THE INITIAL VALUE PROBLEM FOR NONLINEAR BOLTZMANN EQUATION

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FORMULATION OF THE PROBLEM

In the paper the following problem will be studied:

(1) $$\frac{\partial \varphi}{\partial t} + \boldsymbol{v}\,\nabla \varphi(\boldsymbol{x}, \boldsymbol{v}, t) = \int_{\boldsymbol{E}_3} \mathrm{d}\boldsymbol{v}_1 \int_{\boldsymbol{\Omega}} \mathrm{d}\boldsymbol{\varkappa}\sigma^2 \,.\, (\boldsymbol{v}_1 - \boldsymbol{v})\,\boldsymbol{\varkappa} H((\boldsymbol{v}_1 - \boldsymbol{v})\,\boldsymbol{\varkappa})\,.$$
$$.\,[\varphi(\boldsymbol{x}, \boldsymbol{v}', t)\,\varphi(\boldsymbol{\varkappa}, \boldsymbol{v}_1', t) - \varphi(\boldsymbol{x}, \boldsymbol{v}, t)\,\varphi(\boldsymbol{x}, \boldsymbol{v}_1, t)]\,,$$
$$\varphi(\boldsymbol{x}, \boldsymbol{v}, t = 0) = \psi(\boldsymbol{x}, \boldsymbol{v})\,, \quad \varphi \in \boldsymbol{G}_T^{\alpha\beta\gamma}\,, \quad \psi \in \boldsymbol{G}^{\alpha\beta\gamma}\,.$$

Here ψ is a given function, φ the function to be determined, $\boldsymbol{x}$ and $\boldsymbol{v}$ mean three-dimensional vectors, $\boldsymbol{\varkappa}$ the unit one, $\boldsymbol{\Omega}$ the surface of the unit sphere and σ a positive number.

Further,
$$\boldsymbol{v}' = \boldsymbol{v} + \boldsymbol{\varkappa}(\boldsymbol{v}_1 - \boldsymbol{v})\,\boldsymbol{\varkappa}\,,$$
$$\boldsymbol{v}_1' = \boldsymbol{v}_1 - \boldsymbol{\varkappa}(\boldsymbol{v}_1 - \boldsymbol{v})\,\boldsymbol{\varkappa}\,,$$
$$\boldsymbol{M}_0 = \boldsymbol{E}_3 \times \boldsymbol{E}_3\,,$$
$$\boldsymbol{M} = \boldsymbol{M}_0 \times [0, T]\,, \quad T > 0\,.$$

$H(x)$ is the step function, $H(x) = 0$ for $x < 0$, $H(x) = 1$ $x > 0$. $\boldsymbol{G}^{\alpha\beta\gamma}$ denotes the class of nonnegative functions χ defined on $\boldsymbol{M}_0$ and such that for given positive constants α, β and γ the functions

$$\frac{\partial \chi}{\partial x_i}(\beta + \gamma v^2)\,\mathrm{e}^{\frac{1}{2}(\alpha v^2)}, \frac{\partial \chi}{\partial v_i}(\beta + \gamma v^2)\,e^{\frac{1}{2}(\alpha v^2)} \text{ and } \quad \chi(\beta + \gamma v^2)\,e^{\alpha v^2}$$

are continuous and bounded there. Similarly, $\boldsymbol{G}_T^{\alpha\beta\gamma}$ means the class of functions χ defined on $\boldsymbol{M}$ and such that

$$\chi(\beta + \gamma v^2)\,e^{\alpha v^2}\,, \frac{\partial \chi}{\partial x_i}\,.\,(\beta + \gamma v^2)\,e^{\frac{1}{2}(\alpha v^2)}, \frac{\partial \chi}{\partial v_i}(\beta + \gamma v^2)\,e^{\frac{1}{2}(\alpha v^2)}\,, \quad (i = 1, 2, 3) \quad \text{and}$$
$$\frac{\partial \chi}{\partial t}(\beta + \gamma v^2)\,e^{\frac{1}{2}(\alpha v^2)}$$

are continuous and bounded there (for $t = 0$ and $t = T$, the time derivative must be taken from the right and from the left, respectively). The problem just formulated frequently occurs in the theory of gases composed of neutral particles [1, 2, 3]. In that case φ means the density of particles dependent on the spatial coordinate $\mathbf{x}$ velocity $\mathbf{v}$ and time t, σ is the diameter of the particle or, equivalently, the diameter of the respective hard core potential. Problem (1) expresses the time evolution of the system having the following properties:

- the system is composed of particles of one sort,
- the particles interact with each other through binar collisions,
- the velocity of a particle is constant between any two collisions,
- the diameter σ is negligible if compared with the unit volume,
- the system develops in infinite volume (i.e., formulation (1) corresponds to the case of a sufficiently large reservoir and negligible influence of the walls).

