Karel Rektorys; Zdeněk Vospěl

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ON A METHOD OF TWOSIDED EIGENVALUE ESTIMATES FOR ELLIPTIC EQUATIONS OF THE FORM $Au - \lambda Bu = 0$

KAREL REKTORYS and ZDENĚK VOSPĚL

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The well-known Collatz method developed originally for the case of ordinary differential equations was shown to be applicable – at least theoretically – to the case of sufficiently general elliptic equations of the form $Au - \lambda Bu = 0$ by K. Rektorys in his book [1]. From the point of view of its practical use, the main difficulty consists in the fact that in the case of partial differential equations the corresponding boundary value problems are to be solved only approximately, as a rule, so that the estimates of eigenvalues – derived on base of exact solutions of these problems – are no more valid, in general. The aim of this paper is to show how to ensure practical applicability of the method also in this case. At the same time, some results of their own interest are derived.

1. INTRODUCTION. ASSUMPTIONS. SURVEY OF RESULTS

There is a lot of methods yielding twosided eigenvalue estimates in partial differential equations. However, on the whole, they are rather labourious or applicable only to special cases of operators.

A relatively simple method suitable especially for linear ordinary differential equations of the form

$$Au - \lambda Bu = 0 \tag{1.1}$$

with homogeneous boundary conditions not involving the parameter λ, and for the first (= least) eigenvalue λ_1, was suggested by L. Collatz many years ago. It consists in the following:

Let A, or B be linear ordinary differential operators, of order $2k$, or $2l$, respectively, $k > l$, having some properties of symmetry and positiveness on their domains of definition D_A, or D_B. (These domains consist of sufficiently smooth functions satisfying some of the given boundary conditions. We do not go into details here; see [2].) Let $f_0 \in D_B, f_1 \in D_A$ be two (nonzero) functions satisfying

$$Af_1 = Bf_0 \tag{1.2}$$

and denote

$$a_0 = (Bf_0, f_0), \quad a_1 = (Bf_0, f_1) = (Af_1, f_1), \quad a_2 = (Bf_1, f_1) \tag{1.3}$$

(the so-called *Schwarz constants*) and

$$\varkappa_1 = a_0/a_1, \quad \varkappa_2 = a_1/a_2 \tag{1.4}$$

(the so-called *Schwarz quotients*). From the properties of the operators A and B it follows, first, that a_0, a_1, a_2 are positive, and then almost immediately that

$$\varkappa_1 \geqq \varkappa_2 \geqq \lambda_1 . \tag{1.5}$$

Now, provided the first eigenvalue λ_1 is simple and l_2 is a lower bound for the second eigenvalue λ_2, greater than $\varkappa_2$ (thus

$$\varkappa_2 < l_2 \leqq \lambda_2), \tag{1.6}$$

the following twosided eigenvalue estimate is derived by Collatz:

$$\varkappa_2 - \frac{\varkappa_1 - \varkappa_2}{\dfrac{l_2}{\varkappa_2} - 1} \leqq \lambda_1 \leqq \varkappa_2 . \tag{1.7}$$

(An appropriate value for l_2 can be obtained using a propre comparison theorem, see Example 4.1.) Then he improves the accuracy of the estimate (1.7) in the following sense: Starting with the function f_0 again, he constructs the functions $f_1, f_2, \ldots, f_k \in \in D_A$ satisfying

$$\begin{aligned} Af_1 &= Bf_0 , \\ Af_2 &= Bf_1 , \\ &\cdots\cdots\cdots \\ Af_k &= Bf_{k-1} . \end{aligned} \tag{1.8}$$

He than proves that for the corresponding Schwarz quotients we have

$$\varkappa_1 \geqq \varkappa_2 \geqq \ldots \geqq \varkappa_k \geqq \lambda_1 \tag{1.9}$$

and – in consequence of the last relation (1.8) – that

$$\varkappa_{k+1} - \frac{\varkappa_k - \varkappa_{k+1}}{\dfrac{l_2}{\varkappa_{k+1}} - 1} \geqq \lambda_1 \leqq \varkappa_{k+1} . \tag{1.10}$$

On many examples he demonstrates, in [2], the efficiency of his method. Then he extends the obtained results to some simple types of partial differential equations. (Cf. also Collatz [3].)

In the monography [1], K. Rektorys extends the Collatz method to the case of sufficiently general elliptic equations of the form

$$Au - \lambda Bu = 0 \tag{1.11}$$

(with linear boundary conditions, not involving λ), under rather natural assumptions which will be kept throughout the whole paper:

Let Ω be a bounded region in E_N with a Lipschitzian boundary $\dot{\Omega}$. Let A and B be liner differential operators of order $2k$, $2l$, respectively,

$$k > l\,. \tag{1.12}$$

Denote

(1.13) $V_A = \{v;\, v \in W_2^{(k)}(\Omega)$, v satisfies, in the sense of traces, the given (homogeneous) boundary conditions which are stable for the operator $A\}$,

(1.14) $V_B = \{v;\, v \in W_2^{(l)}(\Omega)$, v satisfies, in the sense of traces, the given (homogeneous) boundary conditions which are stable for the operator $B\}$.

In the weak formulation the considered eigenvalue problem consists in finding all values of λ such that to each of them there exists a nonzero function $u \in V_A$ satisfying

$$((v, u))_A - \lambda((v, u))_B = 0 \quad \forall v \in V_A\,. \tag{1.15}$$

Here $((v, u))_A$, $((v, u))_B$ are bilinear forms corresponding, in the usual sense, to the operators A and B, respectively. (Thus we come formally to (1.15) when multiplying (1.11) by $v \in V_A$, integrate over Ω and use the Green theorem in the familiar way.) Throughout this paper, we assume that the forms $((v, u))_A$, $((v, u))_B$ are symmetric on V_A, V_B, respectively, i.e. that

$$((v, u))_A = ((u, v))_A \quad \forall u, v \in V_A\,, \tag{1.16}$$

$$((v, u))_B = ((u, v))_B \quad \forall u, v \in V_B \tag{1.17}$$

(and, consequently, $\forall u, v \in V_A$) and that they are on V_A, V_B, bounded and V_A-, V_B-elliptic, i.e. that there exist such positive constants K_1, K_2, α, β (not depending on u, v) that

$$|((v, u))_A| \leqq K_1 \|v\|_{V_A} \|u\|_{V_A} \quad \forall u, v \in V_A\,, \tag{1.18}$$

$$|((v, u))_B| \leqq K_2 \|v\|_{V_B} \|u\|_{V_B} \quad \forall u, v \in V_B\,, \tag{1.19}$$

$$((v, v))_A \geqq \alpha \|v\|_{V_A}^2 \quad \forall v \in V_A\,, \tag{1.20}$$

$$((v, v))_B \geqq \beta \|v\|_{V_B}^2 \quad \forall v \in V_B\,. \tag{1.21}$$

(Here, $\|v\|_{V_A}$, or $\|v\|_{V_B}$ means $\|v\|_{W_2^{(k)}\Omega}$, or $\|v\|_{W_2^{(l)}\Omega}$ for $v \in V_A$, or $v \in V_B$, respectively.)

When considering the eigenvalue problem for the equation

$$Au - \lambda u = 0 \tag{1.22}$$

(with corresponding boundary conditions) we put, of course, $B = I$ and

$$((v, u))_B = (v, u) \, . \tag{1.23}$$

In the quoted Rektorys monography [1], a thorough treatment of the eigenvalue problem (1.15) is given in Chap. 39: First it is shown that to every $g \in V_B$ there exists precisely one function $u \in V_A$ such that we have

$$((v, u))_A = ((v, g))_B \quad \forall v \in V_A \, . \tag{1.24}$$

At the same time

$$\|u\|_{V_A} \leqq c\|g\|_{V_B} \, , \tag{1.25}$$

where c is independent of v and g. Consequently

$$u = Tg \, , \tag{1.26}$$

where the operator $T: V_B \to V_A$ is linear (as a consequence of linearity of the operators A, B) and bounded (according to (1.25)). Because of the Sobolev immersion theorem, this operator can be shown to be completely continuous as an operator from V_A into V_A.

Denote by $\bar{V}_A$ the space, elements of which consist of elements of the space V_A and in which the scalar product is defined by

$$(v, u)_{\bar{V}_A} = ((v, u))_A \, . \tag{1.27}$$

(Properties of the scalar product are ensured by the properties of the form $((v,u))_A$.) Let us note, at this place, that the metrics in $\bar{V}_A$ and V_A are equivalent because of (1.18) and (1.20). Analogously, let $\bar{V}_B$ be the space of all elements from V_B, with the scalar product

$$(v, u)_{\bar{V}_B} = ((v, u))_B \, . \tag{1.28}$$

The metrics in $\bar{V}_B$ and V_B are equivalent as well.

In [1] it is shown that the operator T considered as an operator from $\bar{V}_A$ into $\bar{V}_A$, thus

$$T: \bar{V}_A \to \bar{V}_A \, , \tag{1.29}$$

is a positive selfadjoint completely continuous operator.

The eigenvalue problem (1.15) can then be written in the following equivalent form:

$$u - \lambda Tu = 0 \quad (u \in \bar{V}_A, u \neq 0) \, . \tag{1.30}$$

The operator T having the just mentioned properties, we have especially ([1], Chap. 39):

The eigenvalue problem (1.15) *has a countable set of* (positive) *eigenvalues*

$$\lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \ldots, \quad \lim_{n\to\infty} \lambda_n = +\infty . \tag{1.31}$$

The corresponding system

$$v_1, v_2, v_3, \ldots^{1)} \tag{1.32}$$

of orthonormalized (in $\overline{V}_A$) *eigenfunctions is complete in* $\overline{V}_A$ (and because of equivalence of the metrics in $\overline{V}_A$ and V_A, also in V_A).

The system of functions

$$\varphi_n = v_n \sqrt{\lambda_n}, \quad n = 1, 2, \ldots \tag{1.33}$$

is then orthonormal and complete in the space $\overline{V}_B$ (and complete in V_B).

