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ERROR ANALYSIS OF THE NONLINEAR MULTI-GRID METHOD OF THE SECOND KIND

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1. INTRODUCTION

The name "*multi-grid algorithm*" is connected with the method of Fedorenko [4], Bachvalov [2], Astrachancev [1], Brandt [3] (further references in [5]) for the fast numerical solution of elliptic problems. We shall call this method "multi-grid iteration of the *first kind*" in contradistinction to the "multi-grid iteration of the *second kind*" that is described by the author in [6] for the fast solving of Fredholm's integral equation of the second kind. The first algorithm has a rate of convergence bounded by a small constant independently of the step size, whereas the second iteration has a convergence rate tending to zero when the step size approaches zero.

In Section 2 we describe the problem, its discretization and the assumptions we need. The multi-grid algorithm of the second kind is explained in Section 3. Section 4 contains the qualitative analysis of the rate of convergence.

2. THE PROBLEM AND ITS DISCRETIZATION

2.1. Equation

We consider the system

$$u = \mathscr{K}(u) \tag{2.1}$$

of nonlinear equations. The function u is an element of a Banach space B_0. Let $U \subset B_0$ be a neighbourhood of a (not necessarily unique) solution of (2.1). If we require that

$$U \text{ be sufficiently small}, \tag{2.2}$$

we may assume that $\mathscr{K}(v)$ is defined for all $v \in U$. Furthermore, $\mathscr{K}$ is assumed to be

Fréchet differentiable:

(2.3) $$K(v) := \mathscr{K}'(v) \quad (\text{Fréchet derivative at } v \in U),$$

where the operator $K(v) : B_0 \to B_0$ is Lipschitz continuous:

(2.4) $$\|K(v) - K(w)\|_{B_0 \to B_0} \leqq C\|v - w\|_{B_0} \quad (v, w \in U).$$

Here and in the sequel C denotes a generic constant. Requirements weaker than (2.3) and (2.4) are discussed in [6].

We introduce the notation

$$K := K(u) \quad (u \in U \text{ a solution of } (2.1)).$$

The multi-grid iteration can be applied to (2.1) only if the range of K belongs to a Banach space $B_1 \subset B_0$ with a finer topology. The essential property of $\mathscr{K}$ is

(2.5) $$\|K\|_{B_0 \to B_1} \leqq C.$$

Here K may be replaced by its power K^m ($m > 1$ fixed; cf. [6]). The estimate

(2.6) $$\|(I - K)^{-1}\|_{B_0 \to B_0} \leqq C \quad (I : \text{identity})$$

ensures that the problem (2.1) is properly posed.

Example 2.1. *Consider a nonlinear integral equation*

$$u(x) = \int_0^1 k(x, y, u(y))\, \mathrm{d}y \quad (x \in [0, 1]),$$

where $k(x, y, u)$ is Lipschitz continuously differentiable. Then $K(v)$ is defined by

$$(K(v)\, w)(x) = \int_0^1 k_u(x, y, v(y))\, w(y)\, \mathrm{d}y.$$

Obviously, the requirements (2.4) and (2.5) are satisfied for the choice of $B_0 = C^0([0, 1])$ and $B_1 = C^m([0, 1])$ $(m \geqq 1)$ provided that $(\partial/\partial x)^m\, k_u(x, y, u)$ is continuous.

Example 2.2. *Consider the elliptic problem $-\Delta u = u^2$ in $\Omega \subset R^n$, $u = 0$ on the boundary Γ of Ω, $1 \leqq n \leqq 3$. Let $\mathscr{K}(v)$ be the solution of $-\Delta u = v^2$, $u \mid_\Gamma = 0$, or in short notation: $\mathscr{K}(v) := -\Delta^{-1} v^2$. Then $K(v)$ defined by $K(v)\, w = -2\Delta^{-1}(vw)$ fulfils (2.4) and (2.5) if Γ is sufficiently smooth and if the Hölder spaces $B_0 = C^\sigma(\bar{\Omega})$, $B_1 = C^{2+\sigma}(\bar{\Omega})$ $(0 < \sigma < 1)$ or the Sobolev spaces $B_0 = L_2(\Omega)$, $B_1 = H_0^\varkappa(\Omega)$ $(0 < \varkappa < 2 - n/2)$ are chosen.*

Proof in the case of $B_0 = L_2(\Omega)$, $B_1 = H_0^\varkappa(\Omega)$. The embedding $H^{2-\varkappa}(\Omega) \subset L_\infty(\Omega)$ yields $L_1(\Omega) \subset L'_\infty(\Omega) \subset (H^{2-\varkappa}(\Omega))'$ for the dual spaces. Therefore, $w \in B_0 = L_2(\Omega) \to vw \in L_1(\Omega) \subset (H^{2-\varkappa}(\Omega))' \to \Delta^{-1}(vw) \in H_0^\varkappa(\Omega) = B_1$ shows (2.5). The continuity of $\Delta^{-1} : L_1(\Omega) \to B_1$ proves (2.4), too. ∎

Further examples are given in [6, 7, 8, 9].

