

which already give the estimates (4.7) and (4.8).

The following part of this section is devoted to the study of the nonlinear problem (P). We shall use the well-known Schauder theorem on a fixed point of a completely continuous mapping (see e.g. [12], Ch. IV, § 3):

Theorem 4.4. *Let $\mathscr{B}$ be a Banach space and let $F : \mathscr{B} \to \mathscr{B}$ be a completely continuous mapping. If $\mathscr{N} \subset \mathscr{B}$ is a nonempty, closed, bounded, convex set such that $F(\mathscr{N}) \subset \mathscr{N}$, then there exists at least one fixed point $u \in \mathscr{N}$ of the mapping F; i.e., $u = F(u)$.*

Let the assumptions (i)–(v) be satisfied. We put here $\mathscr{B} = C^{1,\beta}(\bar{\Omega})$ with $\beta \in (0, \alpha\rangle$. If $u \in C^{1,\beta}(\bar{\Omega})$, then the function $f(., u, \nabla u)$ is an element of the space $C^\gamma(\bar{\Omega})$, where

$$\gamma = \alpha\beta .$$

Let us consider the problem

$$\begin{aligned} Lw(x) &= f(x, u(x), \nabla u(x)) , \quad x \in \Omega , \\ w \mid C_0 &= \psi_0 , \\ w \mid C_i - \psi_i &= q_i = \text{const.} , \quad i = 1, \dots, r , \\ \frac{\partial w}{\partial n} (z_i) &= v_i , \quad i = 1, \dots, r . \end{aligned} \tag{4.12}$$

On the basis of Theorem 4.3 we can assert that to an arbitrary $u \in C^{1,\beta}(\overline{\Omega})$ there exists a unique solution $w_u \in C^{2,\gamma}(\overline{\Omega})$ of the problem (4.12). (The vector v is given and fixed.) In this way the mapping "$u \in C^{1,\beta}(\overline{\Omega}) \to \Phi(u) = w_u \in C^{2,\gamma}(\overline{\Omega})$" is defined. If we take into account that $C^{2,\gamma}(\overline{\Omega}) \subset C^{1,\beta}(\overline{\Omega})$, we can also define the mapping F: $C^{1,\beta}(\overline{\Omega}) \to C^{1,\beta}(\overline{\Omega})$, $F = J \circ \Phi$. where J is the embedding operator of $C^{2,\gamma}(\overline{\Omega})$ into $C^{1,\beta}(\overline{\Omega})$.

Lemma. *The operator F is completely continuous.*

Proof. a) We shall prove that if $\mathscr{M} \subset C^{1,\beta}(\overline{\Omega})$ is an arbitrary bounded set, then the set $F(\mathscr{M})$ is compact.[1]) Let $u \in C^{1,\beta}(\overline{\Omega})$. Then

$$\|f(\cdot, u, \nabla u)\|_{C^\gamma(\overline{\Omega})} \leqq \|f\|_{C(\overline{\Omega} \times E_3)} + \varkappa \langle f \rangle^{(\alpha)}_{\overline{\Omega} \times E_3} (\|u\|^{\alpha}_{C^{1,\beta}(\overline{\Omega})} + 1) , \tag{4.13}$$

where $\varkappa > 0$ is a constant depending only on the domain Ω. This inequality is a consequence of the assumption (v) and the fact that if $\partial\Omega \in C^{2,\alpha}$, then there exists a constant $K > 0$ depending on Ω only and such that every function $u \in C^1(\overline{\Omega})$ satisfies

$$|u(x) - u(y)| \leqq K \max_{\overline{\Omega}} |\nabla u| \, |x - y| \quad \forall x, y \in \overline{\Omega} .$$

(See [3], Ch. IV, § 9.) From the inequality (4.13) and Theorem 4.3 we get an estimate of the solution $w = \Phi(u)$ of the problem (4.12):

$$\|\Phi(u)\|_{C^{2,\gamma}(\overline{\Omega})} \leqq c_5\|v\|_1 + c_4[\|f\|_{C(\overline{\Omega} \times E_3)} +$$
$$+ \varkappa \langle f \rangle^{(\alpha)}_{\overline{\Omega} \times E_3} (\|u\|^{\alpha}_{C^{1,\beta}(\overline{\Omega})} + 1) + \|\varphi_0\|_{C^{2,\gamma}(\partial\Omega)}] .$$

Hence, if $\mathscr{M} \subset C^{1,\beta}(\overline{\Omega})$ is a bounded set, then the set $\Phi(\mathscr{M})$ is bounded in $C^{2,\gamma}(\overline{\Omega})$. The compactness of the embedding J of the space $C^{2,\gamma}(\overline{\Omega})$ into $C^{1,\beta}(\overline{\Omega})$ (see [11]) implies that the set $J(\Phi(\mathscr{M})) = F(\mathscr{M})$ is compact in $C^{1,\beta}(\overline{\Omega})$.

b) Let us show that the mapping F is continuous. Let $\hat{\gamma} \in (0, \gamma)$. Since $C^{2,\gamma}(\overline{\Omega}) \subset C^{2,\hat{\gamma}}(\overline{\Omega})$, the relation "$u \in C^{1,\beta}(\overline{\Omega}) \to w$", where w is a solution of the problem (4.12), defines also a mapping $\hat{\Phi} : C^{1,\beta}(\overline{\Omega}) \to C^{2,\hat{\gamma}}(\overline{\Omega})$. Evidently, $F = \hat{J} \circ \hat{\Phi}$, where $\hat{J}$ is an embedding of the space $C^{2,\hat{\gamma}}(\overline{\Omega})$ into $C^{1,\beta}(\overline{\Omega})$. Since the mapping $\hat{J}$ is continuous, it is sufficient to prove the continuity of $\hat{\Phi}$.