All being prepared in Chap. 39, Rektorys then gets, in Chap. 40, twosided estimates of the Collatz type for the elliptic equation (1.11):

Let $f_0 \in V_B$ be a given nonzero function and let $f_1 \in V_A$ satisfies

$$((v, f_1))_A = ((v, f_0))_B \quad \forall v \in V_A . \tag{1.34}$$

(According to (1.26) it means that

$$f_1 = Tf_0 . \tag{1.35}$$

If all the given data are sufficiently smooth, then f_0 and f_1 satisfy

$$Af_1 = Bf_0 , \tag{1.36}$$

cf. (1.2).) Denote

$$a_0 = ((f_0, f_0))_B , \tag{1.37}$$

$$a_1 = ((f_1, f_0))_B = ((f_1, f_1))_A , \tag{1.38}$$

$$a_2 = ((f_1, f_1))_B , \tag{1.39}$$

$$\varkappa_1 = a_0/a_1 , \quad \varkappa_2 = a_1/a_2 . \tag{1.40}$$

First, Rektorys shows, in a simple way, that

$$a_0 > 0 , \quad a_1 > 0 , \quad a_2 > 0 , \tag{1.41}$$

$$\varkappa_1 \geqq \varkappa_2 \geqq \lambda_1 . \tag{1.42}$$

Then, using the so-called Temple theorem (see [3]), he proves: *If the first eigenvalue*

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[1]) The usual "licence" is chosen for the ordering of eigenvalues in order that the correspondence between (1.31) and (1.32) be one-to-one.

λ_1 *of the problem* (1.15) *is simple and if* l_2 *is such a lower estimate of the second eigenvalue* λ_2 *that* $l_2 > \varkappa_2$ (thus

$$\varkappa_2 < l_2 \leqq \lambda_2), \tag{1.43}$$

then

$$\varkappa_2 - \frac{\varkappa_1 - \varkappa_2}{\dfrac{l_2}{\varkappa_2} - 1} \leqq \lambda_1 \leqq \varkappa_2 . \tag{1.44}$$

In this way, the estimate of the Collatz type is obtained for the elliptic eigenvalue problem (1.15).

As mentioned above, when applying this result practically, the following difficulty arises: While in the case of ordinary differential equations one often succeeds in finding an exact solution of the problem (1.2) (or exact solutions of the problems (1.8)), in the case of partial differential equations the analogous problems should be solved approximately, as a rule, so that estimates of the type (1.44) with $\varkappa_1$, $\varkappa_2$ replaced by the numbers $\tilde{\varkappa}_1$, $\tilde{\varkappa}_2$ constructed with the help of approximate solutions, are no more valid, in general.

To get a better insight into this problematics and to be able to answer the question of applicability of estimates of the form (1.44) in this case, we choose here an other approach than that used in [1], not applying the Temple theorem, but using suitable Fourier expansions. In this way, we come, in Chap. 2, to the following results (p. 224):

Let the first eigenvalue λ_1 *of the problem* (1.15) *be simple. Let* $f_0 \in V_B$ *be not orthogonal, in* V_B, *to the corresponding first eigenfunction* v_1[2] (or, what is the same, to φ_1, cf. (1.33)). *Solve, successively, the following boundary value problems* (cf. (1.8)):

$$\begin{aligned} ((v, f_1))_A &= ((v, f_0))_B , \\ ((v, f_2))_A &= ((v, f_1))_B , \\ &\dots\dots\dots \\ ((v, f_k))_A &= ((v, f_{k-1}))_B , \\ &\dots\dots\dots \end{aligned} \tag{1.45}$$

$\forall v \in V_A$.

Denote

$$\begin{aligned} a_{2n} &= ((f_n, f_n))_B > 0 , \\ a_{2n+1} &= ((f_{n+1}, f_n))_B = ((f_{n+1}, f_{n+1}))_A > 0 , \end{aligned} \tag{1.46}$$

$n = 0, 1, 2, \ldots$,

$$\varkappa_k = a_{k-1}/a_k , \quad k = 1, 2, 3, \ldots \tag{1.47}$$

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[2]) This assumption does not represent a substantial restriction, in practice, see Remark 3.4, p. 235.

Then, first,

(1.48) $$\varkappa_1 \geqq \varkappa_2 \geqq \varkappa_3 \geqq \ldots \geqq \lambda_1,$$

(1.49) $$\lim_{k\to\infty} \varkappa_k = \lambda_1$$

(Theorem 2.1, p. 224.) Thus, λ_1 can be approximated, with an arbitrary accuracy, by $\varkappa_k$, if k is sufficiently large (the problems (1.45) being solved exactly).

Further, *this method yields, in a simple way* (without using the Temple theorem) *estimates of the form* (1.7),

(1.50) $$\varkappa_{k+1} - \frac{\varkappa_k - \varkappa_{k+1}}{\dfrac{l_2}{\varkappa_{k+1}} - 1} \leqq \lambda_1 \leqq \varkappa_{k+1},$$

where $l_2 > \varkappa_{k+1}$ is a lower estimate of the second eigenvalue λ_2. (See the same Theorem 2.1.) Especially, for $k = 1$, we get (1.44).

In Chap. 3, the case of approximate solution of (1.45) is considered. To fix the idea, the Ritz method is chosen. (However, any method, having similar properties, can be applied. Especially, all the main results of this paper remain valid for the finite element method.) Let $\tilde{f}_1$ be the approximate solution of the problem

(1.51) $$((v, u))_A = ((v, f_0))_B \quad \forall v \in V_A,$$

obtained by this method (using N terms of the base), further let $\tilde{f}_2$ be the approximate solution of

(1.52) $$((v, u))_A = ((v, \tilde{f}_1))_B \quad \forall v \in V_A,$$

etc. Denoting

$$\tilde{a}_{2n} = ((\tilde{f}_n, \tilde{f}_n))_B,$$
$$\tilde{a}_{2n+1} = ((\tilde{f}_n, \tilde{f}_{n+1}))_B$$

with $\tilde{f}_0 = f_0$,

(1.53) $$\tilde{\varkappa}_k = \tilde{a}_{k-1}/\tilde{a}_k,$$

we prove (Theorem 3.1, p. 234): *If λ_1 is simple and if f_0 is not orthogonal, in $\overline{V}_B$, to the first eigenfunction φ_1, then*

(1.54) $$\tilde{\varkappa}_1 \geqq \tilde{\varkappa}_2 \geqq \tilde{\varkappa}_3 \geqq \ldots \geqq \lambda_1,$$

(1.55) $$\lim_{\substack{k\to\infty \\ N\to\infty}} \tilde{\varkappa}_k = \lambda_1 .$$

Further it is shown how to modify the estimate (1.50), *or* (1.44) *for this case* (see (3.45), or (3.73) with Remark 3.4).

Although this new estimate is not too complicated, a modification of the present method is given in Chap. 4, enabling to find relatively very simple and accurate estimates (as demonstrated on a numerical example). The idea is the following: Let us perform some steps in solving the problems (1.51), (1.52), ... To fix the idea, let $\tilde{f}_1$ and $\tilde{f}_2$ be found. (The number of steps depends on the required accuracy; an arbitrary accuracy can be obtained by (1.55).) Having $\tilde{f}_2$, it may happen that we succeed in finding such a function $\hat{f}_1$ that the integral identity

$$((v, \tilde{f}_2))_A = ((v, \hat{f}_1))_B \quad \forall v \in V_A \tag{1.56}$$

is satisfied exactly. (This happens very often, because it is much easier, practically, to find $\hat{f}_1$ if $\tilde{f}_2$ is known, than conversely, since the order of the operator B is smaller than that of the operator A; if, especially, the eigenvalue problem $Au - \lambda u = 0$ is solved, so that B is the identity operator, and if the coefficients of the operator A as well as the function $\tilde{f}_2$ are sufficiently smooth, then $\hat{f}_1 = A\tilde{f}_2$. Thus, in this case, the function $\hat{f}_1$ is obtained by operations of the differentiation only.) Now, because (1.56) is fulfilled exactly, the twosided estimate (1.50) is valid with $\varkappa$'s constructed from the functions $\hat{f}_1, \tilde{f}_2$ instead of f_1, f_2. *In this way a very simple method of two-sided eigenvalue estimates is received, giving, moreover, relatively very exact estimates* (cf. Example 4.1, p. 237).

2. CONVERGENCE OF THE SEQUENCE $\varkappa_k$. ESTIMATES OF THE COLLATZ TYPE

Let the forms $((v, u))_A$, $((v, u))_B$ satisfy assumptions (1.16)–(1.21) concerning their symmetry, boundedness and V_A- or V_B-ellipticity. Let

$$v_1, v_2, v_3, \ldots \tag{2.1}$$

be the complete set of eigenfunctions of the problem (1.15), orthonormal in $\bar{V}_A$ (on $\bar{V}_A$ see (1.27), p. 214), i.e.

$$((v_i, v_j))_A = \begin{cases} 0 & \text{if} \quad i \neq j, \\ 1 & \text{if} \quad i = j. \end{cases} \tag{2.2}$$

Then (cf. (1.33))

$$\varphi_1 = v_1 \sqrt{\lambda_1}, \quad \varphi_2 = v_2 \sqrt{\lambda_2}, \quad \varphi_3 = v_3 \sqrt{\lambda_3}, \ldots \tag{2.3}$$

is a complete set, orthonormal in $\bar{V}_B$ i.e.

$$((\varphi_i, \varphi_j))_B = \begin{cases} 0 & \text{if} \quad i \neq j, \\ 1 & \text{if} \quad i = j. \end{cases} \tag{2.4}$$

Let the first eigenvalue λ_1 be simple and let $f_0 \in V_B$ be not orthogonal, in $\bar{V}_B$, to v_1, or, what is the same, to φ_1. Solve, successively, the sequence of problems

$$\begin{aligned} ((v, f_1))_A &= ((v, f_0))_B\,, \\ ((v, f_2))_A &= ((v, f_1))_B\,, \\ &\dots\dots\dots \\ ((v, f_k))_A &= ((v, f_{k-1}))_B \\ &\dots\dots\dots \end{aligned} \tag{2.5}$$

Each of these problems is uniquely solvable (cf. (1.24)–(1.26)). Denote

$$a_{2n} = ((f_n, f_n))_B\,, \tag{2.6}$$

$$a_{2n+1} = ((f_n, f_{n+1}))_B = ((f_{n+1}, f_{n+1}))_A\,,\text{[3]} \tag{2.7}$$

$n = 0, 1, 2, \ldots,$

$$\varkappa_k = a_{k-1}/a_k\,, \quad k = 1, 2, 3, \ldots\,. \tag{2.8}$$

The first purpose of this chapter is to show that

$$\varkappa_1 \geqq \varkappa_2 \geqq \varkappa_3 \geqq \ldots \geqq \lambda_1 \tag{2.9}$$

and that

$$\lim_{k \to \infty} \varkappa_k = \lambda_1\,. \tag{2.10}$$

In the second part of this chapter we show that, for every $k = 1, 2, 3, \ldots$, we have

$$\varkappa_{k+1} - \frac{\varkappa_k - \varkappa_{k+1}}{\dfrac{l_2}{\varkappa_{k+1}} - 1} \leqq \lambda_1 \leqq \varkappa_{k+1} \tag{2.11}$$

where l_2 is a lower estimate for the second eigenvalue λ_2, greater than $\varkappa_{k+1}$.