The nonlinearity of Eq. (1) presents great mathematical difficulties when attempting to solve the problem. This is why methods of linearization (e.g. that of Chapman-Enškog [1]), perturbations [1], approximations (hydrodynamic equations) and numerical ones are mostly used. The main purpose of this paper is to find the exact solution of problem (1) and to prove its uniqueness.

MATHEMATICAL RESULTS

Lemma. *Let $\psi \in \mathbf{G}^{\alpha\beta\gamma}$. Then there exists $T > 0$ such that the function φ_{n+1},*

$$(2)\quad \varphi_{n+1}(\mathbf{x}, \mathbf{v}, t) = \int_0^t \mathrm{d}t_1 \exp\left(-\int_{t_1}^t \mathrm{d}t_2 \pi\sigma^2 \,.\int_{E_3} \mathrm{d}\mathbf{u} |\mathbf{u} - \mathbf{v}|\, \varphi_n(\mathbf{x} - \mathbf{v}(t - t_2), \mathbf{u}, t_2)\right) .$$
$$. \int_{E_3} \mathrm{d}\mathbf{v}_1 \int_{\Omega} \mathrm{d}\boldsymbol{\varkappa}\sigma^2 \,. (\mathbf{v}_1 - \mathbf{v})\, \boldsymbol{\varkappa} H((\mathbf{v}_1 - \mathbf{v})\,\boldsymbol{\varkappa})\, \varphi_n(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}', t_1) \,.$$
$$. \varphi_n(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}'_1, t_1) + \psi(\mathbf{x} - \mathbf{v}t, \mathbf{v}) \,.$$
$$\exp\left(-\int_0^t \mathrm{d}t_1 \pi\sigma^2 \int_{E_3} \mathrm{d}\mathbf{u} |\mathbf{v} - \mathbf{u}|\, \varphi_n(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{u}, t_1)\right),$$
$$\varphi_0(\mathbf{x}, \mathbf{v}, t) = \psi(\mathbf{x} - \mathbf{v}t, \mathbf{v})$$

belongs to the class $\mathbf{G}_T^{\alpha\beta\gamma}$ for any $n = 0, 1, 2, \ldots$. *Furthermore, there exist finite positive constants A_1, A_2, A_3 and A_4 such that*

$$\sup_{\mathbf{M}} \varphi_n(\beta + \gamma v^2)\, e^{\alpha v^2} \leqq A_1 \,, \quad \sup_{\mathbf{M}} \left|\frac{\partial \varphi_n}{\partial x_i}\right| (\beta + \gamma v^2) e^{\frac{1}{2}(\alpha v^2)} \leqq A_2 \,, \quad (i = 1, 2, 3)\,,$$
$$\sup_{\mathbf{M}} \left|\frac{\partial \varphi_n}{\partial v_i}\right| (\beta + \gamma v^2)\, e^{\frac{1}{2}(\alpha v^2)} \leqq A_3 \,, \quad (i = 1, 2, 3) \quad \text{and} \quad \sup_{\mathbf{M}} \left|\frac{\partial \varphi_n}{\partial t}\right| \leqq A_4$$

for all $n = 1, 2, \ldots$.

Proof. The following relations hold:

$$\sup_{E_3}\int_{E_3} \mathrm{d}\mathbf{v}_1 \int_{\Omega} \mathrm{d}\boldsymbol{\varkappa}\sigma^2 \,.\, (\mathbf{v}_1 - \mathbf{v})\,\boldsymbol{\varkappa} H((\mathbf{v}_1 - \mathbf{v})\,\boldsymbol{\varkappa})\, e^{\alpha v^2}(\beta + \gamma v^2) \frac{e^{-\alpha(v'^2 + v_1'^2)}}{(\beta + \gamma v'^2)(\beta + \gamma v_1'^2)} =$$