Let $u_n \in C^{1,\beta}(\overline{\Omega})$, $w_n = \hat{\Phi}(u_n)$, $n = 0, 1, \ldots$. Then $\omega_n = w_n - w_0$ is a solution of the linear problem (P) with the right hand side equal to the function

$$f(x, u_n(x), \nabla u_n(x)) - f(x, u_0(x), \nabla u_0(x)) .$$

We now have $\psi_i = 0$ in the boundary conditions (2.2) and (2.3) and $v_i = 0$ in (2.4). From the estimate (4.7) we get

$$\|\omega_n\|_{C^{2,\hat{\gamma}}(\overline{\Omega})} \leqq \hat{c}_4 \|f(\cdot, u_n, \nabla u_n) - f(\cdot, u_0, \nabla u_0)\|_{C^{\hat{\gamma}}(\overline{\Omega})} . \tag{4.14}$$

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[1]) We mean, of course, relative compactness.

We need to prove that the assumption $u_n \to u_0$ in $C^{1,\beta}(\overline{\Omega})$ (for $n \to +\infty$) implies that the right hand side in (4.14) tends to zero.

If $x \in \overline{\Omega}$ and $n = 0, 1, \ldots$, we denote $\xi_n(x) = (u_n(x), \nabla u_n(x))$. For arbitrary $x, y \in \overline{\Omega}$, $x \neq y$, we have

$$(4.15) \qquad g_n(x, y) := \left|f(x, \xi_n(x)) - f(x, \xi_0(x)) - f(y, \xi_n(y)) + f(y, \xi_0(y))\right| |x - y|^{-\hat{\gamma}} \leqq \min \{\|f\|_{C^\alpha(\overline{\Omega} \times E_3)} [|\xi_n(x) - \xi_0(x)|^\alpha + |\xi_n(y) - \xi_0(y)|^\alpha] |x - y|^{-\hat{\gamma}},\ \varkappa \|f\|_{C^\alpha(\overline{\Omega} \times E_3)} [\|u_n\|^\alpha_{C^{1,\beta}(\overline{\Omega})} + \|u_0\|^\alpha_{C^{1,\beta}(\overline{\Omega})} + 2] |x - y|^{\gamma - \hat{\gamma}}\}$$

with $\varkappa = \varkappa(\Omega)$. Let $u_n \to u_0$ in $C^{1,\beta}(\overline{\Omega})$. Then there exists a constant k such that $\|u_n\|^\alpha_{C^{1,\beta}(\overline{\Omega})} \leqq k$ for $n = 0, 1, \ldots$. Further, $\xi_n \to \xi_0$ uniformly in $\overline{\Omega}$ and thus $f(\cdot, \xi_n) \to f(\cdot, \xi_0)$ in $C(\overline{\Omega})$. With respect to the definition of the norm in the space $C^{\hat{\gamma}}(\overline{\Omega})$, we want to prove that

$$\lim_{n \to +\infty} [\sup_{\substack{x, y \in \overline{\Omega} \\ x \neq y}} g_n(x, y)] = 0 .$$

Let $\varepsilon > 0$; we find $\delta > 0$ and n_0 such that

$$(4.16) \qquad 2\varkappa \|f\|_{C^\alpha(\overline{\Omega} \times E_3)} (k + 1) \delta^{\gamma - \hat{\gamma}} \leqq \varepsilon ,$$

$$2\|f\|_{C^\alpha(\overline{\Omega} \times E_3)} |\xi_n(x) - \xi_0(x)|^\alpha \delta^{-\hat{\gamma}} \leqq \varepsilon \quad \forall x \in \overline{\Omega} , \quad \forall n > n_0 .$$

Now, on the basis of (4.15) and (4.16), we easily find out that

$$\sup_{\substack{x, y \in \overline{\Omega} \\ x \neq y}} g_n(x, y) \leqq \varepsilon \quad \forall n > n_0 .$$

This fact and (4.14) imply that $\omega_n \to 0$ in $C^{2,\hat{\gamma}}(\overline{\Omega})$, which completes the proof.

Now, we shall prove the existence of a solution of the problem (P).

Theorem 4.5. *Let the assumptions* (i)–(v) *be satisfied. Then for an arbitrary given vector* $v \in E_r$ *there exists at least one solution* $u \in C^{2,\alpha}(\overline{\Omega})$, $q \in E_r$ *of the problem* (P).

Proof. Let $v \in E_r$. From the estimate (4.8) it follows that

$$\|F(u)\|_{C^{1,\beta}(\overline{\Omega})} \leqq c_8 = c_6 + c_7 \|v\|_1$$

for an arbitrary $u \in C^{1,\beta}(\overline{\Omega})$. The constant c_8 does not depend on u. Let us denote $\mathscr{N} = \{u \in C^{1,\beta}(\overline{\Omega});\ \|u\|_{C^{1,\beta}(\overline{\Omega})} \leqq c_8\}$. It is evident that $\mathscr{N}$ is a nonempty, closed, bounded, convex subset of the space $\mathscr{B} = C^{1,\beta}(\overline{\Omega})$. The above considerations together with the Schauder theorem 4.4 yield the existence of $u \in C^{1,\beta}(\overline{\Omega})$ which is a solution of the equation $u = F(u)$. This solution is simultaneously an element of the space $C^{2,\gamma}(\overline{\Omega})$. We can consider u as a solution of the linear problem (P) with the equation $Lu = g$, where the right hand side $g = f(\cdot, u, \nabla u)$ belongs to the class

$C^\alpha(\overline{\Omega})$. Putting $\gamma = \alpha$ in Theorem 4.3, we obtain immediately that $u \in C^{2,\alpha}(\overline{\Omega})$.