The proof of (2.9) is simple: We have, for every $t \in (-\infty, +\infty)$,

$$\begin{aligned} 0 \leqq ((f_n + tf_{n+1}, f_n + tf_{n+1}))_B &= ((f_n, f_n))_B + 2t((f_n, f_{n+1}))_B + \\ &+ t^2((f_{n+1}, f_{n+1}))_B = a_{2n} + 2ta_{2n+1} + t^2 a_{2n+2}\,. \end{aligned} \tag{2.12}$$

Because the quadratic expression in t should be nonnegative for all t, its discriminant cannot by positive. Thus

$$a_{2n+1}^2 \leqq a_{2n} a_{2n+2}\,, \tag{2.13}$$

wherefrom, dividing by $a_{2n+1} a_{2n+2}$,

$$\varkappa_{2n+2} \leqq \varkappa_{2n+1}\,. \tag{2.14}$$

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[3]) Because of (2.5). It follows that $a_i > 0$, $i = 0, 1, 2, \ldots$.

Similarly

$$0 \leqq ((f_{n+1} + tf_{n+2}, f_{n+1} + tf_{n+2}))_A = ((f_{n+1}, f_{n+1}))_A +$$

$$+ 2t((f_{n+1}, f_{n+2}))_A + t^2((f_{n+2}, f_{n+2}))_A = a_{2n+1} + 2ta_{2n+2} + t^2 a_{2n+3},$$

wherefrom, in the same way as before,

$$\varkappa_{2n+3} \leqq \varkappa_{2n+2}. \tag{2.15}$$

Moreover, because $f_{n+1} \in V_A$, we have

$$\lambda_1 \leqq \frac{((f_{n+1}, f_{n+1}))_A}{((f_{n+1}, f_{n+1}))_B} = \frac{a_{2n+1}}{a_{2n+2}} = \varkappa_{2n+2}. \tag{2.16}$$

(2.14), (2.15) and (2.16) yield (2.9).

To prove (2.10) and (2.11), we use Fourier expansions of the functions f_n with respect to the orthonormal (in $\bar{V}_B$) system (2.3).

Let

$$f_0 = \sum_{i=1}^{\infty} \alpha_i \varphi_i \tag{2.17}$$

be the Fourier expansion of the function f_0 in $\bar{V}_B$. Thus

$$\alpha_i = ((f_0, \varphi_i))_B, \quad i = 1, 2, 3, \ldots, \tag{2.18}$$

while, according to the assumption,

$$\alpha_1 = ((f_0, \varphi_1))_B \neq 0. \tag{2.19}$$

Moreover,

$$\sum_{i=1}^{\infty} \alpha_i^2 < +\infty. \tag{2.20}$$

Denote, for a while,

$$F_1 = \sum_{i=1}^{\infty} \frac{\alpha_i}{\lambda_i} \varphi_i. \tag{2.21}$$

Because of (2.20) and of $\lambda_i \to +\infty$ for $i \to \infty$, the series (2.21) is convergent (in $\bar{V}_B$). We have (cf. (2.3) and the first of the integral identities (2.5))

$$F_1 = \sum_{i=1}^{\infty} \frac{\alpha_i}{\lambda_i} \varphi_i = \sum_{i=1}^{\infty} \frac{((f_0, v_i \sqrt{\lambda_i}))_B}{\lambda_i} v_i \sqrt{\lambda_i} = \tag{2.22}$$

$$= \sum_{i=1}^{\infty} ((f_0, v_i))_B v_i = \sum_{i=1}^{\infty} ((f_1, v_i))_A v_i = f_1,$$

since $\{v_i\}$ is a complete orthonormal system in $\bar{V}_A$ and the last series in (2.22) is the

Fourier series of the function f_1, thus converging to f_1 in $\bar{V}_A$ (and the more in $\bar{V}_B$).

Similarly, denoting $\alpha_i/\lambda_i = \beta_i$, so that

$$f_1 = \sum_{i=1}^{\infty} \beta_i \varphi_i ,$$

we get

$$F_2 = \sum_{i=1}^{\infty} \frac{\alpha_i}{\lambda_i^2} \varphi_i = \sum_{i=1}^{\infty} \frac{\beta_i}{\lambda_i} \varphi_i = \sum_{i=1}^{\infty} \frac{((f_1, v_i \sqrt{\lambda_i}))_B}{\lambda_i} v_i \sqrt{\lambda_i} =$$

$$= \sum_{i=1}^{\infty} ((f_1, v_i))_B = \sum_{i=1}^{\infty} ((f_2, v_i))_A \, v_i = f_2$$

in $\bar{V}_A$ as well as in $\bar{V}_B$, and, in general

$$f_k = \sum_{i=1}^{\infty} \frac{\alpha_i}{\lambda_i^k} \varphi_i \tag{2.23}$$

(in $\bar{V}_A$ as well as in $\bar{V}_B$).

Putting this result into (2.6) and (2.7), we get

$$a_{2n} = ((f_n, f_n))_B = \left(\left(\sum_{i=1}^{\infty} \frac{\alpha_i}{\lambda_i^n} \varphi_i, \sum_{j=1}^{\infty} \frac{\alpha_j}{\lambda_j^n} \varphi_j\right)\right)_B = \sum_{i=1}^{\infty} \frac{\alpha_i^2}{\lambda_i^{2n}}, \tag{2.24}$$

$$a_{2n+1} = ((f_{n+1}, f_{n+1}))_A = \left(\left(\sum_{i=1}^{\infty} \frac{\alpha_i}{\lambda_i^{n+1}} \varphi_i , \sum_{j=1}^{\infty} \frac{\alpha_j}{\lambda_j^{n+1}} \varphi_j\right)\right)_A = \tag{2.25}$$

$$= \sum_{i,j=1}^{\infty} \frac{\alpha_i}{\lambda_i^{n+1}} \frac{\alpha_j}{\lambda_j^{n+1}} \sqrt{\lambda_i} \sqrt{\lambda_j} \, ((v_i, v_j))_A = \sum_{i=1}^{\infty} \frac{\alpha_i^2}{\lambda_i^{2n+1}} .$$

Thus

$$a_k = \sum_{i=1}^{\infty} \frac{\alpha_i^2}{\lambda_i^k}, \quad k = 0, 1, 2, \ldots, \tag{2.26}$$

$$\varkappa_k = \left(\sum_{i=1}^{\infty} \frac{\alpha_i^2}{\lambda_i^{k-1}}\right) \Big/ \left(\sum_{i=1}^{\infty} \frac{\alpha_i^2}{\lambda_i^k}\right), \quad k = 0, 1, 2, \ldots . \tag{2.27}$$

Now, to prove (2.10) it is sufficient to write

$$\varkappa_k = \frac{\dfrac{1}{\lambda_1^{k-1}} \sum_{i=1}^{\infty} \alpha_i^2 \left(\dfrac{\lambda_1}{\lambda_i}\right)^{k-1}}{\dfrac{1}{\lambda_1^k} \sum_{i=1}^{\infty} \alpha_i^2 \left(\dfrac{\lambda_1}{\lambda_i}\right)^k} = \lambda_1 \frac{\alpha_1^2 + \sum_{i=2}^{\infty} \alpha_i^2 \left(\dfrac{\lambda_1}{\lambda_i}\right)^{k-1}}{\alpha_1^2 + \sum_{i=2}^{\infty} \alpha_i^2 \left(\dfrac{\lambda_1}{\lambda_i}\right)^k} \tag{2.28}$$

whence it immediately follows

$$\lim_{k \to \infty} \varkappa_k = \lambda_1 , \tag{2.29}$$

because $\alpha_1 \neq 0$ and $\lambda_1 < \lambda_2 \leqq \lambda_3 \leqq \ldots$ holds, so that the sums of both the series on the right hand side of (2.28) can be made arbitrarily small if k is sufficiently large.

Remark 2.1. In the preceding text we assumed that the least eigenvalue λ_1 was simple and that f_0 was not orthogonal, in $\overline{V}_B$, to the first eigenfunction φ_1, i.e. that

$$\alpha_1 = ((f_0, \varphi_1))_B \neq 0 . \tag{2.30}$$

To derive both (2.9) and (2.10), the assumption of simplicity of λ_1 is superfluous: When deriving (2.14), (2.15), this assumption was nowhere used. The same holds for (2.16). Moreover, only $f_0 \neq 0$ was sufficient to be required. To examine (2.29), let us assume that the least eigenvalue is of multiplicity s, so that

$$\lambda_1 = \lambda_2 = \ldots \lambda_s < \lambda_{s+1} \tag{2.31}$$

and

$$\varphi_1, \varphi_2, \ldots, \varphi_s \tag{2.32}$$

are corresponding eigenfunctions orthonormalized in $\overline{V}_B$. Let f_0 be not ortogonal to each of the functions (2.32), so that at least one of the numbers

$$\alpha_i = ((f_0, \varphi_i))_B , \quad i = 1, 2, \ldots, s ,$$

is different from zero. Then (cf. (2.28))

$$\varkappa_k = \lambda_1 \frac{\alpha_1^2 + \ldots + \alpha_s^2 + \sum\limits_{i=s+1}^{\infty} \alpha_i^2 \left(\frac{\lambda_1}{\lambda_i}\right)^{k-1}}{\alpha_1^2 + \ldots + \alpha_s^2 + \sum\limits_{i=s+1}^{\infty} \alpha_i^2 \left(\frac{\lambda_1}{\lambda_i}\right)^{k}} , \tag{2.33}$$

whence

$$\lim_{k \to \infty} \varkappa_k = \lambda_1 . \tag{2.34}$$

as before.

If λ_1 is simple and if

$$\alpha_1 = ((f_0, \varphi_1))_B = 0 , \tag{2.35}$$

then (2.9) remains true, but (2.10), i.e. (2.29), does no more hold. For example, if then λ_2 is simple[4]) and

$$\alpha_2 = ((f_0, \varphi_2))_B \neq 0 ,$$

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[4]) Or not, cf. (2.33).

it follows from (2.27) that

$$\varkappa_k = \lambda_1 \frac{\alpha_2^2 + \sum_{i=3}^{\infty} \alpha_i^2 \left(\frac{\lambda_2}{\lambda_i}\right)^{k-1}}{\alpha_2^2 + \sum_{i=3}^{\infty} \alpha_i^2 \left(\frac{\lambda_2}{\lambda_i}\right)^{k}},$$

whence

(2.36) $$\lim_{k\to\infty} \varkappa_k = \lambda_2 .$$

If is easy to examine what happens in other cases.