$$\sup_{E_3}\int_{E_3} \mathrm{d}v_1 \int_{\Omega} \mathrm{d}\boldsymbol{\varkappa}\sigma^2 \,.\, (\mathbf{v}_1 - \mathbf{v})\,\boldsymbol{\varkappa} H((\mathbf{v}_1 - \mathbf{v})\,\boldsymbol{\varkappa}) e^{-\alpha v_1{}^2}\left(\frac{1}{\beta + \gamma(v_1^2 - (\varkappa v_1)^2 + (\varkappa v)^2)} + \right.$$

$$\left. + \frac{1}{\beta + \gamma(v^2 - \varkappa v)^2 + (\varkappa v_1)^2)} \cdot \frac{\gamma(v_1^2 - (\varkappa v_1)^2 + (\varkappa v)^2)}{\beta + \gamma((\varkappa v)^2 - (\varkappa v_1)^2)}\right) \leqq a(\alpha, \beta, \gamma) \equiv a < \infty \,.$$

Thus, denoting by $L_\psi(\varphi_n)$ the right hand side of Rel. (2) and $A = \sup_{M_0} \psi e^{\alpha v^2}(\beta + \gamma v^2)$, we have

$$0 \leqq \varphi_1 = L_\psi(\varphi_0) \leqq (A + A^2 a t)\frac{e^{-\alpha v^2}}{\beta + \gamma v^2} \,.$$

Then, using the recurrent formula (2), we find easily by induction that

$$0 \leqq \varphi_n \leqq \frac{A}{1 - aAt}\frac{e^{-\alpha v^2}}{\beta + \gamma v^2}\,, \quad t \in \left[0, \frac{1}{aA}\right),$$

for all n.

In the following we will take $T < 1/aA$. Then, clearly, $A_1 = A/(1 - aAT)$ satisfies Lemma. The continuity of φ_n in its variables is obvious.

Formal differentiation of (2) with respect to x_i, $(i = 1, 2, 3)$ leads to the relation

$$(3)\quad \frac{\partial}{\partial x_i}\varphi_{n+1}(\mathbf{x}, \mathbf{v}, t) = \int_0^t \mathrm{d}t_1 \exp\left(-\int_{t_1}^t \mathrm{d}t_2 \int_{E_3} \mathrm{d}\mathbf{u} |\mathbf{v} - \mathbf{u}|\, \pi\sigma^2 \varphi_n(\mathbf{x} - \mathbf{v}(t - t_2, \mathbf{u}, t_2)\right).$$

$$.\left\{\int_{E_3} \mathrm{d}\mathbf{v}_1 \int_{\Omega} \mathrm{d}\boldsymbol{\varkappa}\sigma^2 \,.\, \boldsymbol{\varkappa}(\mathbf{v}_1 - \mathbf{v})\, H(\boldsymbol{\varkappa}(\mathbf{v}_1 - \mathbf{v}))\, \varphi_n(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}', t_1)\left[2 \,.\, \frac{\partial}{\partial x_i} \,.\right.\right.$$

$$.\, \varphi_n(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}_1', t_1) - \varphi_n(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}_1', t_1) \,.$$

$$\left.\left. .\int_{t_1}^t \mathrm{d}t_2 \pi\sigma^2 \int_{E_3} \mathrm{d}\mathbf{u} \left|\mathbf{u} - \mathbf{v}\right| \frac{\partial}{\partial x_i}\varphi_n(\mathbf{x} - \mathbf{v}(t - t_2), \mathbf{u}, t_2)\right]\right\} +$$

$$+ \exp\left(-\int_0^t \mathrm{d}t_1 \pi\sigma^2 \int_{E_3} \mathrm{d}\mathbf{u} |\mathbf{u} - \mathbf{v}|\, \varphi_n(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{u}, t_1)\right)\left[\frac{\partial}{\partial x_i}\psi(\mathbf{x} - \mathbf{v}t, \mathbf{v}) - \right.$$

$$\left. - \psi(\mathbf{x} - \mathbf{v}t, \mathbf{v}) \int_0^t \mathrm{d}t_1 \int_{E_3} \mathrm{d}\mathbf{u} \pi\sigma^2 |\mathbf{v} - \mathbf{u}|\, \frac{\partial}{\partial x_i}\varphi_n(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{u}, t_1)\right].$$