The following theorem is devoted to the uniqueness of the solution of the problem (P).

Theorem 4.6. *Let the assumptions* (i)–(v) *be satisfied, let the function* $f(x, \xi_0, \xi_1, \xi_2)$ *have continuous first order derivatives with respect to* ξ_i, $i = 0,1,2$, *in* $\overline{\Omega} \times E_3$ *and let*

$$a(x) + \frac{\partial f}{\partial \xi_0}(x, \xi_0, \xi_1, \xi_2) \geqq 0 \quad \forall (x, \xi_0, \xi_1, \xi_2) \in \overline{\Omega} \times E_3 . \tag{4.17}$$

Then the problem (P) *has exactly one solution* $u \in C^{2,\alpha}(\overline{\Omega})$, $q \in E_r$.

Proof. It is sufficient to prove the uniqueness of the solution. Let $u_1, u_2 \in C^{2,\alpha}(\overline{\Omega})$ be two solutions of the problem (P). Then the function $w = u_1 - u_2$ satisfies the conditions

$$w \mid C_0 = 0 ,$$

$$w \mid C_i = \text{const.} , \quad i = 1, \ldots, r ,$$

$$\frac{\partial w}{\partial n}(z_i) = 0 , \quad i = 1, \ldots, r$$

and is a solution of the equation

$$\tilde{L}w = Lw + \sum_{i=1}^{2} d_i w_{x_i} - d_0 w = 0 ,$$

where

$$d_i(x) = \int_0^1 \frac{\partial f}{\partial \xi_i}(x, t\, u_1(x) + (1 - t)\, u_2(x), t\, \nabla u_1(x) + (1 - t)\, \nabla u_2(x))\, dt ,$$

$$i = 0, 1, 2 .$$

It follows from the assumption (4.17) that $a(x) + d_0(x) \geqq 0$ in $\overline{\Omega}$ so that the operator $\tilde{L}$ satisfies the assumptions of Theorem 3.3, which implies that $w = 0$ in $\overline{\Omega}$.

5. APPLICATIONS IN HYDRODYNAMICS

5.1. **Three-dimensional axially symmetric flows.** Let us suppose that the closure of the domain Ω, defined in Section 2, lies in the upper half-plane $x_2 > 0$, i.e.

$$x_2 > 0 \quad \forall x = (x_1, x_2) \in \overline{\Omega} . \tag{5.1}$$

By rotating the domain Ω round the axis x_1 we get a three-dimensional axially symmetric domain Ω^3. The closure of Ω^3 and the axis x_1 of symmetry are disjoint. The boundary of the domain Ω^3 consists of an outer part, given by the rotation of the curve C_0, and from axially symmetric rings, obtained by the rotation of the curves $C_1, \ldots, C_r$.

We shall consider the steady three-dimensional axially symmetric stream fields of an incompressible fluid in the domain Ω^3. This problem can be solved, in view of the axial symmetry, in the plane domain Ω. Let us denote by $\mathbf{v} = (v_1, v_2, v_3)$ the velocity vector of the fluid with the components in the cylindrical coordinates x_1, x_2, ε, and by H the total energy (called also the generalized enthalpy-see [6]).

The system of the continuity equation and the Euler equations of motion can be written in the form

(5.2) $$\sum_{i=1}^{2} \frac{\partial}{\partial x_i}(x_2 v_i) = 0 ,$$

(5.3) $$\omega = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} ,$$

(5.4) $$\omega v_2 = \frac{\partial H}{\partial x_1} - \frac{1}{2x_2^2} \frac{\partial (x_2 v_3)^2}{\partial x_1} ,$$

$$-\omega v_1 = \frac{\partial H}{\partial x_2} - \frac{1}{2x_2^2} \frac{\partial (x_2 v_3)^2}{\partial x_2} ,$$

$$0 = v_1 \frac{\partial (x_2 v_3)}{\partial x_1} + v_2 \frac{\partial (x_2 v_3)}{\partial x_2} .$$

To these equations we shall add the boundary value condition

(5.5) $$x_2 \,.\, (v_1, v_2) \,.\, \vec{n} \mid \partial\Omega = \varphi .$$

Here $\vec{n}$ denotes the unit vector of the outer normal to $\partial\Omega$ and $\varphi : \partial\Omega \to E_1$ is a given function which satisfies the relations

(5.6) $$\int_{C_0} \varphi \, \mathrm{d}S = 0 , \quad \varphi \mid C_i = 0 , \quad i = 1, \ldots, r .$$

The condition (5.5) defines the flow through the boundary, the assumptions (5.6) mean that the total flow through the component of the boundary given by the curve C_0 is zero and that the rings defined by $C_1, \ldots, C_r$ are impermeable. Further, let us assume that an arc $\Gamma_1 \subset C_0$ is given. We shall call it the inlet. Let

(5.7) $$\varphi \mid \Gamma_1 \leqq \alpha_0 < 0 \quad (\alpha_0 = \text{const.}) .$$

We assume that the total energy H and the angular velocity component v_3 are given at the inlet:

(5.8) $$H \mid \Gamma_1 = h , \quad x_2 v_3 \mid \Gamma_1 = w .$$

Finally, we shall consider the trailing conditions on the curves $C_1, \ldots, C_r$. Let $z_i \in C_i$, $i = 1, \ldots, r$, be the trailing points, at which

$$v_1(z_i) = v_2(z_i) = 0\,, \quad i = 1, \ldots, r\,. \tag{5.9}$$

We shall prove the following theorem on the existence of the flow round the rings given by the curves $C_1, \ldots, C_r$ and inserted in the domain obtained by rotating the curve C_0 round the axis x_1.