2.2. Discretization

The method is named „multi-grid" iteration since we use a sequence of decreasing step sizes:

(2.7) $$h_0 > h_1 > \ldots > h_{\nu-1} > h_\nu > \ldots > 0\,, \quad \bar{\sigma} \geqq h_\nu/h_{\nu-1} \geqq \sigma > 0\,.$$

Usually,

$$h_\nu = 2^{-\nu} h_0 \quad (\nu \in \mathbf{N}_0 := \{0, 1, 2, \ldots\})$$

is chosen. For every $\nu \in \mathbf{N}_0$, the equation (2.1) discretized is

(2.8) $$u_\nu = \mathscr{K}_\nu(u_\nu)\,.$$

In the case of Example 2.1 we may discretize by a quadrature formula. The problem of Example 2.2 can be discretized by replacing Δ by a difference scheme. u_ν belongs to a discrete analogue of B_0 denoted by B_0^ν. The Banach space B_0^ν may consist of grid functions. In the case of Galerkin's procedure B_0^ν is a finite dimensional subspace of B_0. $B_1^\nu \subset B_0^\nu$ is the respective analogue of B_1.

As in Section 2.1 we define the Fréchet derivative

$$K_\nu(v_\nu) := \mathscr{K}'_\nu(v_\nu)\,, \quad K_\nu := K_\nu(u_\nu) \quad (u_\nu \text{ a solution of } (2.8))\,,$$

which is assumed to be defined for $v_\nu \in U_\nu \subset B_0^\nu$, where

$$U_\nu = \{v_\nu \in B_0^\nu : P_\nu v_\nu \in U\}$$

is defined by means of the prolongation $P_\nu : B_0^\nu \to B_0$ explained in Section 2.3. The definition of K_ν requires $u_\nu \in U_\nu$. Since $P_\nu u_\nu \to u$ is expected, $u_\nu \in U_\nu$ holds if we assume that

(2.9) h_0 *be sufficiently small.*

$K_\nu(v_\nu)$ has to satisfy the analogues of (2.4), (2.5), (2.6):

(2.10) $$\|K_\nu(v_\nu) - K_\nu(w_\nu)\|_{B_0^\nu \to B_0^\nu} \leqq C \|v_\nu - w_\nu\|_{B_0^\nu} \quad (v_\nu, w_\nu \in U_\nu;\ \nu \in \mathbf{N}_0)\,,$$

(2.11) $$\|K_\nu\|_{B_0^\nu \to B_1^\nu} \leqq C \quad (\nu \in \mathbf{N}_0)\,,$$

(2.12) $$\|(I_\nu - K_\nu)^{-1}\|_{B_0^\nu \to B_0^\nu} \leqq C \quad (\nu \in \mathbf{N}_0;\ I_\nu\text{: identity on } B_0^\nu)\,.$$

All constants are independent of ν.

2.3. Restrictions and Prolongations

The Banach spaces B_i and B_i^ν $(i = 0, 1;\ \nu \in \mathbf{N}_0)$ are connected by the restrictions

$$R_\nu : B_i \to B_i^\nu\,, \quad r_{\nu-1,\nu} : B_i^\nu \to B_i^{\nu-1} \quad (i = 0, 1)$$

with

(2.13a) $$\|R_\nu\|_{B_i \to B_i^\nu} \leqq C\,, \quad \|r_{\nu-1,\nu}\|_{B_i^\nu \to B_i^{\nu-1}} \leqq C\,, \quad R_{\nu-1} = r_{\nu-1,\nu} R_\nu \quad (i = 0, 1)$$

and by the prolongations

$$P_\nu : B_0^\nu \to B_0 \,, \quad p_{\nu,\nu-1} : B_0^{\nu-1} \to B_0^\nu$$

with

$$\text{(2.13b)} \qquad \|P_\nu\|_{B_0{}^\nu \to B_0} \leqq C \,, \quad \|p_{\nu,\nu-1}\|_{B_0{}^{\nu-1} \to B_0{}^\nu} \leqq C \,, \quad P_\nu p_{\nu,\nu-1} = P_{\nu-1} \,.$$

Furthermore, we assume the existence of $\hat{P}_\nu : B_1^\nu \to B_1$ with

$$\text{(2.13c)} \qquad R_\nu \hat{P}_\nu = I_\nu = \text{identity} \,, \quad \|\hat{P}_\nu\|_{B_1{}^\nu \to B_1} \leqq C \,.$$

The finer topology of B_1 is needed for the *approximation property*

$$\text{(2.13d)} \qquad \|I_\nu - p_{\nu,\nu-1} r_{\nu-1,\nu}\|_{B_1{}^\nu \to B_0{}^\nu} \leqq C h_{\nu-1}^\alpha \quad (\alpha > 0;\ \nu \geqq 1)$$

and the condition of *consistency*

$$\text{(2.14)} \qquad \|K_\nu R_\nu - R_\nu K\|_{B_1 \to B_0{}^\nu} \leqq C h_\nu^\beta \quad (\beta > 0;\ \nu \in N_0) \,.$$

The assumptions (2.5), (2.6), (2.13c) can be omitted if (2.14) is replaced by the relative consistency condition (cf. [10]):

$$\|K_{\nu-1} r_{\nu-1,\nu} - r_{\nu-1,\nu} K_\nu\|_{B_1{}^\nu \to B_0{}^{\nu-1}} \leqq C h_{\nu-1}^\beta \,.$$

3. MULTI-GRID ALGORITHM OF THE SECOND KIND

3.1. Preliminaries

The multi-grid algorithm depends on the choice of the step sizes (2.7), on the discretizations (2.8), on $r_{\nu-1,\nu}$ and $p_{\nu,\nu-1}$ and on the method used for solving (3.1) on the level $\nu = 0$. The mappings R_ν, P_ν, $\hat{P}_\nu$ and the derivatives K_ν are used only for the theoretical discussion.