This is evidently correct for $n = 0$. Denoting

$$B = \sup_{M_0}\left|\frac{\partial \psi}{\partial x_i}\right| e^{\frac{1}{2}(\alpha v^2)}(\beta + \gamma v^2)\,, \quad B_1 = A_1 \,.\, a(\tfrac{1}{2}\alpha, \beta, \gamma)\,,$$

$$B_2 = \sup_{E_3} A_1^2 \,.\, a(\alpha, \beta, \gamma) \frac{e^{-\frac{1}{2}(\alpha v^2)}}{\beta + \gamma v^2} \int_{E_3} d\mathbf{u} \pi \sigma^2 |\mathbf{u} - \mathbf{v}| \frac{e^{-\frac{1}{2}(\alpha u^2)}}{\beta + \gamma u^2}$$

and

$$B_3 = \frac{B_2 A}{A_1^2 \,.\, a(\alpha, \beta, \gamma)}$$

we obtain from (3) that

$$\left|\frac{\partial \varphi_1}{\partial x_i}\right| \leqq B \frac{e^{-\frac{1}{2}(\alpha v^2)}}{\beta + \gamma v^2} \left[\int_0^t dt_1 \left(\int_{t_1}^t dt_2 B_2 + 2B_1 + B_3\right) + 1\right] =$$

$$= B \,.\, \frac{e^{-\frac{1}{2}(\alpha v^2)}}{\beta + \gamma v^2} \left[B_2 \frac{t^2}{2} + (2B_1 + B_3)\, t + 1\right] \leqq$$

$$\leqq B \frac{e^{-\frac{1}{2}(\alpha v^2)}}{\beta + \gamma v^2} \exp\left(\frac{B_2 t^2}{2} + (2B_1 + B_3)\, t\right)$$

$(i = 1, 2, 3)$,

i.e. Rel. (3) is valid for $n = 1$ and an easy calculation yields

$$\left|\frac{\partial \varphi_2}{\partial x_i}\right| \leqq B \frac{e^{-\frac{1}{2}(\alpha v^2)}}{\beta + \gamma v^2} \exp\left(\frac{B_2 t^2}{2} + (2B_1 + B_3)\, t\right) \qquad (i = 1, 2, 3)\,.$$

Clearly, in this manner the validity of (3), the continuity of $\partial\varphi_n/\partial x_i$ in its variables and the inequality

$$\left|\frac{\partial \varphi_n}{\partial x_i}\right| \leqq B \left|\frac{e^{-\frac{1}{2}(\alpha v^2)}}{\beta + \gamma v^2}\right| \exp\left(\frac{B_2 t^2}{2} + (2B_1 + B_3)\, t\right) \qquad (i = 1, 2, 3)$$

can be proved for all n. Therefore, we can put

$$A_2 = B \exp\left(\frac{B_2 T^2}{2} + 2\left(B_1 + \frac{B_3}{2}\right) T\right) \qquad (i = 1, 2, 3).$$

Similarly, applying the operation $\partial/\partial v_i$ $(i = 1, 2, 3)$ on both sides of (2) we obtain a recurrent formula for $\partial\varphi_{n+1}/\partial v_i$. Using this formula and taking into account the properties of the functions φ_n, $\partial\varphi_n/\partial x_i$ and $\partial\psi/\partial v_i$ just proved we arrive at the relation

$$\left|\frac{\partial \varphi_n}{\partial v_i}\right| \leqq A_3 \frac{e^{-\frac{1}{2}(\alpha v^2)}}{\beta + \gamma v^2} \qquad (i = 1, 2, 3)$$

independently of n. In the same way, the continuity of $\partial\varphi_n/\partial v_i$ $(i = 1, 2, 3)$ in its variables can be shown.

Now it is seen that Rel. (2) is equivalent to the relation

$$(4) \qquad \frac{\partial \varphi_{n+1}}{\partial t} + \mathbf{v}\, \nabla_{n+1} + \varphi_{n+1} \int_{E_3} d\mathbf{u} |\mathbf{v} - \mathbf{u}|\, \pi \sigma^2\, \varphi_n(\mathbf{x}, \mathbf{u}, t) =$$

$$= \int_{E_3} \mathrm{d}\mathbf{v}_1 \int_{\Omega} \mathrm{d}\varkappa \sigma^2 \varkappa(\mathbf{v}_1 - \mathbf{v}) \, H(\varkappa(\mathbf{v}_1 - \mathbf{v})) \, \varphi_n(\mathbf{x}, \mathbf{v}', t) \, \varphi_n(\mathbf{x}, \mathbf{v}_1', t) \,,$$

$$\varphi_{n+1}(\mathbf{x}, \mathbf{v}, t = 0) = \psi(\mathbf{x}, \mathbf{v})$$

and. therefore, $\partial \varphi_{n+1}/\partial t$ ($n = 0, 1, 2, \ldots$) satisfies the assertions of Lemma as well.