Theorem 5.1. *Let us assume that*

1) $\alpha \in (0, 1)$, $\partial\Omega \in C^{2,\alpha}$,

2) $\Gamma_1 \subset C_0$ *is a closed arc,*

3) $\varphi \in C^{1,\alpha}(\partial\Omega)$, $h, w \in C^{1,\alpha}(\Gamma_1)$.

Further, let (5.1), (5.6) *and* (5.7) *hold. Then there exist functions* $v_1, v_2, v_3, H \in C^{1,\alpha}(\overline{\Omega})$ *which solve the equations* (5.2)–(5.4) *in* Ω *and satisfy the conditions* (5.5), (5.8) *and* (5.9) *on the boundary* $\partial\Omega$.

Proof. We shall prove this theorem on the basis of results contained in the preceding sections and in the paper [6]. After introducing the stream function, we shall transform our problem to the problem (P). By integrating the function φ from (5.5) along the curves C_i, we get the functions ψ_i that appear in the boundary value conditions (2.2) and (2.3). Therefore, $\psi_0 \in C^{2,\alpha}(C_0)$ and $\psi_i = 0$ for $i = 1, \ldots, r$. It follows from (5.5) that $\psi_0 \mid \Gamma_1$ is a one-to-one mapping of the arc Γ_1 onto a closed interval $\langle Q_1, Q_2 \rangle$ $(Q_1 < Q_2)$. We denote the inverse function to $\psi_0 \mid \Gamma_1$ by $(\psi_0 \mid \Gamma_1)_{-1}$ and define the functions A, B: $\langle Q_1, Q_2 \rangle \to E_1$ by the relations

$$A = h \circ (\psi_0 \mid \Gamma_1)_{-1}\,,$$

$$B = w \circ (\psi_0 \mid \Gamma_1)_{-1}\,.$$

It is evident that $A, B \in C^{1,\alpha}(\langle Q_1, Q_2 \rangle)$. We can easily extend the functions A and B to the whole interval E_1 so that $A', BB' \in C^{\alpha}(E_1)$ (it means that $\|A'\|_{C^\alpha(E_1)} := \sup_{E_1} |A'| + \sup_{\substack{t,t^* \in E_1 \\ t \neq t^*}} |A'(t) - A'(t^*)|\,|t - t^*|^{-\alpha} < +\infty$, similarly for BB'; A' denotes the derivative of A).

Let us now consider the partial differential equation

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i} (a(x)\, u_{x_i}) = f(x, u) \quad \text{in} \quad \Omega \tag{5.10}$$

with

$$a(x) = x_2^{-1}\,, \quad f(x, t) = x_2\, A'(t) - (2x_2)^{-1}\, (B^2)'\,(t)\,,$$

and the boundary value conditions

(5.11) $$u \mid C_0 = \psi_0 ,$$

(5.12) $$u \mid C_i = q_i , \quad i = 1, \ldots, r ,$$

(5.13) $$\frac{\partial u}{\partial n}(z_i) = 0 , \quad i = 1, \ldots, r ,$$

for an unknown stream function u and unknown constants q_i. This problem is a special case of the problem (P). We obviously have $a \in C^\infty(\overline{\Omega})$ and $f \in C^\alpha(\overline{\Omega} \times E_1)$ (see (5.1)). Theorem 4.5 immediately implies the existence of a solution $u \in C^{2,\alpha}(\overline{\Omega})$, $q = (q_1, \ldots, q_r) \in E_r$ of the problem (5.10)–(5.13). In [6] it was proved that the functions

$$v_1 = \frac{1}{x_2} \frac{\partial u}{\partial x_2} , \quad v_2 = - \frac{1}{x_2} \frac{\partial u}{\partial x_1} ,$$

$$H = A \circ u , \quad v_3 = \frac{1}{x_2} B \circ u$$

form a solution of the equations (5.2)–(5.4) and satisfy the conditions (5.5) and (5.8). It is evident that the trailing conditions (5.9) are also satisfied. Moreover, $v_1, v_2, v_3, H \in C^{1,\alpha}(\overline{\Omega})$.

5.2. Rotational flow through cascades of profiles. The mathematical theory of plane cascades of profiles is a basis for the theoretical investigation of the flow through blade rows of turbomachines, compressors and other stream machines. Up to now, the irrotational incompressible flow has been studied in detail. We can mention e.g. the papers [10, 13, 15]. We shall propose here a mathematical formulation and prove the existence of rotational, incompressible, non-viscous stream fields through plane cascades.