Let us turn to the proof of (2.11).

Let λ_1 be simple and $\alpha_1 = ((f_0, \varphi_1))_B \neq 0$. Let l_2 be a lower estimate of λ_2 greater then $\varkappa_{k+1}$. According to (2.9) the right-hand side inequality in (2.11) is ensured. We thus have to prove the validity of the left-hand one.

Let

(2.37) $$\varkappa_{k+1} > \lambda_1 .$$

(If $\varkappa_{k+1} = \lambda_1$, there is nothing to prove, because $\varkappa_k - \varkappa_{k+1} \geqq 0$ and $l_2/\varkappa_{k+1} > 1$.) Then (cf. (2.26))

(2.38) $$\varkappa_{k+1} \frac{\varkappa_k - \lambda_1}{\varkappa_{k+1} - \lambda_1} = \frac{a_{k-1} - \lambda_1 a_k}{a_k - \lambda_1 a_{k+1}} =$$

$$= \frac{\sum_{i=1}^{\infty} \frac{\alpha_i^2}{\lambda_i^{k-1}} - \lambda_1 \sum_{i=1}^{\infty} \frac{\alpha_i^2}{\lambda_i^k}}{\sum_{i=1}^{\infty} \frac{\alpha_i^2}{\lambda_i^k} - \lambda_1 \sum_{i=1}^{\infty} \frac{\alpha_i^2}{\lambda_i^{k+1}}} = \frac{\sum_{i=2}^{\infty} \frac{\alpha_i^2}{\lambda_i^{k-1}} \left(1 - \frac{\lambda_1}{\lambda_i}\right)}{\sum_{i=2}^{\infty} \frac{\alpha_i^2}{\lambda_i^{k}} \left(1 - \frac{\lambda_1}{\lambda_i}\right)} .$$

Now, for $i \geqq 2$, we have $\lambda_i^k \geqq \lambda_2 \lambda_i^{k-1}$. Thus

(2.39) $$\varkappa_{k+1} \frac{\varkappa_k - \lambda_1}{\varkappa_{k+1} - \lambda_1} \geqq \frac{\sum_{i=2}^{\infty} \frac{\alpha_i^2}{\lambda_i^{k-1}} \left(1 - \frac{\lambda_1}{\lambda_i}\right)}{\sum_{i=2}^{\infty} \frac{\alpha_i^2}{\lambda_2 \lambda_i^{k-1}} \left(1 - \frac{\lambda_1}{\lambda_i}\right)} = \lambda_2 \geqq l_2 .$$

Further

(2.40) $$\frac{\varkappa_k - \lambda_1}{\varkappa_{k+1} - \lambda_1} = \frac{\varkappa_k - \varkappa_{k+1}}{\varkappa_{k+1} - \lambda_1} + 1 .$$

(2.39) and (2.40) yield

$$\frac{\varkappa_k - \varkappa_{k+1}}{\varkappa_{k+1} - \lambda_1} \geqq \frac{l_2}{\varkappa_{k+1}} - 1$$

and finally,

$$\frac{\varkappa_k - \varkappa_{k+1}}{\frac{l_2}{\varkappa_{k+1}} - 1} \geqq \varkappa_{k+1} - \lambda_1 , \tag{2.41}$$

what is nothing else than the left-hand side inequality in (2.11).

In this way, (2.11) is proved. Note that here the assumption on λ_1 to be simple is essential (we require that $\varkappa_{k+1} < l_2$ and, at the same time, we have $\lambda_1 \leqq \varkappa_{k+1}$ and $l_2 \leqq \lambda_2$).

Let us summarize the results received in this chapter into the following

Theorem 2.1. *Let the forms* $((v, u))_A$, $((v, u))_B$ *satisfy assumptions* (1.16)–(1.21) (concerning their symmetry, boundedness and ellipticity in V_A, resp. V_B). *Let the first eigenvalue* λ_1 *of the problem* (1.15) *be simple*[5]). *Let* $f_0 \in V_B$ *be not ortogonal, in* $\bar{V}_B$,[6]) *to the first eigenfunction* φ_1. *Let* $f_1, f_2, f_3, \ldots$ *be solutions of the boundary value problems* (2.5), *let* a_k, $\varkappa_k$ *be defined by* (2.6)–(2.8). *Then we have*

$$\varkappa_1 \geqq \varkappa_2 \geqq \varkappa_3 \geqq \ldots \geqq \lambda_1 , \tag{2.42}$$

$$\lim_{k \to \infty} \varkappa_k = \lambda_1 , \tag{2.43}$$

and the following twosided estimate for λ_1 *is valid*:

$$\varkappa_{k+1} - \frac{\varkappa_k - \varkappa_{k+1}}{\frac{l_2}{\varkappa_{k+1}} - 1} \leqq \lambda_1 \leqq \varkappa_{k+1} , \tag{2.44}$$

where l_2 *is a lower bound*[7]) *for the second eigenvalue* λ_2, *greater then* $\varkappa_{k+1}$, *thus satisfiing*

$$\varkappa_{k+1} < l_2 \leqq \lambda_2 .$$

In this way, a twosided eigenvalue estimate of the Collatz typ is received. No Temple theorem has been used.

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[5]) See, however, Remark 2.1.

[6]) On $\bar{V}_A$ and $\bar{V}_B$ see p. 213.

[7]) Received, as a rule, using an appropriate comparison theorem.

3. APPROXIMATE SOLUTIONS. THE SEQUENCE $\{\tilde{\varkappa}_k\}$ AND ITS PROPERTIES

The aim of this chapter is to make clear what happens if the "iterative" problems (2.5) are solved approximately. To be concrete, let us use the Ritz method. The same results are obtained when using some other method (e.g. the finite element method) with properties similar to those which are summarized at the beginning of this chapter for the Ritz method.

Let

$$w_1, w_2, w_3, \ldots \tag{3.1}$$

be a complete (linearly independent) system in $\overline{V}_A$. Let us choose the first N functions

$$w_1, w_2, \ldots, w_N \tag{3.2}$$

from this system and denote by S_N the N-dimensional subspace of $\overline{V}_A$, constituted by these functions. Let us solve the problem of finding such a function $u \in \overline{V}_A$ that

$$((v, u))_A = ((v, f))_B \quad v \in \overline{V}_A \,. \tag{3.3}$$

The solution u minimizes, in $\overline{V}_A$, the functional

$$Fv = ((v, v))_A - 2((v, f))_B \,. \tag{3.4}$$

As well known, the Ritz method consist in finding such a function

$$\tilde{u}_N = \sum_{i=1}^{N} a_{Ni} w_i \tag{3.5}$$

which minimizes this functional in S_N. It can be shown that $\tilde{u}_N$ is the orthogonal projection, in $\overline{V}_A$, of u into S_N, i.e. that

$$\tilde{u}_N = P_N u \,, \tag{3.6}$$

where P_N is the corresponding projector. Obviously,

$$\|\tilde{u}_N\|_{V_A} \leqq \|u\|_{V_A} \,. \tag{3.7}$$

(Remind that $\|h\|_{V_A} = \sqrt{((h, h))_A}$.) If an orthonormalized in V_A system $\{z_i\}$ is used instead of (3,1), i.e. if

$$((z_i, z_k))_A = \delta_{ik} \,, \tag{3.8}$$

then

$$u = \sum_{i=1}^{\infty} ((z_i, u))_A \, z_i = \sum_{i=1}^{\infty} ((z_i, f))_B \, z_i \tag{3.9}$$

(according to (3.3)), while

$$\tilde{u}_N = \sum_{i=1}^{N} ((z_i, f))_B \, z_i \,. \tag{3.10}$$

The system $\{z_i\}$ can be obtained from the system (3.1) using the familiar Schmidt orthonormalization in $\overline{V}_A$. The function (3.5) with a_{Ni} found by the Ritz method can then be written in the form (3.10)[8]).

Now, let us solve the problems (2.5), p. 219, approximately using the Ritz method with the base functions (3.2). Let

$$\{\tilde{f}_n\}, \tag{3.11}$$

$$\{\bar{f}_n\} \tag{3.12}$$

be sequences of such functions that $\tilde{f}_{n+1}$ is the approximate solution of the problem

$$((v, u))_A = ((v, \tilde{f}_n))_B \quad \forall v \in V_A \tag{3.13}$$

($n = 0, 1, 2, \ldots, \tilde{f}_0 = f_0$), obtained by the Ritz method, while $\bar{f}_{n+1}$ is its exact solution, i.e.

$$((v, \bar{f}_{n+1}))_A = ((v, \tilde{f}_n))_B \quad \forall v \in V_A \tag{3.14}$$

($n = 0, 1, 2, \ldots, \bar{f}_0 = f_0, \bar{f}_1 = f_1$).

We shall assume, as before, that λ_1 is simple and that f_0 is not orthogonal, in $\overline{V}_B$, to the first eigenfunction φ_1, i.e. that

$$((f_0, \varphi_1))_B \neq 0\,. \tag{3.15}$$

(Cf. (2.19).) In particular, $f_0 \neq 0$. It will follow from Remark 3.3 (see also Theorem 3.1) that for all sufficiently large N, the functions (3.11) and (3.12) are different from zero. We shall always assume N so large to quarantee this property.

Denote

$$\tilde{a}_{2n} = ((\tilde{f}_n, \tilde{f}_n))_B\,, \tag{3.16}$$

$$\tilde{a}_{2n+1} = ((\tilde{f}_n, \tilde{f}_{n+1}))_B\,, \tag{3.17}$$

$n = 0, 1, 2, \ldots$. We have, according to (3.14), written for $v = \tilde{f}_{n+1}$:

$$\tilde{a}_{2n+1} = ((\tilde{f}_n, \tilde{f}_{n+1}))_B = ((\bar{f}_{n+1}, \tilde{f}_{n+1}))_A\,.$$

Now, $\bar{f}_{n+1}$ is of the form (3.9), $\tilde{f}_{n+1}$ of the form (3.10), both with $f = \tilde{f}_n$. Thus

$$((\bar{f}_{n+1}, \tilde{f}_{n+1}))_A = ((\tilde{f}_{n+1}, \tilde{f}_{n+1}))_A \tag{3.18}$$

because z_i are orthonormal in $\overline{V}_A$. From $\tilde{f}_n \neq 0$, $\tilde{f}_{n+1} \neq 0$ and from (3.16)–(3.18) it follows

$$\tilde{a}_{2n} > 0\,, \quad \tilde{a}_{2n+1} > 0\,, \quad n = 0, 1, 2, \ldots\,. \tag{3.19}$$

We thus can define

$$\tilde{\varkappa}_k = \tilde{a}_{k-1}/\tilde{a}_k\,, \quad k = 1, 2, 3, \ldots\,. \tag{3.20}$$

[8]) This fact is mentioned from theoretical reasons only, see (3.18), etc.