Theorem. *For any* $\psi \in \mathbf{G}^{\alpha\beta\gamma}$, *there exist* $T > 0$ *and just one solution* $\varphi \in \mathbf{G}_T^{\alpha\beta\gamma}$ *of the problem* (1). *It satisfies* $\varphi(\mathbf{x}, \mathbf{v}, t) = \lim_{n \to \infty} \varphi_n(\mathbf{x}, \mathbf{v}, t)$, *where* φ_n *is given by the formula* (2).

Proof. First of all, let us choose the constant T in the same manner as in Lemma, i.e. $T \in (0, 1/aA)$, and let us keep the above notation. Now consider the recurrent formula (2) and put $\chi_{n+1} = \varphi_{n+1} - \varphi_n$.

Apparently, according to Lemma this function satifies

$$(5) \quad \chi_{n+1}(\mathbf{x}, \mathbf{v}, t) = \int_0^t \mathrm{d}t_1 \exp\left(- \int_{t_1}^t \mathrm{d}t_2 \pi \sigma^2 \int_{E_3} \mathrm{d}\mathbf{u} |\mathbf{u} - \mathbf{v}| \, \varphi_n(\mathbf{x} - \mathbf{v}(t - t_2), \mathbf{u}, t_2) \right) .$$

$$. \left\{ \int_{E_3} \mathrm{d}\mathbf{v}_1 \int_{\Omega} \mathrm{d}\varkappa \sigma^2 \, \varkappa(\mathbf{v}_1 - \mathbf{v}) \, H(\varkappa(\mathbf{v}_1 - \mathbf{v})) \left[\varphi_n(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}', t_1) \, . \right.\right.$$

$$\left. . \, \chi_n(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}_1', t_1) + \chi_n(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}', t_1) \, \varphi_{n-1}(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}_1', t_1) \right] -$$

$$\left. - \, \varphi_n(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}, t_1) \, \pi \sigma^2 \int_{E_3} \mathrm{d}\mathbf{u} |\mathbf{v} - \mathbf{u}| \, \chi_n(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{u}, t_1) \right\}$$

($n = 0, 1, 2, \ldots$). Obviously,

$$|\chi_1| \leqq B_0 t e^{-\frac{1}{2}(\alpha v^2)} \, . \, \frac{1}{\beta + \gamma v^2} \,,$$

where B_0 is a positive constant. Then we have from (5)

$$|\chi_2| \leqq \frac{t^2}{2} B_0 \frac{e^{-\frac{1}{2}(\alpha v^2)}}{\beta + \gamma v^2} \left[2A_1 a\left(\frac{\alpha}{2}, \beta, \gamma\right) + \frac{B_3 A_1}{A} \right] \equiv \frac{t^2}{2} C \, . \, \frac{e^{-\frac{1}{2}(\alpha v^2)}}{\beta + \gamma v^2} B_0 \,,$$

$$|\chi_3| \leqq \frac{t^3}{3!} C^2 \frac{e^{-\frac{1}{2}(\alpha v^2)}}{\beta + \gamma v^2} B_0 \,,$$

$$\vdots$$

$$|\chi_n| \leqq \frac{t^n}{n!} C^{n-1} \frac{e^{-\frac{1}{2}(\alpha v^2)}}{\beta + \gamma v^2} B_0 \,.$$

$$\vdots$$

Hence the sequence $\{e^{\frac{1}{2}(\alpha v^2)}(\beta + \gamma v^2) \, \varphi_n\}_{n=0}^{\infty}$ is uniformly convergent. Rel. (5) and Lemma imply

$$(6) \quad \frac{\partial \chi_{n+1}}{\partial x_i} = \int_0^t \mathrm{d}t_1 \exp\left(- \int_{t_1}^t \mathrm{d}t_2 \pi \sigma^2 \int_{E_3} \mathrm{d}\mathbf{u} |\mathbf{v} - \mathbf{u}| \, \varphi_n(\mathbf{x} - \mathbf{v}(t - t_2), \mathbf{u}, t_2) \right) .$$