Let us denote the set of all integer numbers by $\mathscr{Z}$. We define a plane cascade of profiles with a pitch $s > 0$ as a set $\boldsymbol{L} \subset E_2$ that consists of infinitely many disjoint geometric images of Jordan curves $K^{(n)}$, $n = 0, \pm 1, \pm 2, \ldots$ (i.e. , $n \in \mathscr{Z}$), which will be called profiles. Each profile $K^{(n)} \subset \boldsymbol{L}$ is obtained by translating the fundamental profile $K^{(0)}$ by the distance ns in the direction x_2, so that $K^{(n)} = \{(x_1, x_2 + ns); (x_1, x_2) \in K^{(0)}\}$.

Let us consider $r - 1$ ($r \geqq 2$) mutually disjoint plane cascades $\boldsymbol{L}_i$, $i = 1, \ldots, r - 1$, with a pitch $s > 0$ that consist of profiles $K_i^{(n)}$, $i = 1, \ldots, r - 1$, $n \in \mathscr{Z}$[2]). Let us suppose that all cascades $\boldsymbol{L}_1, \ldots, \boldsymbol{L}_{r-1}$ lie in a (sufficiently wide) strip $P = \{(x_1, x_2); -\infty < \sigma_1 < x_1 < \sigma_2 < +\infty, x_2 \in E_1\}$ and let us denote $\Omega_c = P - \bigcup_{i=1}^{r-1} \bigcup_{n \in \mathscr{Z}} (K_i^{(n)} \cup \operatorname{Int} K_i^{(n)})$. ($\overline{\Omega}_c$ denotes the closure of the domain Ω_c.)

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[2]) We assume that $(K_i^{(m)} \cup \operatorname{Int} K_i^{(m)}) \cap (K_j^{(n)} \cup \operatorname{Int} K_j^{(n)}) = \emptyset \; \forall i, j = 1, \ldots, r - 1, \forall m, n \in \mathscr{Z}, i \neq j$.

On each profile $K_i^{(n)}$ we consider the point $Z_i^{(n)}$ obtained by translating a given point $Z_i^{(0)} \in K_i^{(0)}$ by the distance ns in the direction x_2. On the line $x_1 = \sigma_1$ we consider the points $Z_r^{(n)} = (\sigma_1, x_2^{(0)} + ns)$, $n \in \mathscr{Z}$. We shall denote the lines $x_1 = \sigma_1$ and $x_1 = \sigma_2$ by K_r and K_0, respectively.

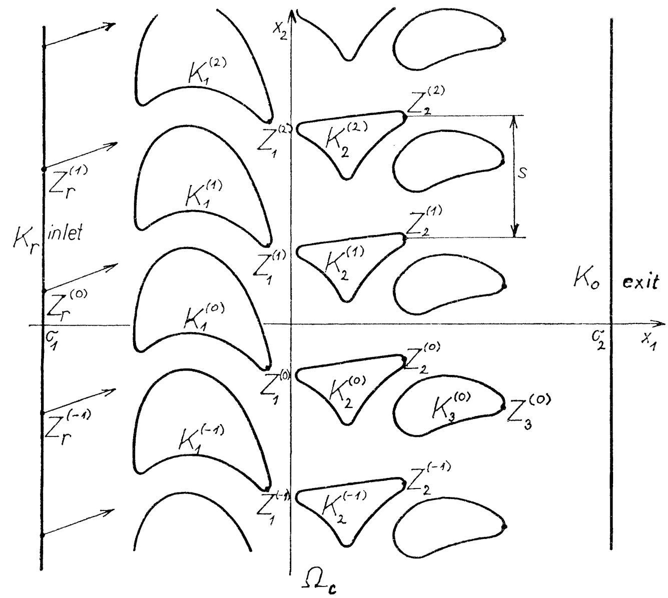

Fig. 2.

The set Ω_c is periodic in the direction x_2 with the period s, which means that

$$(x_1, x_2) \in \overline{\Omega}_c \Leftrightarrow (x_1, x_2 + s) \in \overline{\Omega}_c\,.$$

We shall say that a function $f : \overline{\Omega}_c \to E_1$ is periodic in the direction x_2 with the period s, if

$$f(x_1, x_2 + s) = f(x_1, x_2) \quad \forall (x_1, x_2) \in \overline{\Omega}_c\,.$$

Let $\alpha \in (0, 1)$. We define $C^{1,\alpha}_{s,x_2}(\overline{\Omega}_c)$ as the space of all functions defined in $\overline{\Omega}_c$ which are periodic in the direction x_2 with the period s, and have the first-order derivatives satisfying the Hölder condition with the exponent α in $\overline{\Omega}_c$. Similarly, $C^{2,\alpha}_{s,x_2}(\overline{\Omega}_c)$ is the space of all functions which are periodic in the direction x_2 with the period s in $\overline{\Omega}_c$ and their second-order derivatives satisfy the Hölder condition with the exponent α in $\overline{\Omega}_c$. Finally, we denote by the symbol $C^{1,\alpha}_Q(E_1)$ $(Q > 0)$ the space of all functions periodic in E_1 with the periodic Q, the first-order derivative of which satisfies the Hölder condition with the exponent α in E_1.

Now, let us direct our attention to the mathematical formulation of the plane flow of an ideal incompressible fluid in the domain Ω_c round the cascades L_i, $i = 1, \ldots, r - 1$. The lines K_r and K_0 represent the inlet and the exit of the cascades, respectively. We suppose that all profiles are smooth enough, fixed and impermeable. We consider the points $Z_i^{(n)}$, $n \in \mathscr{Z}$, $i = 1, \ldots, r - 1$, as the trailing points, which means that the velocity vector $\vec{V}$, with the components v_1, v_2 in the direction of the Cartesian coordinates x_1, x_2, is zero at these points. At the inlet K_r and exit K_0 the velocity component v_1 is given. Moreover, at K_r the distribution of the total energy H is given, which determines the vorticity of the stream field. At the periodically spaced points $Z_r^{(n)}$, $n \in \mathscr{Z}$, let both velocity components v_1, v_2 be given. We shall suppose that the velocity components and the total energy are periodic functions in the direction x_2 with the period s.