Now, it is easy to prove that

$$\tilde{\varkappa}_1 \geqq \tilde{\varkappa}_2 \geqq \tilde{\varkappa}_3 \geqq \ldots \geqq \lambda_1 . \tag{3.21}$$

In fact, the inequality

$$\tilde{\varkappa}_{2n+2} \leqq \tilde{\varkappa}_{2n+1} ,$$

$n = 0, 1, 2, \ldots$ can be derived in the same way as the inequality (2.14), p. 219. Further,

$$\tilde{a}_{2n+1} = ((\tilde{f}_{n+1}, \tilde{f}_{n+1}))_A$$

as we have just proved, and

$$\tilde{a}_{2n+2} = ((\tilde{f}_{n+1}, \tilde{f}_{n+1}))_B = ((\tilde{f}_{n+1}, \tilde{f}_{n+2}))_A = ((\tilde{f}_{n+1}, \tilde{f}_{n+2}))_A ,$$

because $\tilde{f}_{n+1}$ is of the form (3.10) with $f = \tilde{f}_n$ and $\bar{f}_{n+2}$, or $\tilde{f}_{n+2}$ is of the form (3.9), or (3.10), respectively, with $f = \tilde{f}_{n+1}$. Thus we can use precisely the same procedure as when proving (2.15). Because

$$\tilde{\varkappa}_{2n+2} = \frac{\tilde{a}_{2n+1}}{\tilde{a}_{2n+2}} = \frac{((\tilde{f}_{n+1}, \tilde{f}_{n+1}))_A}{((\tilde{f}_{n+1}, \tilde{f}_{n+1}))_B} \geqq \lambda_1 ,$$

the proof of (3.21) is finished.

Further, we shall prove that not only the analogue (3.21) of (2.9) is valid, but that also the analogue of (2.10) holds:

$$\lim_{\substack{k \to \infty \\ N \to \infty}} \tilde{\varkappa}_k = \lambda_1 . \tag{3.22}$$

However, the proof of this assertion is not so simple as the proof of (3.21), and it is left to Remark 3.3.

We proceed now to give an analogue of the twosided estimate (2.11), p. 219. First we show that, k being fixed, it is possible to make $|\varkappa_k - \tilde{\varkappa}_k|$ arbitrarily small if N in the Ritz method is sufficiently large. To this purpose, we show the same assertion for $\|f_p - \tilde{f}_p\|_{V_B}$ if p is fixed.

Let us remind, first, that when solving the problem (1.24), p. 221,

(i) a constant $c > 0$ exists, independent of g and v, such that we have

$$\|u\|_{V_A} \leqq c\|g\|_{V_B} \tag{3.23}$$

(cf. (1.25), p. 221);

(ii) using the Ritz method, to every $\eta > 0$ an N_0 can be found such that

$$\|\tilde{u}_N - u\|_{V_A} < \eta \tag{3.24}$$

for every $N > N_0$. (Here, $\tilde{u}_N$ is the approximate solution received by the Ritz method when using the functions (3.2).)

Further, note that (see (1.18), (1.20), (3.7))

$$(3.25)\quad \|\tilde{u}_N\|_{V_A} \leqq \frac{1}{\sqrt{\alpha}} \|\tilde{u}_N\|_{V_A} \leqq \frac{1}{\sqrt{\alpha}} \|u\|_{V_A} \leqq \frac{\sqrt{K_1}}{\sqrt{\alpha}} \|u\|_{V_A} \leqq \frac{\sqrt{K_1}}{\sqrt{\alpha}} c\|g\|_{V_B} = \bar{c}\|g\|_{V_B},$$

where the constant

$$(3.26)\qquad \bar{c} = \frac{\sqrt{K_1}}{\sqrt{\alpha}} c$$

does not depend on g.

Let, as before, λ_1 be simple, let f_0 satisfy (3.15). Let, again, $f_1, f_2, f_3, \ldots, f_p$ be (exact) solutions of the first p "iterative" problems (2.5). Let $\hat{f}_i$ be the Ritz approximation of f_i, i.e. the approximate solution of the problem

$$((v, u))_A = ((v, f_{i-1}))_B \quad \forall v \in V_A$$

received by the Ritz method, taking the first N_i terms of the base (3.1). (Thus

$$(3.27)\qquad \hat{f}_i = P_{N_i} T f_{i-1}, \quad i = 1, \ldots, p,$$

in the sense of (3.6.)[9]) Let, as before, $\tilde{f}_i$, $i = 1, \ldots, p$, be the Ritz approximation of the solution of the problem

$$((v, u))_A = ((v, \tilde{f}_{i-1}))_B \quad \forall v \in V_A .$$

Let $\eta > 0$ be given and let N_i, $i = 1, \ldots, p$, be so large that

$$(3.28)\qquad \|f_i - \hat{f}_i\|_{V_A} < \eta .$$

Such N_i always exist (see (3.24)). Moreover, if we denote

$$(3.29)\qquad N = \max(N_1, \ldots, N_p),$$

(3.28) remains true if we substitute N for each of N_i. (This is a well-known property of the Ritz method, based on (3.9).)

Performing the Ritz method with this N, we get according to (3.28) and (3.25) (note that $\tilde{f}_1 = \hat{f}_1$)

$$(3.30)\qquad \|f_p - \tilde{f}_p\|_{V_B} \leqq \|f_p - \tilde{f}_p\|_{V_A} \leqq \|f_p - \hat{f}_p\|_{V_A} + \|\hat{f}_p - \tilde{f}_p\|_{V_A} \leqq$$
$$\leqq \eta + \bar{c}\|f_{p-1} - \tilde{f}_{p-1}\|_{V_B} \leqq \eta + \bar{c}(\eta + \bar{c}\|f_{p-2} - \tilde{f}_{p-2}\|_{V_B}) \leqq \ldots \leqq$$
$$= \eta + \bar{c}\{\eta + \bar{c}[\eta + \ldots + \bar{c}\|f_1 - \tilde{f}_1\|_{V_B}]\} =$$
$$= \eta(1 + \bar{c} + \bar{c}^2 + \ldots + \bar{c}^{p-1}) = \eta C_p,$$

with

$$(3.31)\qquad C_p = 1 + \bar{c} + \bar{c}^2 + \ldots + \bar{c}^{p-1} .$$

[9]) The functions $\hat{f}_i$ play only an auxiliary role here, they are not constructed, actually.

Thus p being given, to arbitrary $\varrho > 0$ such an N can be found that

$$\|f_q - \tilde{f}_q\|_B < \varrho \tag{3.32}$$

(even that

$$\|f_q - \tilde{f}_q\|_A < \varrho) \tag{3.33}$$

for all $q = 1, 2, \ldots, p$.

Now, from the form of $a_{2n}, a_{2n+1}, \varkappa_k$ and $\tilde{a}_{2n}, \tilde{a}_{2n+1}, \tilde{\varkappa}_k$ (cf. (3.16), (3.17), (3.20)) it immediately follows (choosing ϱ sufficiently small) that k being given, to every $\zeta > 0$ such an N can be found that

$$|\varkappa_k - \tilde{\varkappa}_k| < \zeta . \tag{3.34}$$

Now, the sequence of $\varkappa_k$ satisfies (2.43),

$$\lim_{k\to\infty} \varkappa_k = \lambda_1 . \tag{3.35}$$

Thus choosing $\varepsilon > 0$ and denoting $\varepsilon/2 = \zeta$, it is possible, first, to find such a k that

$$|\varkappa_k - \lambda_1| < \zeta ,$$

and then to find such an N that

$$|\varkappa_k - \tilde{\varkappa}_k| < \zeta .$$

Thus we have:

Proposition 3.1. *If k is given, then to every $\varepsilon > 0$ it is possible to find such an N that using the Ritz method with the first N terms of the base, we have*

$$|\tilde{\varkappa}_k - \lambda_1| < \varepsilon . \tag{3.36}$$

Remark 3.1. In Remark 3.3 we give a substantially stronger assertion, even that

$$\lim_{\substack{k\to\infty \\ N\to\infty}} \tilde{\varkappa}_k = \lambda_1$$

(see also Theorem 3.1).

Remark 3.2. When deriving inequality (3.34), it was possible, using (3.30) and the formulae for $a_{2n}, a_{2n+1}, \varkappa_k, \tilde{a}_{2n}, \tilde{a}_{2n+1}, \tilde{\varkappa}_k$, to give an explicite estimate for $|\varkappa_k - \tilde{\varkappa}_k|$. We did not do it, because our aim was to show only that $\tilde{\varkappa}_k$ can be made arbitrarily close to λ_1 if k and N are sufficiently large, and to establish in this way the usefulness of carrying out our iterational process. To get the error-estimate when stopping the process after k steps we can go on in a considerably simpler way:

Thus let the functions $\tilde{f}_0 = f_0, \tilde{f}_1, \ldots, \tilde{f}_{k-1}$ be known. As described above, $\tilde{f}_k$ is the approximate solution of the problem

$$((v, u))_A = ((v, \tilde{f}_{k-1}))_B \quad \forall v \in V_A ,$$

while $\bar{f}_k$ is its exact solution, thus satisfying

$$((v, \bar{f}_k))_A = ((v, \tilde{f}_{k-1}))_B \quad \forall v \in V_A .$$

If $\bar{f}_k$ be known, we could write down the tvosided eigenvalue estimate in the form (1.44),

$$\varkappa_2 - \frac{\varkappa_1 - \varkappa_2}{\dfrac{l_2}{\varkappa_2} - 1} \leqq \lambda_1 \leqq \varkappa_2 \,, \tag{3.37}$$

taking $\bar{f}_{k-1}$ for the outcoming function f_0 and $\bar{f}_k$ for f_1, i.e. writing

$$\varkappa_1 = a_0/a_1 \,, \quad \varkappa_2 = a_1/a_2 \tag{3.38}$$

with

$$a_0 = ((\tilde{f}_{k-1}, \tilde{f}_{k-1}))_B \,, \quad a_1 = ((\tilde{f}_{k-1}, \bar{f}_k))_B \,, \quad a_2 = ((\bar{f}_k, \bar{f}_k))_B \,. \tag{3.39}$$