$$\cdot \left\{ \int_{E_3} \mathrm{d}\mathbf{v}_1 \int_{\Omega} \mathrm{d}\varkappa \pi \sigma^2 \, \varkappa(\mathbf{v}_1 - \mathbf{v}) \, H(\varkappa(\mathbf{v}_1 - \mathbf{v})) \, . \right.$$

$$\cdot \left[\varphi_n(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}', t_1) \frac{\partial}{\partial x_i} \chi_n(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}_1', t_1) + \right.$$

$$\left. + \frac{\partial}{\partial x_i} \chi_n(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}', t_1) \, \varphi_{n-1}(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}_1', t_1) \right] -$$

$$\left. - \varphi_n(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}, t_1) \int_{E_3} \mathrm{d}\mathbf{u} \pi \sigma^2 |\mathbf{v} - \mathbf{u}| \frac{\partial}{\partial x_i} \chi_n(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{u}, t_1) \right\} + F_{n+1}^{(i)}(\mathbf{x}, \mathbf{v}, t)$$

$(i = 1, 2, 3)$. The form of the function $F_{n+1}^{(i)}$ is rather complicated but obvious. It is homogeneous in χ_n and on the basis of the previous result, it can be shown that

$$\left| F_{n+1}^{(i)} \right| \leqq \frac{(tC_1)^{n+1}}{(n + 1)!} \frac{e^{-\frac{1}{4}(\alpha v^2)}}{\beta + \gamma v^2}, \; (i = 1, 2, 3, n = 1, 2, \ldots),$$

where C_1 is a finite and nonnegative constant. Clearly,

$$\left| \frac{\partial \chi_1}{\partial x_i} \right| \leqq C_2^{(1)} \, . \, t e^{-\frac{1}{4}(\alpha v^2)} \frac{1}{\beta + \gamma v^2}, (i = 1, 2. 3),$$

where $C_2^{(1)}$ is a finite constant.

The use of Rel. (6) gives

$$e^{\frac{1}{4}(\alpha v^2)}(\beta + \gamma v^2) \left| \frac{\partial \chi_2}{\partial x_i} \right| \leqq \frac{t^2}{2} \left[C_2^{(1)} \left(2A_1 a \left(\frac{\alpha}{4}, \beta, \gamma \right) + A_1 D \right) + C_1^2 \right] \equiv \frac{t^2}{2} C_2^{(2)},$$

where $D = \sup_{E_3} \exp\left(- \frac{3\alpha v^2}{4} \right) . \frac{\pi \tau^2}{\beta + \gamma v^2} . \int_{E_3} \mathrm{d}\mathbf{u} |\mathbf{v} - \mathbf{u}| \frac{e^{-\frac{1}{4}(\alpha u^2)}}{\beta + \gamma u^2}$

and in general

$$e^{\frac{1}{4}(\alpha v^2)}(\beta + \gamma v^2) \left| \frac{\partial \chi_n}{\partial x_i} \right| \leqq \frac{t^n}{n!} \left[C_2^{(n-1)} A_1 \left(2a \left(\frac{\alpha}{4}, \beta, \gamma \right) + D \right) + C_1^n \right] \equiv \frac{t^n}{n!} C_2^{(n)}$$

$(i = 1, 2, 3)$.

From the definition of $C_2^{(n)}$ it is seen that $C_2^{(n)} \geqq C_1^n$ so that

$$\frac{C_2^{(n)}}{C_2^{(n-1)}} \leqq A_1 \left(2a \left(\frac{\alpha}{4}, \beta, \gamma \right) + D \right) + C_1 \, .$$

Therefore the sequence $\{ e^{\frac{1}{4}(\alpha v^2)} (\beta + \gamma v^2) \, \partial \varphi_n / \partial x_i \}_{n=0}^{\infty}$ is uniformly convergent.