Problem (C). Let us consider the following assumptions:

1) $\alpha \in (0, 1)$,

2) $K_i^{(n)} \in C^{2,\alpha}, \quad i = 1, \ldots, r - 1, \quad n \in \mathscr{Z}$,

3) $\varphi_0, \varphi_r, h \in C_s^{1,\alpha}(E_1)$ are given functions,

$$\varphi_r \geqq \varphi > 0 , \quad \varphi = \text{const.},$$

$$\int_{x_2}^{x_2+s} \varphi_0(\vartheta) \, d\vartheta = \int_{x_2}^{x_2+s} \varphi_r(\vartheta) \, d\vartheta = Q \quad \forall x_2 \in E_1 .$$

4) The constant $v \in E_1$ is given.

We shall say that functions $v_1, v_2, H \in C_{s,x_2}^{1,\alpha}(\overline{\Omega}_c)$ are a solution of the problem (C), if they satisfiy the following equations and conditions:

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0 \quad \text{in} \quad \Omega_c \tag{5.14}$$

(continuity equation),

$$\begin{aligned} v_2 \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) &= \frac{\partial H}{\partial x_1} , \\ -v_1 \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) &= \frac{\partial H}{\partial x_2} \quad \text{in} \quad \Omega_c \end{aligned} \tag{5.15}$$

(Euler equations),

$$\text{a)} \quad v_1(\sigma_1, x_2) = \varphi_r(x_2) \quad \text{b)} \quad v_1(\sigma_2, x_2) = \varphi_0(x_2) , \quad x_2 \in E_1 , \tag{5.16}$$

$$H(\sigma_1, x_2) = h(x_2) , \quad x_2 \in E_1 \tag{5.17}$$

(inlet and exit conditions),

$$\vec{V} . \vec{n} \mid K_i^{(n)} = 0 , \quad n \in \mathscr{Z} , \quad i = 1, \ldots, r - 1 \tag{5.18}$$

(condition of the flow round the profiles),

(5.19) $$v_2(Z_r^{(n)}) = v\,, \quad n \in \mathscr{Z}$$

(supplementary condition to (5.16) a) for the determination of the inlet velocity),

(5.20) $$v_t(Z_i^{(n)}) = 0\,, \quad n \in \mathscr{Z}\,, \quad i = 1, \ldots, r - 1$$

(trailing conditions).

Here $\vec{n}$ denotes the unit vector of the outer normal to $\partial\Omega_c$, v_t is the tangential velocity component to $\partial\Omega_c$.

We shall prove the following existence theorem:

Theorem 5.2. *Let the assumptions* 1)–4) *of Paragraph* 5.2 *be satisfied. Then there exists at least one solution* $v_1, v_2, H \in C^{1,\alpha}_{s,x_2}(\overline{\Omega}_c)$ *of the problem* (*C*).

Proof. We shall transform the problem (C) with the use of the stream function, i.e., the function $\psi : \overline{\Omega}_c \to E_1$ that satisfies the relations $\psi_{x_2} = v_1$, $\psi_{x_1} = -v_2$ in Ω_c. Let us denote

(5.21) $$\Psi_0(x_2) = \int_0^{x_2} \varphi_0(\vartheta)\,d\vartheta\,, \quad \Psi_r(x_2) = \int_0^{x_2} \varphi_r(\vartheta)\,d\vartheta\,, \quad x_2 \in E_1\,.$$

The assumption 3) implies that the functions Ψ_0 and Ψ_r are continuous and their first-order derivatives satisfy $\Psi_0', \Psi_r' \in C_s^{1,\alpha}(E_1)$. Further,

(5.22) $$\Psi_j(x_2 + s) = \Psi_j(x_2) + Q \quad \forall x_2 \in E_1\,, \quad j = 0, r\,.$$

The function Ψ_r is increasing and maps E_1 onto E_1. By the symbol $(\Psi_r)_{-1}$ we denote the inverse function to Ψ_r and define the function $A : E_1 \to E_1$ by the relation

(5.23) $$A = h \circ (\Psi_r)_{-1}\,.$$

This function is periodic with the period Q:

$$A(t + Q) = h((\Psi_r)_{-1}\,(t + Q)) = h((\Psi_r)_{-1}\,(t) + s) =$$
$$= h((\Psi_r)_{-1}\,(t)) = A(t) \quad \forall t \in E_1\,.$$

From the properties of the functions h, φ_r and Ψ_r it follows that $A \in C_Q^{1,\alpha}(E_1)$.