In Section 3.2 we study the one-stage iteration which uses only one auxiliary grid. In general it is of no practical use. Nevertheless, its rate of convergence is nearly the same as that of the final algorithm. By a recursive application of the one-stage method the iteration of Section 3.3 is obtained. The recursive method needs the solutions of (2.8) for coarser grid widths. The algorithm of Section 3.4 provides for these values.

3.2. One-stage Method

Let F_ν be the range of $I_\nu - \mathscr{K}_\nu$:

$$F_\nu = \{f_\nu \in B_0^\nu : f_\nu = v_\nu - \mathscr{K}_\nu(v_\nu) \text{ and } v_\nu \in U_\nu\} \,.$$

Thanks to (2.12), F_ν is a neighbourhood of zero. Consider the generalized equation

$$\text{(3.1)} \qquad v_\nu = \mathscr{K}_\nu(v_\nu) + f_\nu \quad (f_\nu \in F_\nu)$$

and denote its solution by

$$v_\nu = \Phi_\nu(f_\nu)\,.$$

The one-stage iteration $v_\nu^{(\mu)} \to v_\nu^{(\mu+1)}$ is defined by

$$v_\nu^{(\mu+1/2)} = \mathscr{K}_\nu(v_\nu^{(\mu)}) + f_\nu\,, \tag{3.2a}$$

$$d_\nu^{(\mu)} = v_\nu^{(\mu+1/2)} - \mathscr{K}_\nu(v_\nu^{(\mu+1/2)}) - f_\nu = \mathscr{K}_\nu(v_\nu^{(\mu)}) - \mathscr{K}_\nu(v_\nu^{(\mu+1/2)})\,, \tag{3.2b}$$

$$v_\nu^{(\mu+1)} = v_\nu^{(\mu+1/2)} - p_{\nu,\nu-1}[\Phi_{\nu-1}(r_{\nu-1,\nu}d_\nu^{(\mu)}) - u_{\nu-1}]\,, \tag{3.2c}$$

where $u_{\nu-1} = \Phi_{\nu-1}(0)$ is the solution of (2.8). In the following we justify some modifications of the iteration (3.2).

Consider Example 2.2. $\mathscr{K}_\nu(v_\nu)$ has the representation $\Delta_\nu^{-1} v_\nu^2$, where Δ_ν is the difference analogue of Δ. Therefore, $\mathscr{K}_\nu(v_\nu)$ can be computed exactly only if a direct method is applicable. Otherwise, the inversion of Δ_ν is approximated by an iterative process as a secondary iteration. We assume

$$\mathscr{K}_\nu(v_\nu) = (I_\nu - A_\nu)^{-1}\,\mathscr{B}_\nu(v_\nu)\,, \quad \|A_\nu^\varrho\|_{B_0{}^\nu \to B_0{}^\nu} \leqq C_\nu \varepsilon_\nu^\varrho\,, \quad \varepsilon_\nu < 1\,,$$

i.e., the iteration

$$w_\nu^{(\mu+1)} = A_\nu w_\nu^{(\mu)} + \mathscr{B}_\nu(v_\nu)$$

converges to $\mathscr{K}_\nu(v_\nu)$. By $\mathscr{K}_\nu(v_\nu, w_\nu^{(0)}, \varrho)$ we denote the result of ϱ iteration steps starting with $w_\nu^{(0)}$:

$$\mathscr{K}_\nu(v_\nu, w_\nu, \varrho) = A_\nu^\varrho w_\nu + \sum_{\chi=0}^{\varrho-1} A_\nu^\chi\, \mathscr{B}_\nu(v_\nu) = \mathscr{K}_\nu(v_\nu) + A_\nu^\varrho[w_\nu - \mathscr{K}_\nu(v_\nu)]\,. \qquad (\varrho \geqq 0) \tag{3.3}$$

Example 3.1. *Consider the nonlinear boundary value problem of Example* 2.2 *and solve the linear problems* $-\Delta_\nu^{-1} v_\nu^2$ *by means of the multi-grid iteration of the first kind. In* [5] *we proved* $\|A_\nu^\varrho\|_{B_0{}^\nu \to B_0{}^\nu} \leqq \varepsilon^\varrho < 1$ *for* B_0^ν *being the discrete analogue of* $B_0 = L_2(\Omega)$. *Thus, neither* $C_\nu = 1$ *nor* $\varepsilon_\nu = \varepsilon$ *depend on* ν.

Eq. (3.2c) involves $u_{\nu-1}$. Since this solution is not known exactly, it is replaced by an approximation $\tilde{u}_{\nu-1}$. Let

$$\tilde{\delta}_{\nu-1} = \tilde{u}_{\nu-1} - \mathscr{K}_{\nu-1}(\tilde{u}_{\nu-1}, \tilde{u}_{\nu-1}, \tilde{\varrho}_{\nu-1}) \quad (\tilde{\varrho}_{\nu-1} \geqq 0)$$

be an approximation of the defect of $\tilde{u}_{\nu-1}$: $\tilde{u}_{\nu-1} \approx \Phi_{\nu-1}(\tilde{\delta}_{\nu-1})$. In the case of $\tilde{\varrho}_{\nu-1} = 0$, (3.3) yields $\tilde{\delta}_{\nu-1} = 0$.