Similarly, the operation $\partial / \partial v_i$ applied to (5) gives

$$(7) \quad \frac{\partial}{\partial v_i} \chi_{n+1} = \int_0^t \mathrm{d}t_1 \exp\left(- \int_{t_1}^t \mathrm{d}t_2 \pi \sigma^2 \int_{E_3} \mathrm{d}\mathbf{u} |\mathbf{v} - \mathbf{u}| \, \varphi_n(\mathbf{x} - \mathbf{v}(t - t_2), \mathbf{u}, t_2) \right) .$$

$$\cdot \int_{E_3} \mathrm{d}\mathbf{v}_1 \int_{\Omega} \mathrm{d}\varkappa\sigma^2\, \varkappa(\mathbf{v}_1 - \mathbf{v})\, H(\varkappa(\mathbf{v}_1 - \mathbf{v})) \sum_{j=1}^{3} \left[\varphi_n(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}', t_1)\,\cdot\right.$$

$$\cdot \frac{\partial v'_{1j}}{\partial v_i} \frac{\partial}{\partial v'_{1j}} \chi_n(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}_1, t_1) + \frac{\partial v'_1}{\partial v_i} \frac{\partial}{\partial v_j}\,\cdot$$

$$\left.\cdot\, \chi_n(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}', t_1)\, \varphi_{n-1}(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}'_1, t_1)\right] + G^{(i)}_{n+1} \qquad (i = 1, 2, 3)\,.$$

Here $G^{(i)}_{n+1}$ is an expression containing combinations of χ_n and $\partial\chi_n/\partial x_i$ $(i = 1, 2, 3)$ and it can be easily shown that

$$\left|G^{(i)}_{n+1}\right| \leqq \frac{R^{n+1}}{(n + 1)!}\, t^{n+1}\,, \quad (i = 1, 2, 3, n = 1, 2, \ldots)\,,$$

where R is a finite constant. It is also apparent that

$$\left|\frac{\partial\chi_1}{\partial v_i}\right| \leqq S^{(1)}\,.\, t\,.\exp\left(-\frac{\alpha v^2}{4}\right) \frac{1}{\beta + \gamma v^2}\,, \quad (i = 1, 2, 3)\,,$$

where $S^{(1)}$ is a finite constant. Then, using (7), we conclude

$$e^{\frac{1}{4}(\alpha v^2)}(\beta + \gamma v^2) \left|\frac{\partial\chi_2}{\partial v_i}\right| \leqq \frac{t^2}{2}\left(12A_1 a\left(\frac{\alpha}{4}, \beta, \gamma\right) S^{(1)} + R^2\right) \equiv S^{(2)} \frac{t^2}{2}\,,$$

$$\vdots$$

$$e^{\frac{1}{4}(\alpha v^2)}(\beta + \gamma v^2) \left|\frac{\partial\chi_n}{\partial v_i}\right| \leqq \frac{t^n}{n!}\left(12A_1 a\left(\frac{\alpha}{4}, \beta, \gamma\right) S^{(n-1)} + R^n\right) \equiv S^{(n)} \frac{t^n}{n!}\,,$$

$$\vdots$$

$(i = 1, 2, 3)$. Clearly, $S^{(n)}/S^{(n-1)} \leqq 12A_1 a(\alpha/4, \beta, \gamma) + R$ and, therefore, the sequence $\{e^{\frac{1}{4}(\alpha v^2)}(\beta + \gamma v^2)\, \partial\varphi_n/\partial v_i\}_{n=0}^{\infty}$ is uniformly convergent for any i.

Finally, the uniform convergence of $\{\partial\varphi_n/\partial t\}_{n=0}^{\infty}$ immediately follows from Rel. (4). Thus we have proved: the function φ, $\varphi = \lim_{n\to\infty} \varphi_n(\mathbf{x}, \mathbf{v}, t)$ exists and is continuous together with its first derivatives on the set $\mathbf{E}_3 \times \mathbf{E}_3 \times [0, T] \equiv \boldsymbol{M}$. This function is a solution of the problem (1) (see Rel. (4) and Ref. [4]). It remains to prove the uniqueness.