Let us consider the following boundary value problem for the stream function:

Problem (C_ψ): We seek a function $\psi : \overline{\Omega}_c \to E_1$ and constants $q_1, \ldots, q_r$ such that

(5.24) $$\psi_{x_1}, \psi_{x_2} \in C^{1,\alpha}_{s,x_2}(\overline{\Omega}_c)\,,$$

(5.25) $$\text{a) } \Delta\psi = A'(\psi) \quad \text{in} \quad \Omega_c\,(\Delta \text{ is the Laplace operator})\,,$$

$$\text{b) } \psi\,\big|\,K_0 = \Psi_0\,, \qquad \psi\,\big|\,K_r = \Psi_r + q_r\,,$$

$$\text{c) } \psi\,\big|\,K_i^{(n)} = q_i + nQ\,, \quad n \in \mathscr{Z}\,, \quad i = 1, \ldots, r - 1\,,$$

d) $\dfrac{\partial\psi}{\partial n}(Z_r^{(n)}) = v\,,\quad n \in \mathscr{Z}\,,$

$\dfrac{\partial\psi}{\partial n}(Z_i^{(n)}) = 0\,,\quad n \in \mathscr{Z}\,,\quad i = 1, \ldots, r-1\,.$

Let us show that if the function ψ and the constants q_i, $i = 1, \ldots, r$, form a solution of the problem (C_ψ), then the functions

(5.26) $$v_1 = \psi_{x_2}\,,\quad v_2 = -\psi_{x_1}\,,\quad H = A \circ \psi$$

represent a solution of the problem (C).

First, it is evident that $v_1, v_2 \in C^{1,\alpha}_{s,x_2}(\overline{\Omega}_c)$. The properties of the function ψ and (5.22) imply

(5.27) $$\psi(x_1, x_2 + s) = \psi(x_1, x_2) + Q \quad \forall (x_1, x_2) \in \overline{\Omega}_c\,.$$

In view of (5.27) and $A \in C^{1,\alpha}_Q(E_1)$, we can easily prove that $H \in C^{1,\alpha}_{s,x_2}(\overline{\Omega}_c)$. From (5.25) b)–d), it follows immediately that the conditions (5.16) and (5.18)–(5.20) are satisfied. Further, we have

$$H(\sigma_1, x_2) = A(\psi(\sigma_1, x_2)) = A(\Psi_r(x_2)) = h((\Psi_r)_{-1}(\Psi_r(x_2))) = h(x_2)$$

for all $x_2 \in E_1$, which proves the validity of the condition (5.17). By substitution, similarly as in [6], we can verify that the continuity equation (5.14) and the Euler equations (5.15) are satisfied.

Let us denote

(5.28) $$\tilde{\psi}(x_1, x_2) = \Psi_r(x_2) + \frac{x_1 - \sigma_1}{\sigma_2 - \sigma_1}(\Psi_0(x_2) - \Psi_r(x_2))\,.$$

It is easy to see that $\tilde{\psi}_{x_1}, \tilde{\psi}_{x_2} \in C^{1,\alpha}_{s,x_2}(\overline{\Omega}_c)$, $\tilde{\psi} \mid K_0 = \Psi_0$, $\tilde{\psi} \mid K_r = \Psi_r$ and

(5.29) $$\tilde{\psi}(x_1, x_2 + s) = \tilde{\psi}(x_1, x_2) + Q \quad \forall (x_1, x_2) \in \overline{\Omega}_c\,.$$

We shall seek a solution of the problem (C_ψ) in the form $\psi = \mathscr{U} + \tilde{\psi}$ and get thus a new problem, equivalent to the problem (C_ψ), for an unknown function $\mathscr{U} \in C^{2,\alpha}_{s,x_2}(\overline{\Omega}_c)$ and constants $q_1, \ldots, q_r$:

(5.30) a) $\Delta\mathscr{U} = \tilde{f}(x, \mathscr{U})$ in Ω_c,

b) $\mathscr{U} \mid K_0 = 0\,,\quad \mathscr{U} \mid K_r = q_r\,,$

c) $\mathscr{U} \mid K_i^{(n)} = \Phi_i^{(n)} + q_i\,,\quad n \in \mathscr{Z}\,,\quad i = 1, \ldots, r-1\,,$

d) $\dfrac{\partial\mathscr{U}}{\partial n}(Z_i^{(n)}) = \mu_i\,.\quad n \in \mathscr{Z}\,,\quad i = 1, \ldots, r\,.$

Here,

$$\tilde{f}(x, t) = A'(t + \tilde{\psi}(x)) - \Delta\tilde{\psi}(x)\,, \quad x \in \overline{\Omega}_c\,, \quad t \in E_1\,,$$

$$\Phi_i^{(n)}(x_1, x_2) = \Phi_i^{(0)}(x_1, x_2 - ns) = -\tilde{\psi}(x_1, x_2) + nQ\,, \quad (x_1, x_2) \in K_i^n\,,$$

$$n \in \mathscr{Z}\,, \quad i = 1, \ldots, r - 1\,,$$

$$\mu_i = -\frac{\partial\tilde{\psi}}{\partial n}(Z_i^{(0)})\,, \quad i = 1, \ldots, r - 1\,, \quad \mu_r = v - \frac{\partial\tilde{\psi}}{\partial n}(Z_r^{(0)})\,.$$

The function $\tilde{f}$ is periodic in the direction x_2 with the period s in $\overline{\Omega}_c \times E_1$, $\tilde{f} \in C^\alpha(\overline{\Omega}_c \times E_1)$; $\Phi_i^{(n)} \in C^{2,\alpha}(K_i^{(n)})$ for $n \in \mathscr{Z}$ and $i = 1, \ldots, r - 1$.