With the help of (3.38), the inequalities (3.37) can be rewritten in the form

$$\varkappa_2 \frac{l_2 - \varkappa_1}{l_2 - \varkappa_2} \leqq \lambda_1 \leqq \varkappa_2 \,,$$

or

$$\frac{a_1 l_2 - a_0}{a_2 l_2 - a_1} \leqq \lambda_1 \leqq \frac{a_1}{a_2} \,. \tag{3.40}$$

Now, we do not know $\bar{f}_k$, but only its Ritz approximation $\tilde{f}_k$, received with an error ε,

$$\|\bar{f}_k - \tilde{f}_k\|_{V_B} \leqq \varepsilon \,. \tag{3.41}$$

(How to get error estimates using the Ritz method see e.g. [1], especially Chaps 11 and 21.) Before substituting $\tilde{f}_k$ for $\bar{f}_k$ into (3.39) note that

$$|((\tilde{f}_{k-1}, \bar{f}_k))_B - ((\tilde{f}_{k-1}, \tilde{f}_k))_B| = |((\tilde{f}_{k-1}, \bar{f}_k - \tilde{f}_k))_B| \leqq \|\tilde{f}_{k-1}\|_{V_B} \varepsilon \,, \tag{3.42}$$

$$|((\bar{f}_k, \bar{f}_k))_B - ((\tilde{f}_k, \tilde{f}_k))_B| = |((\bar{f}_k, \bar{f}_k - \tilde{f}_k))_B - ((\tilde{f}_k, \tilde{f}_k - \bar{f}_k))_B| \leqq \tag{3.43}$$

$$\leqq (\|\tilde{f}_k\|_{V_B} + \varepsilon)\, \varepsilon + \|\tilde{f}_k\|_{V_B}\, \varepsilon = (2\|\tilde{f}_k\|_{V_B} + \varepsilon)\, \varepsilon \,.$$

Thus denoting

$$\tilde{a}_0 = ((\tilde{f}_{k-1}, \tilde{f}_{k-1}))_B \,, \ \tilde{a}_1 = ((\tilde{f}_{k-1}, \tilde{f}_k))_B, \ \tilde{a}_2 = ((\tilde{f}_k, \tilde{f}_k))_B \,, \tag{3.44}$$

we have

$$\tilde{a}_0 = a_0 \,, \quad a_1 \geqq \tilde{a}_1 - \|\tilde{f}_{k-1}\|_{V_B} \varepsilon \,, \quad a_2 \leqq \tilde{a}_2 + (2\|\tilde{f}_k\|_{V_B} + \varepsilon)\, \varepsilon \,.$$

Substituting into (3.40) and noting that

$$\tilde{a}_1/\tilde{a}_2 \geqq \lambda_1$$

(cf. (3.21)), we get finally the following relatively simple *twosided eigenvalue estimate*

$$\frac{(\tilde{a}_1 - \|\tilde{f}_{k-1}\|_{V_B} \varepsilon)\, l_2 - \tilde{a}_0}{[\tilde{a}_2 + (2\|\tilde{f}_k\|_{V_B} + \varepsilon)\, \varepsilon]\, l_2 + \|\tilde{f}_{k-1}\|_{V_B} \varepsilon - \tilde{a}_1} \leqq \lambda_1 \leqq \tilde{a}_1/\tilde{a}_2 \,. \tag{3.45}$$

Here, $\tilde{a}_0$, $\tilde{a}_1$, $\tilde{a}_2$ are given by (3.44), l_2 is a lower estimate of the second eigenvalue

λ_2, greater then $\tilde{\varkappa}_2 = \tilde{a}_1/\tilde{a}_2$, ε is the error caused by using the Ritz method when solving the problem

$$((v, u))_A = ((v, \tilde{f}_{k-1}))_B \quad \forall v \in V_A$$

(see (3.41); how to get ε, see the text following (3.41)), $\|\cdot\|_{V_B} = \sqrt{((\cdot,\cdot))_B}$.

Remark 3.3. (Proof of the relation

$$\lim_{\substack{k\to\infty \\ N\to\infty}} \tilde{\varkappa}_k = \lambda_1 .) \tag{3.46}$$

Let us remind, first, that $\tilde{f}_p, \tilde{f}_{p+1}$ being known, we have

$$\tilde{a}_{2p} = ((\tilde{f}_p, \tilde{f}_p))_B , \quad \tilde{a}_{2p+1} = ((\tilde{f}_p, \tilde{f}_{p+1}))_B , \quad \tilde{a}_{2p+2} = ((\tilde{f}_{p+1}, \tilde{f}_{p+1}))_B \tag{3.47}$$

and

$$\tilde{\varkappa}_{2p+1} = \frac{\tilde{a}_{2p}}{\tilde{a}_{2p+1}} , \quad \tilde{\varkappa}_{2p+2} = \frac{\tilde{a}_{2p+1}}{\tilde{a}_{2p+2}} . \tag{3.48}$$

From (3.47) and (3.48) we have immediately:

Proposition 3.2. *To every* $\varepsilon > 0$ *there exists such an* $\eta > 0$ *that if*

$$\tilde{f}_p = c(\varphi_1 + \sigma_1) \quad \textit{with} \quad \sigma_1 \perp \varphi_1 \ \textit{in} \ \overline{V}_B \ \textit{and} \ \|\sigma_1\|_{V_B} < \eta , \tag{3.49}$$

$$\tilde{f}_{p+1} = \frac{cK}{\lambda_1} (\varphi_1 + \sigma_2) \quad \textit{with} \quad \sigma_2 \perp \varphi_1 \ \textit{in} \ \overline{V}_B \ \textit{and} \ \|\sigma_2\|_{V_B} < \eta , \tag{3.50}$$

$$|K - 1| < 2\eta , \tag{3.51}$$

then

$$|\tilde{\varkappa}_{2p+1} - \lambda_1| < \varepsilon , \quad |\tilde{\varkappa}_{2p+2} - \lambda_1| < \varepsilon . \tag{3.52}$$ [10]

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[10]) Roughly speaking, if $\tilde{f}_p \approx c\varphi_1, \tilde{f}_{p+1} \approx (c/\lambda_1)\,\varphi_1$, then $\tilde{\varkappa} \approx \lambda_1$. In fact, we have

$$\tilde{\varkappa}_{2p+1} = \frac{\tilde{a}_{2p}}{\tilde{a}_{2p+1}} = \frac{((\tilde{f}_p, \tilde{f}_p))_B}{((\tilde{f}_p, \tilde{f}_{p+1}))_B} = \frac{c^2((\varphi_1 + \sigma_1, \varphi_1 + \sigma_1))_B}{\frac{c^2 K}{\lambda_1}((\varphi_1 + \sigma_1, \varphi_1 + \sigma_2))_B} = \frac{\lambda_1}{K} \frac{\|\varphi_1\|^2_{V_B} + \|\sigma_1\|^2_{V_B}}{\|\varphi_1\|^2_{V_B} + ((\sigma_1, \sigma_2))_B} ,$$

$$\tilde{\varkappa}_{2p+1} - \lambda_1 = \lambda_1 \left(\frac{1}{K} \frac{1 + \|\sigma_1\|^2_{V_B}}{1 + ((\sigma_1, \sigma_2))_B} - 1 \right)$$

(because $\|\varphi_1\|_{V_B} = 1$) and

$$\lambda_1 \left(\frac{1}{1 + 2\eta} \frac{1}{1 + \eta^2} - 1 \right) \leqq \lambda_1 \left(\frac{1}{K} \frac{1 + \|\sigma_1\|^2_{V_B}}{1 + ((\sigma_1, \sigma_2))_B} - 1 \right) \leqq \lambda_1 \left(\frac{1}{1 - 2\eta} \frac{1 + \eta^2}{1 - \eta^2} - 1 \right).$$

Similar inequalities are received for $\tilde{\varkappa}_{2p+2} - \lambda_1$. Wherefrom Proposition 3.2 immediately follows.

Here, φ_1 is the first of the eigenfunctions orthonormalized in $\bar{V}_B$ (see (1.33)).

Because $\lambda_1 < \lambda_2$ is always assumed, it is well to be seen that η may be choosen so small that, moreover,

(3.53) $$1 - \sqrt{(\lambda_1/\lambda_2)} - 2\eta > 0 .$$

First, we can really achieve that $\tilde{f}_p$ be of the form (3.49) choosing p and N in the Ritz method sufficiently large: In fact, as shown in the preceding chapter, starting with a function $f_0 \in V_B$ such that

(3.54) $$f_0 = \sum_{i=1}^{\infty} \alpha_i \varphi_i ,$$

we have

(3.55) $$f_1 = Tf_0 = \sum_{i=1}^{\infty} (\alpha_i/\lambda_i)\, \varphi_i .$$

The assumption that f_0 is not orthogonal, in $\bar{V}_B$, to φ_1 ensures that

(3.56) $$\alpha_i = ((f_0, \varphi_1))_B \neq 0 .$$

It follows, because $\lambda_1 < \lambda_2$ and

(3.57) $$\sum_{i=1}^{\infty} \alpha_i^2 = \|f_0\|^2_{\bar{V}_B} < +\infty ,$$

that f_p is arbitrarily close (in $\bar{V}_B$) to

(3.58) $$\frac{\alpha_1}{\lambda_1^p}\, \varphi_1$$

if p is sufficiently large. Now, this p being kept fixed, the N in the Ritz method can be choosen so large that f_p and $\tilde{f}_p$ are arbitrarily close (see (3.32), p. 229). Thus $\tilde{f}_p$ can be made arbitrarily close to the function (3.58). More precisely, $\eta > 0$ being given, p and N can be found such that f_p is of the form (3.49). (With $c \approx \alpha_1/\lambda_1^p$; however, this is unessential for what follows.) This result remains true (for p fixed) even if we increase N, because from the well-known property of the Ritz method it follows that the more will then f_p and $\tilde{f}_p$ be close.

Now it is sufficient to prove that $\tilde{f}_p$ being of the form (3.49), $\tilde{f}_{p+1}$ will be of the form (3.50).