Finally, we note that the argument of $\Phi_{\nu-1}$ must belong to $F_{\nu-1}$. This is ensured if $d_\nu^{(\mu)}$ is replaced by $\lambda_{\nu\mu} d_\nu^{(\mu)}$, where $\lambda_{\nu\mu} \neq 0$ is chosen suitably. The modified one-stage method takes the form

$$v_\nu^{(\mu+1/2)} = \mathscr{K}_\nu(v_\nu^{(\mu)}, v_\nu^{(\mu)} - f_\nu, \varrho_\nu) + f_\nu\,, \tag{3.4a}$$

$$d_\nu^{(\mu)} = v_\nu^{(\mu+1/2)} - \mathscr{K}_\nu(v_\nu^{(\mu+1/2)}, v_\nu^{(\mu+1/2)} - f_\nu, \varrho_\nu) - f_\nu\,, \tag{3.4b}$$

$$v_\nu^{(\mu+1)} = v_\nu^{(\mu+1/2)} - \lambda_{\nu\mu}^{-1} p_{\nu,\nu-1}[\Phi_{\nu-1}(r_{\nu-1,\nu}\lambda_{\nu\mu}d_\nu^{(\mu)} + \tilde{\delta}_{\nu-1}) - \tilde{u}_{\nu-1}]\,. \tag{3.4c}$$

3.3. Recursive Method

Eq. (3.4c) requires the exact evaluation of $\Phi_{\nu-1}$, i.e. the solving of an equation of the form (3.1). Starting with $\tilde{u}_{\nu-1}$, we approximate $\Phi_{\nu-1}(f_{\nu-1})$ by two iterations of the one-stage method for the levels $\nu-1$, $\nu-2$ and treat $\Phi_{\nu-2}(f_{\nu-2})$ similarly, etc. On the level $\nu = 0$, Eq. (3.1) is to be solved by any other method. We assume that $\Phi_0(f_0)$ is approximated by $\tilde{\Phi}_0(f_0)$ satisfying

$$(3.5) \qquad \|\Phi_0(f_0) - \tilde{\Phi}_0(f_0)\|_{B_0^0} \leqq C_0 \quad (f_0 \in F_0;\ C_0 \textit{ sufficiently small}).$$

The recursive method is defined by the following procedure similar to ALGOL.

procedure $rm(i, \nu, v, f)$; **value** ν; **integer** i, ν; **array** v, f;

comment i: number of iterations.

v: input $v = v_\nu^{(\mu)}$. output: $v = v_\nu^{(\mu+i)}$.

f: $f = f_\nu$ of Eq. (3.1);

if $\nu = 0$ **then** $v := \tilde{\Phi}_0(f)$ **else**

begin integer j; **array** w, d; **real** λ;

for $j := 1$ **step** 1 **until** i **do**

begin $w := \mathscr{K}_\nu(v, v - f, \varrho_\nu)$; $v := w + f$; $d := w - \mathscr{K}_\nu(v, w, \varrho_\nu)$;

$\lambda := \lambda_\nu(d)$; **comment** choice of $\lambda = \lambda_{\nu\mu}$ depending on d;

$d := \tilde{\delta}[\nu - 1] + \lambda * r_{\nu-1,\nu} * d$; $w := \tilde{u}[\nu - 1]$;

$rm(2, \nu - 1, w, d)$; $v := v - p_{\nu,\nu-1} * (w - \tilde{u}[\nu - 1])/\lambda$

end end i iterations on the level ν;

The variables $\tilde{u}[\nu - 1]$ and $\tilde{\delta}[\nu - 1]$ denote $\tilde{u}_{\nu-1}$ and $\tilde{\delta}_{\nu-1}$. The function $\lambda_\nu(d)$ is to be chosen accordingly to the discussion of Section 4.

3.4. The complete Algorithm

The following procedure calls rm for $\mu = 0, 1, \ldots, \nu$ and determines $\tilde{u}_0, \tilde{u}_1, \ldots, \tilde{u}_\nu$. The prescribed number of iterations per level μ is i_μ.

procedure $multigrid$ $(\nu, \tilde{u})$; **integer** ν; **array** $\tilde{u}$;

comment input: ν = maximal level.

output: $\tilde{u}[0 : \nu]$. $\tilde{u}[\mu]$ approximates the solution u_μ of (2.8);

begin integer μ; **array** $\tilde{\delta}[0 : \nu - 1]$;

for $\mu := 0$ **step** 1 **until** ν **do**

begin if $\mu = 0$ **then** $\tilde{u}[0] := \tilde{\Phi}_0(0)$ **else**

begin $\tilde{u}[\mu] := p_{\mu,\mu-1} * \tilde{u}[\mu - 1]; rm(i_\mu, \mu, \tilde{u}[\mu], 0)$

end computation of $\tilde{u}[\mu]$;

if $\mu < \nu$ **then** $\tilde{\delta}[\mu] := \tilde{u}[\mu] - \mathscr{K}_\mu(\tilde{u}[\mu], \tilde{u}[\mu], \tilde{\varrho}_\mu)$;

comment This statement can be omitted if $\tilde{\varrho}_\mu = 0$;

end end multi-grid iteration of the second kind;

In Section 4 we analyse this procedure. To obtain a practical algorithm, we have to add checks. For example, if one states divergence (or convergence to another solution of the problem), the condition (2.9) is violated and one has to refine the coarsest step size h_0. Another check should terminate the calculation as soon as the discretization error of $\tilde{u}[\mu]$ is small enough.