Let $\varphi_1 \in \boldsymbol{G}_T^{\alpha\beta\gamma}$ be another solution of (1). Then $\chi \equiv \varphi - \varphi_1 \in \boldsymbol{G}_T^{\alpha\beta\gamma}$ and, therefore

$$(8) \qquad |\varphi - \varphi_1| \leqq C_3 \frac{e^{-\alpha v^2}}{\beta + \gamma v^2}$$

$(C_3 = \text{const.} < \infty)$. From (1) it follows that

$$(9) \quad \chi(\mathbf{x}, \mathbf{v}, t) = \int_0^t \mathrm{d}t_1 \exp\left(-\int_{t_1}^t \mathrm{d}t_2 \pi\sigma^2 \int_{E_3} \mathrm{d}\mathbf{u}|\mathbf{v} - \mathbf{u}|\, \varphi(\mathbf{x} - \mathbf{v}(t - t_2), \mathbf{u}, t_2)\right)\,.$$

$$\cdot \left\{\int_{E_3} \mathrm{d}\mathbf{v}_1 \int_{\Omega} \mathrm{d}\varkappa\sigma^2\, \varkappa(\mathbf{v}_1 - \mathbf{v})\, H(\varkappa(\mathbf{v}_1 - \mathbf{v}))\,.\right.$$

$$\cdot\, [\varphi(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}', t_1)\, \chi(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}'_1, t_1) +$$

$$+ \chi(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}', t_1)\, \varphi_1(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}_1', t_1)] -$$

$$- \varphi_1(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}, t_1) \int_{E_3} \mathrm{d}\mathbf{u} \pi\, \sigma^2 |\mathbf{v} - \mathbf{u}|\, \chi(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{u}, t_1)\}$$

(see also [5]). Substituting the estimate (8) into Rel. (9) we obtain

$$|\varphi - \varphi_1| \leqq t C_3 C_4 \frac{e^{-\frac{1}{4}(\alpha v^2)}}{\beta + \gamma v^2}$$

($C_4 = \text{const.} < \infty$) and, recurrently,

$$|\varphi - \varphi_1| \leqq C_3 \frac{(C_4 t)^n}{n!} \frac{e^{-\frac{1}{4}(\alpha v^2)}}{\beta + \gamma v^2}.$$

Therefore $\varphi = \varphi_1$.

Concluding remarks

1. The class $\mathbf{G}_T^{\alpha\beta\gamma}$ was defined so as to enable us to study the majority of physical situations. Such a restriction has a certain mathematical advantage: problem (1) is equivalent to the problem

$$(10)\quad \varphi(\mathbf{x}, \mathbf{v}, t) = \int_0^t \mathrm{d}t_1 \exp\Big(- \int_{t_1}^t \mathrm{d}t_2 \pi\sigma^2 \int_{E_3} \mathrm{d}\mathbf{u} |\mathbf{u} - \mathbf{v}|\, \varphi(\mathbf{x} - \mathbf{v}(t - t_2), \mathbf{u}, t_2)\Big) .$$

$$. \int_{E_3} \mathrm{d}\mathbf{v}_1 \int_{\Omega} \mathrm{d}\boldsymbol{\varkappa} \sigma^2\, \boldsymbol{\varkappa}(\mathbf{v}_1 - \mathbf{v})\, H(\boldsymbol{\varkappa}(\mathbf{v}_1 - \mathbf{v})) .$$

$$. \varphi(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}', t_1)\, \varphi(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{v}_1', t_1) +$$

$$+ \psi(\mathbf{x} - \mathbf{v}t, \mathbf{v}) \exp\Big(- \int_0^t \mathrm{d}t_1 \pi\sigma^2 \int_{E_3} \mathrm{d}\mathbf{u} |\mathbf{u} - \mathbf{v}|\, \varphi(\mathbf{x} - \mathbf{v}(t - t_1), \mathbf{u}, t_1)\Big),$$

$$\psi \in \mathbf{G}^{\alpha\beta\gamma}, \quad \varphi \in \mathbf{G}_T^{\alpha\beta\gamma}$$

[5], and its solution can be found by an iterative method (see Rel. (2)).

2. It is seen that Theorem remains valid also in the case when σ^2 is replaced by a function of the variables $\mathbf{x}$, $\boldsymbol{\varkappa}$, $\mathbf{v}$, $\mathbf{v}_1$ and t, which is continuous and bounded together with its first derivatives in the arguments mentioned above. Such a modification is useful especially in the case of scattering which is anisotropic in the centre of mass system [1].

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Souhrn

ÚLOHA S POČÁTEČNÍ PODMÍNKOU PRO NELINEÁRNÍ BOLTZMANNOVU ROVNICI

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Článek se zabývá problémem časového vývoje hustoty v systému částic jednoho druhu v závislosti na prostorových souřadnicích a rychlosti v nekonečném objemu. V konečném časovém intervalu je nalezeno přesné řešení úlohy a dokázána jeho jednoznačnost.

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