Let us choose such an N_0 in the Ritz method ($N_0 \geqq N$) that the orthogonal (in $\bar{V}_A$) projection $P_{N_0}\varphi_1$ on the subspace $\bar{V}_{N_0}$ spanned by the first N_0 functions (3.1) be sufficiently close to φ_1. More precisely, that we have

(3.59) $$P_{N_0}\varphi_1 = k\varphi_1 + \gamma ,$$

where

(3.60) $$|k - 1| < \eta , \quad \|\gamma\|_{\bar{V}_B} \leqq c = (1 - \sqrt{(\lambda_1/\lambda_0)} - 2\eta)\, \eta$$

(cf. (3.53)). Solving the problem

$$((v, u))_A = ((v, \tilde{f}_p))_B \quad \forall v \in V_A$$

by the Ritz method, we have (cf. (3.6))

$$\tilde{f}_{p+1} = P_{N_0} T \tilde{f}_p \,. \tag{3.61}$$

However,

$$T\varphi_1 = \frac{\varphi_1}{\lambda_1}$$

and

$$P_{N_0} T\varphi_1 = \frac{1}{\lambda_1} (k\varphi_1 + \gamma)$$

according to (3.59). Further (we have $\sigma_1 \perp \varphi_1$ in $\bar{V}_B$)

$$\sigma_1 = \sum_{i=2}^{\infty} \delta_i \varphi_i \,, \quad \delta_i = ((\sigma_1, \varphi_i))_B \,, \quad i = 2, 3, \ldots, \quad \sum_{i=2}^{\infty} \delta_i^2 < \eta^2$$

according to (3.49). Thus

$$T\sigma_1 = \sum_{i=2}^{\infty} \frac{\delta_i}{\lambda_i} \varphi_i = \sum_{i=2}^{\infty} \frac{\delta_i}{\sqrt{\lambda_i}} v_i \,,$$

$$\|T\sigma_1\|_{V_A}^2 = \sum_{i=2}^{\infty} \frac{\delta_i^2}{\lambda_i} \leqq \frac{\eta^2}{\lambda_2} \,,$$

$$\|P_{N_0} T\sigma_1\|_{V_A} \leqq \|T\sigma_1\|_{V_A} \leqq \frac{\eta}{\sqrt{\lambda_2}} \,,$$

$$\|P_{N_0} T\sigma_1\|_{V_B} \leqq \frac{1}{\sqrt{\lambda_1}} \|P_{N_0} T\sigma_1\|_{V_A} \leqq \frac{\eta}{\sqrt{(\lambda_1 \lambda_2)}} \,, \tag{3.62}$$

because for all $v \in V_A$ we have

$$\frac{((v, v))_A}{((v, v))_B} \geqq \lambda_1 \,.$$

Thus if we denote $P_{N_0} T\sigma_1 = \tau$, we have according to (3.49)

$$\tilde{f}_{p+1} = P_{N_0} T \tilde{f}_p = c(P_{N_0} T\varphi_1 + P_{N_0} T\sigma_1) = c \left(\frac{1}{\lambda_1} (k\varphi_1 + \gamma) + \tau \right) = \tag{3.63}$$

$$= \frac{c}{\lambda_1} (k\varphi_1 + \gamma + \lambda_1 \tau) \,,$$

where

$$(3.64) \qquad \|\lambda_1 \tau\|_{V_B} \leqq \eta \sqrt{\frac{\lambda_1}{\lambda_2}}$$

in consequence of (3.62).

Denote by γ_1, resp. τ_1 the orthogonal projection, in $\bar{V}_B$, of γ, resp. τ on the subspace generated by the function φ_1, thus

$$\gamma = \gamma_1 + \gamma_2\,, \quad \tau = \tau_1 + \tau_2\,, \quad \gamma_1 \perp \gamma_2\,, \quad \tau_1 \perp \tau_2 \quad \text{in} \quad \bar{V}_B\,.$$

Because of (3.60), (3.64) we have

$$(3.65) \qquad \begin{aligned} \|\gamma_1 + \lambda_1 \tau_1\|_{V_B} &\leqq (1 - 2\eta)\,\eta\,, \\ \|\gamma_2 + \lambda_1 \tau_2\|_{V_B} &\leqq (1 - 2\eta)\,\eta\,. \end{aligned}$$

According to (3.63) we then get

$$(3.66) \qquad \tilde{f}_{p+1} = \frac{c}{\lambda_1}\left[(k\varphi_1 + \gamma_1 + \lambda_1 \tau_1) + (\gamma_2 + \lambda_1 \tau_2)\right] = \frac{c}{\lambda_1} K(\varphi_1 + \sigma_2)\,,$$

where

$$(3.67) \qquad \sigma_2 \perp \varphi_1 \quad \text{in} \quad \bar{V}_B \quad \text{and} \quad \|\sigma_2\|_{V_B} < \eta\,,$$

since

$$(3.68) \qquad 1 - 2\eta < K < 1 + 2\eta$$

according to the first inequality (3.60) and the first inequality of (3.65), and because of the second inequality (3.65).

Thus if $\tilde{f}_p$ is of the form (3.49), $\tilde{f}_{p+1}$ is of the form (3.50). It follows that a similar result is obtained for $\tilde{f}_{p+1}$ and $\tilde{f}_{p+2}$, etc.

In consequence of Proposition 3.2, the proof of (3.45) is finished.

At the same time it follows that all the $\tilde{f}_i$ are nonzero functions if the number of the terms in the Ritz method is sufficiently large.

Let us summarize the results of Chap. 3 into the following theorem:

Theorem 3.1. *Let the forms* $((v, u))_A$, $((v, u))_B$ *satisfy assumptions* (1.16)–(1.21) (concerning their symmetry, boundedness end ellipticity in V_A, resp. V_B). *Let the first eigenvalue* λ_1 *be simple. Let* $f_0 \in V_B$ *be not orthogonal, in* $\bar{V}_B$, *to the first eigenfunction* φ_1 *and let* $\tilde{f}_i$ $(i = 1, 2, \ldots)$ *be approximate solutions of the problems*

$$(3.69) \qquad ((v, u))_A = ((v, \tilde{f}_{i-1}))_B \quad \forall v \in V_A$$

$(\tilde{f}_0 = f_0)$ *received by the Ritz method*[11]), *taking the first* N *terms of the base* (3.1).

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[11]) Or by an other method with similar properties, e.g. by the finite element method.

Then:

(i) *For all N sufficiently large no* $\tilde{f}_i$ *is a zero function,*

$$\tilde{a}_{2p} = ((\tilde{f}_p, \tilde{f}_p))_B > 0\,, \quad \tilde{a}_{2p+1} = ((\tilde{f}_p, \tilde{f}_{p+1}))_B > 0\,,$$

so that the quotients

$$\tilde{\varkappa}_k = \frac{\tilde{a}_{k-1}}{\tilde{a}_k}$$

are well-defined and positive for all $k = 1, 2, \dots$.

(ii) *We have*

$$\tilde{\varkappa}_1 \geqq \tilde{\varkappa}_2 \geqq \dots \geqq \lambda_1\,, \tag{3.70}$$

$$\lim_{\substack{k \to \infty \\ N \to \infty}} \tilde{\varkappa}_k = \lambda_1\,. \tag{3.71}$$

(iii) *Let us stop the process* (3.6) *after i steps and denote, for this case only,*

$$\tilde{a}_0 = ((\tilde{f}_{i-1}, \tilde{f}_{i-1}))_B\,, \quad \tilde{a}_1 = ((\tilde{f}_{i-1}, \tilde{f}_i))_B\,, \quad \tilde{a}_2 = ((\tilde{f}_i, \tilde{f}_i))_B\,.$$

Then if for the exact solution $\bar{f}_i$ *of the problem* (3.69) *and its Ritz approximation* $\tilde{f}_i$ *the relation*

$$\|\bar{f}_i - \tilde{f}_i\|_{V_B} < \varepsilon \tag{3.72}$$

holds[12]) *and if* l_2 *is a lower estimate of the eigenvalue* λ_2, *greater then* $\tilde{\varkappa}_2 = \tilde{a}_1/\tilde{a}_2$, *the following twosided estimate is valid:*

$$\frac{(\tilde{a}_1 - \|\tilde{f}_{i-1}\|_{V_B}\,\varepsilon)\, l_2 - \tilde{a}_0}{[\tilde{a}_2 + (2\|\tilde{f}_i\|_{V_B} + \varepsilon)\,\varepsilon]\, l_2 + \|\tilde{f}_{i-1}\|_{V_B}\,\varepsilon - \tilde{a}_1} \leqq \lambda_1 \leqq \frac{\tilde{a}_1}{\tilde{a}_2}\,. \tag{3.73}$$

Remark 3.4. (A remark of practical nature.) The assumptions of the theorem are rather natural. To get the needed information, one applies, as a rule, a suitable comparison theorem: The given problem is compared with a "similar" problem, or with "similar" problems; most often the given differential equation is compared with differential equations of the same kind, but with constant coefficients (and with the same boundary conditions). If these "similar" problems are directly solvable – and this is often the case – we get, on base of the comparison theorem, the needed (rough) estimates for λ_1 and λ_2. (See Example 4.1, p. 237.) In this way, the simplicity of λ_1 follows immediately, as a rule. If, moreover, the system of eigenfunctions ψ_n of such a "similar" problem is known, it is convevient to use it as a base for the Ritz method. (For details and for the theoretical background see, e.g., [1].) Choosing then $f_0 = \psi_1$, it is sufficient, as a rule, to take only a few terms of this system to ensure the functions $\tilde{f}_i$ to be nonzero ones.

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[12]) (3.72) can always be achieved for an arbitrary $\varepsilon > 0$ if only N is sufficiently large.

The number of steps in the iterative process depends on the accuracy required. The relation (3.71) shows that it is advisable to carry out more then one step. After stopping this process, (3.73) gives the desired twosided estimate. How to determine ε in (3.72), see e.g. [1], especially Chaps 11 and 21.

4. A MODIFICATION OF THE METHOD. AN EXAMPLE

Although the twosided eigenvalue estimate (3.73) is relatively simple, its construction may appear rather labourious in some cases. This concerns especially the often tedious way how to get ε in (3.72). We thus show a modification of the described method which may appear considerably more suitable to get the desired twosided estimate.