A practical choice of the first step size h_0 is to define h_0 as large as possible. For uncritical problems this value suffices. We illustrate this comment by some examples. In [5] we solved the linear Fredholm integral equation

$$u(x) = \lambda \int_0^1 \cos(\pi x s)\, u(s)\, \mathrm{d}s + f(x) \quad (0 \leqq x \leqq 1). \tag{3.6}$$

It turned out that $h_0 = 1$ suffices for $\lambda = 1$. In the case of $\lambda = 10$ the step size h_0 of the quadrature formula must be $\leqq 1/4$. From [9] we cite the nonlinear boundary value problem

$$-\Delta\, u(x, y) = e^{u(x,y)} \quad \text{in} \quad \Omega = (0, 1) \times (0, 1), \quad u = 0 \quad \text{on} \quad \Gamma = \partial\Omega, \tag{3.7}$$

(cf. Example 2.2). Also in this case the coarsest grid width $h_0 = 1/2$ is sufficient. Example 2.2 with $\Omega = (0, 1) \times (0, 1)$ has the trivial solution $u = 0$ and another solution $u > 0$. The computation of the latter solution requires $h_0 \leqq 1/4$.

For considerations about the amount of computational work we refer to [6, 7, 8, 9].

4. ANALYSIS OF RATE OF CONVERGENCE

4.1. One-stage Iteration (3.4)

In the sequel the norm $\|\cdot\|_{B_0^\nu}$ is abbreviated by $\|\cdot\|$. We represent the starting vector $v_\nu^{(\mu)}$ by

$$v_\nu^{(\mu)} = v_\nu + \Delta_\nu^{(\mu)}, \quad \text{where} \quad v_\nu = \Phi_\nu(f_\nu) \quad \text{is a solution of (3.1)}.$$

Then

$$v_\nu^{(\mu+1/2)} = v_\nu + \Delta_\nu^{(\mu+1/2)}, \tag{4.1}$$

$$\Delta_\nu^{(\mu+1/2)} = K_\nu(v_\nu)\, \Delta_\nu^{(\mu)} + A_\nu^{\varrho\nu}[I_\nu - K_\nu(v_\nu)]\, \Delta_\nu^{(\mu)} + O(\|\Delta_\nu^{(\mu)}\|^2)$$

and

$$(4.2) \qquad d_\nu^{(\mu)} = [I_\nu - A_\nu^{\varrho_\nu}]\,[I_\nu - K_\nu(v_\nu)]\,\Delta_\nu^{(\mu+1/2)} + O(\|\Delta_\nu^{(\mu+1/2)}\|^2)$$

hold. The symbol $O(\cdot)$ denotes the estimation of the remainder with respect to $\|\cdot\| = \|\cdot\|_{Bo^\nu}$.

Define $\delta_{\nu-1} := \tilde{u}_{\nu-1} - \mathscr{K}_{\nu-1}(\tilde{u}_{\nu-1})$, i.e. $\tilde{u}_{\nu-1} = \Phi_{\nu-1}(\delta_{\nu-1})$. Then

$$C^{-1}\|\delta_{\nu-1}\| \leqq \|\tilde{u}_{\nu-1} - u_{\nu-1}\| \leqq C\|\delta_{\nu-1}\| \quad (u_{\nu-1} := \Phi_{\nu-1}(0))$$

is valid. By definition of $\delta_{\nu-1}$,

$$(4.3) \qquad \|\delta_{\nu-1} - \tilde{\delta}_{\nu-1}\| = \|A_{\nu-1}^{\tilde{\varrho}_{\nu-1}}\delta_{\nu-1}\| \leqq C_{\nu-1}\varepsilon_{\nu-1}^{\tilde{\varrho}_{\nu-1}}\|\delta_{\nu-1}\|$$

follows.

The Fréchet derivative of Φ_μ is $\Phi_\mu'(f_\mu) = [I_\mu - K_\mu(\Phi_\mu(f_\mu))]^{-1}$. Using

$$\Phi_\mu(g_\mu) = \Phi_\mu(\delta_\mu) + [I_\mu - K_\mu(\tilde{u}_\mu)]^{-1}\,(g_\mu - \delta_\mu) + O(\|g_\mu - \delta_\mu\|^2),$$

we obtain

$$(4.4) \qquad \Phi_{\nu-1}(\lambda_{\nu\mu}r_{\nu-1,\nu}d_\nu^{(\mu)} + \tilde{\delta}_{\nu-1}) - \Phi_{\nu-1}(\delta_{\nu-1}) =$$
$$[I_{\nu-1} - K_{\nu-1}(\tilde{u}_{\nu-1})]^{-1}\,[\lambda_{\nu\mu}r_{\nu-1,\nu}d_\nu^{(\mu)} + \tilde{\delta}_{\nu-1} - \delta_{\nu-1}] +$$
$$O([\lambda_{\nu\mu}\|d_\nu^{(\mu)}\| + \|\tilde{\delta}_{\nu-1} - \delta_{\nu-1}\|]^2).$$

From (2.10) one concludes

$$(4.5) \quad \|K_\nu(v_\nu) - K_\nu\|_{Bo^\nu\to Bo^\nu} \leqq C\|f_\nu\|, \quad \|K_{\nu-1}(\tilde{u}_{\nu-1}) - K_{\nu-1}\|_{Bo^\nu\to Bo^\nu} \leqq C\|\delta_{\nu-1}\|.$$