Finally, we shall transform the problem (5.30) into the problem (P) by means of the mapping

$$F(Z) = F(x_1 + ix_2) = \exp(2\pi Z/s) \quad (i^2 = -1)\,.$$

The image of the unbounded, infinitely multiply connected, periodic domain Ω_c is the set $\Omega = F(\Omega_c)$, which is bounded, $(r + 1)$-multiply connected. Its boundary is formed by $r + 1$ disjoint Jordan curves $C_0, \ldots, C_r$, where $C_i = F(K_i^{(n)})$, $n \in \mathscr{Z}$, $i = 1, \ldots, r - 1$, $C_0 = F(K_0)$, $C_r = F(K_r)$. The curves C_0 and C_r are concentric circumferences with the centre at the origin. The curves $C_1, \ldots, C_{r-1}$ lie in the annulus bounded by C_0 and C_r. We denote $z_i = F(Z_i^{(n)})$, $n \in \mathscr{Z}$, $i = 1, \ldots, r$.

If $\mathscr{U} \in C^{2,\alpha}_{s,x_2}(\overline{\Omega}_c)$ and $q = (q_1, \ldots, q_r)$ form a solution of the problem (5.30), let us define the function $u = \mathscr{U} \circ F_{-1} : \overline{\Omega} \to E_1$. Although the inverse F_{-1} to F is an infinitely valued analytic function, the properties of $\mathscr{U}$ and F imply that the function u is single-valued. Since F locally has the property of a conformal mapping, we can easily transform the problem (5.30) into the equivalent problem in the domain Ω for the unknown function u and unknown constants q_i:

(5.31)

$$\text{a)}\ \Delta u = f(\cdot, u) \quad \text{in} \quad \Omega\,,$$

$$\text{b)}\ u \mid C_0 = 0\,, \quad u \mid C_r = q_r\,,$$

$$\text{c)}\ u \mid C_i = \psi_i + q_i\,, \quad i = 1, \ldots, r - 1\,,$$

$$\text{d)}\ \frac{\partial u}{\partial n}(z_i) = v_i\,, \quad i = 1, \ldots, r\,.$$

The functions f, ψ_i and the constants v_i are defined by the relations

$$f(F(x), t) = \tilde{f}(x, t)\,|F'(x)|^{-2}\,, \quad x \in \overline{\Omega}_c\,, \quad t \in E_1\,,$$

$$\psi_i(F(x)) = \Phi_i^{(0)}(x)\,, \quad x \in K_i^{(0)}\,, \quad i = 1, \ldots, r - 1\,,$$

$$v_i = \mu_i|F'(Z_i^{(0)})|^{-1}\,, \quad i = 1, \ldots, r\,.$$

Moreover, $\partial\Omega \in C^{2,\alpha}$, $f \in C^\alpha(\overline{\Omega} \times E_1)$, $\psi_i \in C^{2,\alpha}(C_i)$ for $i = 1, \ldots, r - 1$.

We see that the solution of our original problem (C) has been transformed into the solution of the problem (5.31), which is a special case of the problem (P). Theorem 4.5 guarantees the existence of at least one solution $u \in C^{2,\alpha}(\overline{\Omega})$, $q_1, \ldots, q_r \in E_1$ of the problem (5.31). From this fact and from the above considerations, it immediately follows that there exists at least one solution v_1, v_2, $H \in C^{1,\alpha}_{s,x_2}(\overline{\Omega}_c)$ of the problem (C).

Remark 5.1. The problem (C) can be modified a little in such a way that the condition (5.17) is replaced by the given distribution of the vorticity $\omega = \partial v_2/\partial x_1 - \partial v_1/\partial x_2$ at the inlet K_r. If we assume that $\omega(\sigma_1, x_2) \in C^\alpha_s(E_1)$, then we are able to define the first-order derivative of the function A by the formula

$$A' = (\omega \mid K_r) \circ (\Psi_r)_{-1}$$

and to obtain A by integration. It is necessary to mention that in this case the generalized enthalpy $H = A \circ \psi$ need not be periodic in the direction x_2. To satisfy this demand, we have to consider the additional condition

$$\int_{x_2}^{x_2+s} \omega(\sigma_1, \vartheta)\, \varphi_r(\vartheta)\, d\vartheta = 0 \quad \forall x_2 \in E_1 .$$

We leave the detailed calculation to the reader.

Remark 5.2. In the same way as in this section, we can study the existence of the stream fields through cascades of profiles in a layer of variable thickness; the only difference is in the continuity equation, which has the form $\operatorname{div}(\sigma \vec{V}) = 0$ in this case. The function σ, which is continuous, periodic in the direction x_2 with the period s and bounded from below by a positive constant, characterizes the thickness of the layer of the fluid. Without any difficulty, we can also apply our approach to the solution of a rotational flow in radial plane cascades.

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Souhrn

MATEMATICKÉ VYŠETŘOVÁNÍ ZAVÍŘENÉHO PROUDĚNÍ NESTLAČITELNÉ NEVAZKÉ TEKUTINY VE VÍCENÁSOBNĚ SOUVISLÝCH OBLASTECH

MILOSLAV FEISTAUER

V článku byla dokázána existence řešení okrajové úlohy pro eliptickou kvazilineární parciální diferenciální rovnici druhého řádu v rovinné, vícenásobně souvislé, omezené oblasti za předpokladu, že dirichletovské okrajové podmínky jsou na jednotlivých komponentách hranice dány až na aditivní konstanty. Tyto konstanty je třeba najít spolu s řešením uvažované rovnice tak, aby byly splněné dodatečné, tzv. odtokové, podmínky. Výsledky mají bezprostřední aplikace při vyšetřování zavířeného obtékání skupiny profilů a lopatkových mříží ideální nestlačitelnou tekutinou.

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