Thus let us have the functions $f_0, \tilde{f}_1, \ldots, \tilde{f}_i$[13]. Now, instead of constructing the estimate (3.73), let us try to find a function $\hat{f}_{i-1}$ which is the exact solution of the problem

$$((v, \tilde{f}_i))_A = ((v, \hat{f}_{i-1}))_B \quad \forall v \in V_A . \tag{4.1}$$

We often succeed in finding such a function, because, roughly speaking, (4.1) means to solve the problem

$$B\hat{f}_{i-1} = A\tilde{f}_i \tag{4.2}$$

with $\tilde{f}_i$ known, and this problem is considerably simpler to solve than the "inverse" problem, because the order of the operator B is smaller than that of the operator A[14]. If, especially, B is the identity operator and if all the data are sufficiently smooth, $\hat{f}_{i-1}$ is received only by differentiation[15]. Now, because (4.1) is fulfilled exactly, not only approximately, we can apply the estimate (1.44) with $f_0 = \hat{f}_{i-1}$, $f_1 = \tilde{f}_i$. Thus denoting

$$\hat{a}_0 = ((\hat{f}_{i-1}, \hat{f}_{i-1}))_B , \quad \hat{a}_1 = ((\hat{f}_{i-1}, \tilde{f}_i))_B , \quad \hat{a}_2 = ((\tilde{f}_i, \tilde{f}_i))_B , \tag{4.3}$$

$$\hat{\varkappa}_1 = \hat{a}_0/\hat{a}_1 , \quad \hat{\varkappa}_2 = \hat{a}_1/\hat{a}_2 ,$$

we have under the same assumptions as before (λ_1 simple, $\hat{\varkappa}_2 < l_2 \leqq \lambda_2$)

$$\hat{\varkappa}_2 - \frac{\hat{\varkappa}_1 - \hat{\varkappa}_2}{\dfrac{l_2}{\hat{\varkappa}_2} - 1} \leqq \lambda_1 \leqq \hat{\varkappa}_2 . \tag{4.4}$$

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[13]) In practice, $i = 1$ or $i = 2$ is often sufficient.

[14]) Moreover, the corresponding twosided estimate obtained in this way is very accurate, see [4]. See also Example 4.1.

[15]) If finite element method is used, this step requires application of smoother spline functions.

Example 4.1. Let us solve the eigenvalue problem

$$Au - \lambda Bu = 0\,, \quad u \neq 0\,, \tag{4.5}$$

with

$$A = (9 + \cos y)\frac{\partial^4}{\partial x^4} + 18\frac{\partial^4}{\partial x^2\,\partial y^2} + (9 - \cos x)\frac{\partial^4}{\partial y^4}\,, \tag{4.6}$$

$$B = -\Delta = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{4.7}$$

on the square

$$G = (0, \pi) \times (0, \pi)$$

with the boundary conditions

$$u = 0\,, \quad \frac{\partial^2 u}{\partial v^2} = 0 \quad \text{on the boundary } \Gamma\,. \tag{4.8}$$

Here (the second of the conditions (4.8) being unstable)

$$V_A = \{v;\, v \in W_2^{(2)}(G),\ v = 0 \text{ on } \Gamma \text{ in the sence of traces}\}\,, \tag{4.9}$$

$$V_B = \{v;\, v \in W_2^{(1)}(G),\ v = 0 \text{ on } \Gamma \text{ in the sense of traces}\}\,. \tag{4.10}$$

If we multiply (4.5) by $v \in V_A$, integrate over G and use formally the Green theorem in the usual way (to get symmetric forms), (4.5) turns into

$$((v, u))_A = \lambda((v, u))_B \quad \forall v \in V_A\,, \quad u \neq 0\,, \tag{4.11}$$

with

$$((v, u))_A = \int_G \left\{(9 + \cos y)\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + 18\frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial^2 v}{\partial x\,\partial y} + (9 - \cos x)\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2}\right\} \mathrm{d}x\,\mathrm{d}y\,, \tag{4.12}$$

$$((v, u))_B = \int_G \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y\,. \tag{4.13}$$

The form (4.12), resp. (4.13) is on V_A, resp. V_B symmetric, bounded and V_A-, resp. V_B-elliptic. This can be established precisely in the same way as in [1], Chaps 22 and 23.

Use of the comparison theorem:

The problem

$$A_1 u - \alpha B_1 u = 0\,, \tag{4.14}$$

where

$$(4.15) \qquad A_1 = 8 \frac{\zeta^4}{\partial x^4} + 18 \frac{\partial^4}{\partial x^2 \, \partial y^2} + 8 \frac{\partial^4}{\partial y^4},$$

$$(4.16) \qquad B_1 = -\Delta ,$$

with the same boundary conditions (4.8) on Γ, has smaller eigenvalues. Here the system of eigenfunctions (not orthonormalized) is known:

$$(4.17) \qquad \begin{aligned} \psi_1 &= \sin x \sin y , \\ \psi_2 &= \sin 2x \sin y , \\ \psi_3 &= \sin x \sin 2y , \\ &\dots\dots\dots\dots\dots\dots \end{aligned}$$

The value of α_1 follows immediately:

$$(8 + 18 + 8) \sin x \sin y - 2\alpha_1 \sin x \sin y = 0 , \quad \alpha_1 = 17 .$$

Similarly, we get $\alpha_2 = 41.6$.

If we compare our problem with a similar problem

$$A_2 u - \beta B_2 u = 0 ,$$

where

$$A_2 = 10 \frac{\partial^4}{\partial x^4} + 18 \frac{\partial^4}{\partial x^2 \, \partial y^2} + 10 \frac{\partial^4}{\partial y^4}$$

$$B_2 = -\Delta ,$$

with the same boundary conditions, we get in a quite analogous way upper estimates for λ_1 and λ_2:

$$\beta_1 = 19 , \quad \beta_2 = 48.4 .$$

Thus we have

$$(4.18) \qquad 17 \leqq \lambda_1 \leqq 19 , \quad 41.6 \leqq \lambda_2 \leqq 48.4 .$$

Especially it follows that λ_1 is simple. At the same time, we have a lower bound 41·6 for λ_2.

Choose in the Ritz method the system (4.17) for the base (this is possible, see [1], Chap. 20) and choose $f_0 = \psi_1 = \sin x \sin y$. Let, first, $N = 1$, so that we look for $\tilde{f}_i$ in the form

$$\tilde{f}_i = k\psi_1 = k \sin x \sin y .$$

The Ritz system reduces here to

$$((\psi_1, \psi_1))_A \, k_{11} = ((f_0, \psi_1))_B = ((\psi_1, \psi_1))_B .$$

A simple computation leads to the equation

$$9k_{11} = 1/2 ,$$

i.e.

$$\tilde{f}_1 = \frac{1}{18} \sin x \sin y .$$

Now, it is possible to find the error ε (see (3.72)) and use the estimate (3.73). Instead of this, we show that it is possible to find in a simple way such a function $\hat{f}_0$ for which

$$B\hat{f}_0 = A\tilde{f}_1 .$$

Performing $A\tilde{f}_1$, we get

$$(4.19)\quad B\hat{f}_0 = \frac{1}{18}\left[(9 + \cos y) \sin x \sin y + 18 \sin x \sin y + (9 - \cos x) \sin x \sin y\right] =$$
$$= 2 \sin x \sin y + \tfrac{1}{36} \sin x \sin 2y - \tfrac{1}{36} \sin 2x \sin y .$$

From (4.19) and from the form of the operator B it follows that $\hat{f}_0$ will be of the form

$$\hat{f}_0 = k_1\psi_1 + k_2\psi_2 + k_3\psi_3 .$$

A simple computation then leads to the result

$$\hat{f}_0 = \sin x \sin y + 1/180 \sin x \sin 2y - 1/180 \sin 2x \sin y .$$

Now, we apply (4.3) with $i = 1$. We get

$$\hat{\varkappa}_1 = 18.0027 , \quad \hat{\varkappa}_2 = 18 .$$

We can take $l_2 = \alpha_2 = 41.6$ (see 4.18)) and get finally

$$18 - \frac{0{\cdot}002\bar{7}}{\dfrac{41.6}{18} - 1} \leqq \lambda_1 \leqq 18 ,$$

wherefrom

$$17.997\bar{8} \leqq \lambda_1 \leqq 18 .$$

Thus we have obtained by our method a very satisfactory result performing only one iterative step and taking even one term of the base in the Ritz method.

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O JEDNÉ METODĚ DVOUSTRANNÝCH ODHADŮ VLASTNÍCH ČÍSEL ELIPTICKÝCH DIFERENCIÁLNÍCH ROVNIC TVARU $Au - \lambda Bu = 0$

KAREL REKTORYS a ZDENĚK VOSPĚL

Collatzova metoda dvoustranných odhadů vlastních čísel typu (1.7), resp. (1.10), koncipovaná pro obyčejné diferenciální rovnice, byla rozšířena na případ dostatečně obecných eliptických rovnic tvaru $Au - \lambda Bu = 0$ K. Rektorysem v jeho monografii [1]. Zatímco však v případě obyčejných rovnic se často podaří řešit okrajové problémy (1.8) přesně, je v případě parciálních rovnic zpravidla třeba užít přibližných metod (Ritzovy metody, metody konečných prvků apod.). Tím ovšem nedostaneme přesné hodnoty Schwarzových kvocientů $\varkappa_k$, nýbrž jen přibližné hodnoty $\tilde{\varkappa}_k$. Zmíněné dvoustranné odhady s těmito $\tilde{\varkappa}_k$, dosazenými za $\varkappa_k$, pak obecně neplatí. Cílem práce je ukázat použitelnost uvedené metody i v tomto případě.

Proto autoři použili poněkud jiného postupu než jakého se užívá v knize [1] a dokázali nejprve větu 2.1 (str. 224), týkající se vlastností Schwarzových kvocientů a dvoustranných odhadů Collatzova typu (odvozených bez užití Templeovy věty). V kap. 3 pak ukázali (viz větu 3.1, str. 234), že „přibližné" Schwarzovy kvocienty $\tilde{\varkappa}_k$ mají podobné vlastnosti a že při přibližném řešení zmíněných okrajových problémů lze místo odhadu (2.44) (s přesnými Schwarzovými kvocienty) použít odhadu (3.73).

Přestože tento odhad není nijak komplikovaný, činí někdy potíže praktické určení odhadu čísla ε (chyby přibližného řešení). Proto je v kap. 4 uvedena určitá modifikace uvažované metody, která vede často k cíli podstatně jednodušší cestou. Zároveň je uveden numerický příklad ukazující praktické užití této metody a demonstrující její přesnost.

Author's addresses: Prof. RNDr. *Karel Rektorys*, DrSc., Ing. *Zdeněk Vospěl*, CSc., katedra matematiky stavební fakulty ČVUT, 16699 Praha 6, Thákurova 7.