If $\mathscr{K}_\nu$ is affine and if $\tilde{u}_{\nu-1} = u_{\nu-1}$, $\tilde{\delta}_{\nu-1} = \delta_{\nu-1}$ and $\varepsilon_\nu = 0$ are assumed, then

$$\Delta_\nu^{(\mu+1)} := v_\nu^{(\mu+1)} - v_\nu = M_\nu\Delta_\nu^{(\mu)}$$

holds with

$$M_\nu = [I_\nu - p_{\nu,\nu-1}(I_{\nu-1} - K_{\nu-1})^{-1}\,r_{\nu-1,\nu}(I_\nu - K_\nu)]\,K_\nu.$$

In [6] we proved

Lemma 4.1. *If* (2.5), (2.6), (2.7), (2.11), (2.12), (2.13a–d) *and* (2.14) *are valid, then the estimate*

$$\|M_\nu\|_{Bo^\nu\to Bo^\nu} \leqq Ch_\nu^\gamma, \quad \textit{where} \quad \gamma := \min(\alpha, \beta),$$

holds. Therefore, convergence follows from (2.9).

In the general case the estimates (4.1)–(4.5) yield

$$(4.6) \qquad \|\Delta_\nu^{(\mu+1)}\| \leqq C\{[h_\nu^\gamma + C_\nu\varepsilon_\nu^{\varrho_\nu} + \|\delta_{\nu-1}\| + \|f_\nu\| + \|\Delta_\nu^{(\mu)}\|]\,\|\Delta_\nu^{(\mu)}\| +$$
$$+ \lambda_{\nu\mu}^{-1}\,C_{\nu-1}\varepsilon_{\nu-1}^{\tilde{\varrho}_{\nu-1}}\|\delta_{\nu-1}\|\}.$$

Note 4.2. *Let all the assumptions of Section 2 be valid. If* $C_\nu \varepsilon_\nu^{\varrho_\nu}$, $\|f_\nu\|$, $\|\delta_{\nu-1}\|$, *and* $\|\Delta_\nu^{(0)}\|$ *are sufficiently small, the argument of* $\Phi_{\nu-1}$ *in Eq.* (3.4c) *belongs to* $F_{\nu-1}$. *Therefore, the estimate* (4.6) *holds. The iteration* (3.4) *converges to* $\tilde{v}_\nu$ *with*

$$\|\tilde{v}_\nu - v_\nu\| = O(\lambda_{\min}^{-1} C_{\nu-1} \varepsilon_{\nu-1}^{\tilde{\varrho}_{\nu-1}} \|\delta_{\nu-1}\|), \quad where \quad \lambda_{\min} = \min_\mu |\lambda_{\nu\mu}| .$$

A suitable choice of ϱ_ν, $\tilde{\varrho}_\nu$, $\|\delta_\nu\|$ is characterized by

$$\|\delta_\nu\| \leqq C h_\nu^\eta (\eta \geqq \gamma), \quad \lambda_{\nu\mu} \geqq \lambda_{\min} > 0 , \tag{4.7a}$$

$$C_\nu \varepsilon_\nu^{\varrho_\nu} \leqq C h_\nu^\gamma , \quad C_{\nu-1} \varepsilon_{\nu-1}^{\tilde{\varrho}_{\nu-1}} \leqq \varepsilon \cdot \lambda_{\min} . \tag{4.7b}$$

We recall that $\gamma > 0$ is defined in Lemma 4.1.

Note 4.3. *If* (4.7a,b) *and* $\|f_\nu\| \leqq C h_\nu^\gamma$ *hold, the estimate* (4.6′) *follows*:

$$\|\Delta_\nu^{(\mu+1)}\| \leqq C[h_\nu^\gamma \|\Delta_\nu^{(\mu)}\| + \|\Delta_\nu^{(\mu)}\|^2 + \varepsilon h_\nu^\eta] . \tag{4.6′}$$

Note 4.4. *There exists a number* ε_F *such that* $\|f_\nu\| \leqq 2\varepsilon_F$ *implies* $f_\nu \in F_\nu$ *for all* $\nu \in N_0$. *A suitable choice of* $\lambda_{\nu\mu}$ is

$$\lambda_{\nu\mu} \approx \min (C h_{\nu-1}^\gamma, \varepsilon_F) / \|r_{\nu-1,\nu} d_\nu^{(\mu)}\| . \tag{4.8}$$

Then the arguments of $\Phi_{\nu-1}$ *always belong to* $F_{\nu-1}$. *Moreover, their magnitude is less than* $C h_{\nu-1}^\gamma$. *The assumption* $\|f_\nu\| \leqq C h_\nu^\gamma$ *implies* $\lambda_{\nu\mu} \geqq \lambda_{\min} > 0$ *as required in* (4.7a). *It is evident that* $\lambda_{\nu\mu}$ *allows an estimation of the iteration error, if* $\|r_{\nu-1,\nu} d_\nu^{(\mu)}\|$ *is replaced by* $C\|d_\nu^{(\mu)}\|$. *If* $\|r_{\nu-1,\nu} d_\nu^{(\mu)}\|$ *is too small, Eq.* (3.4c) *can be omitted. If* $\|d_\nu^{(\mu)}\|$ *is small enough, the iteration can be terminated.*

4.2. Recursive Method *rm*

The recursive iteration can be obtained from (3.4) by substituting $\Phi_{\nu-1}$ by $\tilde{\Phi}_{\nu-1}$, where $\tilde{\Phi}_{\nu-1}$ is defined as follows. $\tilde{\Phi}_0$ is mentioned in Section 3.3. $\tilde{\Phi}_\mu(f_\mu)$ ($\mu \geqq 1$) is the result of $rm(2, \mu, v, f_\mu)$ with the starting vector $v := \tilde{u}_\mu$ (i.e. two iterations of (3.4) with $\tilde{\Phi}_{\mu-1}$ instead of $\Phi_{\mu-1}$).

By induction we show:

Lemma 4.5. *Under the conditions of Note* 4.3 *and with* $\lambda_{\nu\mu}$ *from Note* 4.4, *the following estimate holds*:

$$\|\Phi_\nu(f_\nu) - \tilde{\Phi}_\nu(f_\nu)\| \leqq C[h_\nu^{2\gamma} \|f_\nu\| + \varepsilon h_\nu^\eta] . \tag{4.9}$$

Proof. (4.9) follows from (3.5) for $\nu = 0$. Assume that (4.9) holds for $0, 1, \ldots, \nu - 1$. Replacing $\Phi_{\nu-1}$ by $\tilde{\Phi}_{\nu-1}$, we obtain the additional term $C\{h_\nu^{2\gamma}[\|\Delta_\nu^{(\mu)}\| + \|\delta_{\nu-1}\|] + \varepsilon h_\nu^\eta\}$ on the right-hand side of (4.6').

Then

$$\|\Delta_\nu^{(0)}\| \leqq C\|f_\nu - \delta_\nu\| \leqq C'[\|f_\nu\| + h_\nu^\eta] \leqq C'' h_\nu^\gamma ,$$

$$\|\Delta_\nu^{(\mu)}\| \leqq C\{h_\nu^\gamma \|\Delta_\nu^{(\mu-1)}\| + \varepsilon h_\nu^\eta + h_\nu^{2\gamma}[\|\Delta_\nu^{(\mu-1)}\| + C h_{\nu-1}^\eta] + \varepsilon h_{\nu-1}^\eta\}$$

yields

$$\|\Delta_\nu^{(2)}\| \leqq C\left\{h_\nu^{2\gamma}\|f_\nu\| + [\varepsilon + (Ch_\nu^{2\gamma} + \varepsilon)\left(\frac{h_{\nu-1}}{h_\nu}\right)^\eta h_\nu^\eta\right\}.$$

The inequalities $Ch_\nu^{2\gamma} \leqq \varepsilon$ (cf. (2.9)) and $h_{\nu-1}/h_\nu \leqq 1/\bar{\sigma}$ imply (4.9)). ■

Note 4.6. *Under the conditions of Section* 2 *and* (3.5), (4.7), (4.8), *the estimates* $\|f_\nu\| \leqq Ch_\nu^\gamma$ *and* $\|\Delta_\nu^{(0)}\| \leqq Ch_\gamma^\nu$ *imply*

$$\|\Delta_\nu^{(\mu+1)}\| \leqq C[h_\nu^\gamma\|\Delta_\nu^{(\mu)}\| + \varepsilon h_\nu^\eta]. \tag{4.10}$$

4.3. Complete Algorithm *multigrid*

In the procedure *multigrid* $p_{\mu,\mu-1}\,\tilde{u}_{\mu-1}$ is used as the starting value for $u_\mu^{(0)}$. The difference of $p_{\mu,\mu-1}\,\tilde{u}_{\mu-1}$ and $\boldsymbol{u}_\mu$ consists of a discretization error and an approximation error of $p_{\mu,\mu-1}$. Assume that the first error is of order $O(h_\mu^d)$, while the second is $O(h_\mu^a)$. Usually $a = \alpha$ holds (cf. (2.13d)). Therefore,

$$\|u_\mu - p_{\mu,\mu-1}\tilde{u}_{\mu-1}\| \leqq C[h_\mu^d + h_\mu^a + \|\delta_{\mu-1}\|] \leqq C' h_\mu^{\min(d,a,\eta)} \tag{4.11}$$
$$(u_\mu = \Phi_\mu(0))$$

(cf. (4.7a)) is the error estimate of $u_\mu^{(0)}$.

We want to obtain $\tilde{u}_\nu$ with $\|\tilde{u}_\nu - u_\nu\| \leqq Ch_\nu^\varkappa$ for given $\varkappa$ and ν. The usual choice of $\varkappa$ is $\varkappa = d$, i.e. iteration error $\approx$ discretization error.

Proposition 4.7. *Let* $\varkappa \geqq \gamma$ *and assume that all the conditions of Section* 2 *are satisfied. We propose the following choice of parameters*:

a) $i_\mu = i$ *with* $i \geqq 1$ *such that* $i \cdot \gamma + \min(d, a, \varkappa) > \varkappa$;

b) ϱ_μ *and* $\tilde{\varrho}_\mu$ *according to* (4.7b) *with* ε *sufficiently small*[1]);

c) $\lambda_{\nu\mu}$ *defined by* (4.8).

Then the procedure multigrid *of Section* 3.4 *produces* $\tilde{u}_\mu$ $(\mu = 0, \ldots, \nu)$ *with the desired accuracy*:

$$\|u_\mu - \tilde{u}_\mu\| \leqq Ch_\nu^\varkappa \quad (0 \leqq \mu \leqq \nu). \tag{4.12}$$

Proof. (4.12) is equivalent to the first estimate of (4.7a) if we equate η and $\varkappa$ (note that $\varkappa \geqq \gamma$). We prove (4.12) by induction. (3.5) results in $\|u_0 - \tilde{u}_0\| \leqq C'$. Since C' is assumed to be sufficiently small, (4.12) follows for $\mu = 0$. If (4.12) holds on the level $\mu - 1$, (4.11) yields $\|\Delta_\mu^{(0)}\| \leqq Ch_\mu^{\min(d,a,\varkappa)}$. Note 4.6 shows

$$\|\Delta_\mu^{(i)}\| \leqq C_1 h_\mu^{i\cdot\gamma+\min(d,a,\varkappa)} + C_2\varepsilon h_\mu^\varkappa = [C_1 h_\mu^\chi + C_2\varepsilon]\, h_\mu^\varkappa.$$

Since $\chi > 0$ and since ε is sufficiently small, (4.12) is valid for μ. ■

[1]) The proof will show that there exists $\varepsilon_{\max}$ such that $\varepsilon \lesssim \varepsilon_{\max}$ and $\|\delta_\mu\| \lesssim Ch_\mu^\varkappa$ imply $\|\delta_{\mu+1}\| \lesssim Ch_{\mu+1}^\varkappa$ with the same constant C.

We conclude the discussion with the special case of a linear equation, i.e. $\mathscr{K}_\nu(v_\nu) = K_\nu v_\nu + q_\nu$. In this case the result $v_\nu^{(\mu)}$ is independent of the choice of $\tilde{u}_\mu (\mu < \nu)$. Therefore, the linear multi-grid method is obtained from the procedure *rm* by setting formally $\tilde{u}_\mu := 0$, since in this case δ_μ and $\tilde{\delta}_\mu$ vanish. Thus, all terms of (4.9), (4.10), (4.11) containing ε or $\|\delta_\mu\|$ can be omitted.

For the linear case it is not necessary to implement the nested iteration of Section 3.4. On the other hand, the use of the algorithm *multigrid* has many advantages. It might be less expensive to provide for good starting values $u_\nu^{(o)}$ by computations on the lower levels. Furthermore, the computation may fail if (2.9) is violated. It is advantageous to check this condition by observing the convergence during the performance of the procedure *multigrid*.

4.4. Examples

In order to give an idea of the fast convergence of the multi-grid method we cite the results of the problems (3.6) and (3.7) from [5, 9]. Consider the integral equation (3.6). Discretizing the integral by the trapezoidal formula for $h_\nu = 2^{-\nu} h_0$ and defining P_ν by piecewise linear interpolation, we obtain $B_0 = C^0([0, 1])$, $B_1 = C_L^1([0, 1]$ (Lipschitz continuous derivatives), and $\alpha = \beta = 2$, hence $\gamma = 2$. The observed rates of convergence of the linear recursive method *rm* are listed below for $\lambda = 1, 10$ and varying sizes h:

	$h = 1/32$	$h = 1/64$	$h = 1/128$	$h = 1/256$
$\lambda = 1$	$6{\cdot}2_{10} - 4$	$1{\cdot}6_{10} - 4$	$3{\cdot}6_{10} - 5$	$9{\cdot}1_{10} - 6$
$\lambda = 10$	$8_{10} - 3$	$2_{10} - 3$	$6_{10} - 4$	$1{\cdot}4_{10} - 4$

Therefore, it suffices to perform the procedure *multigrid* with $i_\mu = 1$. The error $|\tilde{u}_\mu(x) - u(x)|$ is almost equal to the discretization error $|u_\mu(x) - u(x)|$.

The nonlinear problem (3.7) is reported in [9]. The rates of convergence of the recursive procedure *rm* are approximately:

step size	$h_1 = 1/4$	$h_2 = 1/8$	$h_3 = 1/16$	$h_4 = 1/32$	$h_5 = 1/64$
rates	0·06	0·008	0·002	0·0009	0·0006

Choosing $\tilde{\varrho}_\mu = i_\mu = 1$ in procedure *multigrid* one obtains the following results at $x = y = 1/2$:

$h_0 = 1/2$: $\tilde{u}_0 = 0{\cdot}\mathbf{066\ 819}$ $\quad$ $h_3 = 1/16$: $\tilde{u}_3 = 0{\cdot}\mathbf{077\ 872}\,65$

$h_1 = 1/4$: $\tilde{u}_1 = 0{\cdot}\mathbf{074\ 715}\,05$ $\quad$ $h_4 = 1/32$: $\tilde{u}_4 = 0{\cdot}\mathbf{078\ 043}\,72$

$h_2 = 1/8$: $\tilde{u}_2 = 0{\cdot}\mathbf{077\ 200}\,48$ $\quad$ $h_5 = 1/64$: $\tilde{u}_5 = 0{\cdot}\mathbf{078\ 086}\,69$

(bold-face figures indicate correct digits). Quadratic extrapolation of $\tilde{u}_3$, $\tilde{u}_4$ and $\tilde{u}_5$ results in 0·**078 101 022** 6. The corresponding computation time (CDC Cyber 70/76, Rechenzentrum der Universität zu Köln) amounts to 0·51 s CPU.

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Souhrn

ANALÝZA CHYB NELINEÁRNÍ MNOHOSÍŤOVÉ METODY DRUHÉHO DRUHU

WOLFGANG HACKBUSCH

Mnohosíťová metoda druhého druhu je rychlý numerický algoritmus pro řešení problémů, které lze formálně vyjádřit ve tvaru Fredholmovy integrální rovnice druhého druhu. Příklady takových problémů jsou Fredholmovy integrální rovnice, speciální problémy optimální regulace, nelineární eliptické rovnice atd. Metoda vyžaduje provedení jen několika iterací pro posloupnost zmenšujících se kroků. V článku se diskutuje vliv různých parametrů na rychlost konvergence.

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