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CONVERGENCE OF DUAL FINITE ELEMENT APPROXIMATIONS FOR UNILATERAL BOUNDARY VALUE PROBLEMS

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1. INTRODUCTION

In the papers [1], [2], [3] a dual finite element approach for some unilateral boundary value problems has been proposed. A priori error estimate have been proven under the assumption that the exact solution is sufficiently regular. This condition, however, is not satisfied in the case of general data – see e.g. [7] [8], [9]. Hence the following problem remained open: do the dual finite element approximations converge if the solution is not regular enough?

It was the aim of the present study to prove the convergence without any superfluous regularity assumption. Assuming that the domain is in a subclass of convex polygons (cf. Theorem 2.2 for the definition of the subclass) the convergence can be proven for the following model problem:

$$-\Delta u = f \quad \text{in} \quad \Omega\,, \tag{1.1}$$
$$u \geqq 0\,, \quad \frac{\partial u}{\partial \nu} \geqq 0\,, \quad u \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega\,.$$

The problem (1.1) represents a class of semi-coercive unilateral problems of Signorini's type.

Let Ω be a bounded domain with a polygonal boundary $\partial\Omega \equiv \Gamma$, $f \in L_2(\Omega)$, $\partial u/\partial \nu$ denotes the derivative with respect to the outward normal to Γ.

We use the Sobolev spaces $H^k(\Omega)$ $(\equiv W_2^{(k)}(\Omega))$ with the usual norm $\|\cdot\|_k$ and the following notation:

$$\mathbf{x} = (x_1, x_2)\,, \quad (u, v)_0 = \int_\Omega uv \,\mathrm{d}\mathbf{x}\,,$$
$$((\mathbf{q}, \mathbf{p})) = \sum_{i=1}^{2} (q_i, p_i)_0\,, \quad \|\mathbf{q}\| = ((\mathbf{q}, \mathbf{q}))^{1/2}\,.$$

$H^{1/2}(\Gamma)$ will denote the space of traces γv of the functions $v \in H^1(\Omega)$ on the boundary Γ,

$H^{-1/2}(\Gamma)$ the space of linear continuous functionals over $H^{1/2}(\Gamma)$.

We introduce the space

$$Q = \{\boldsymbol{q} \in [L_2(\Omega)]^2 \mid \operatorname{div} \boldsymbol{q} \in L_2(\Omega)\}$$

and for any $\boldsymbol{q} \in Q$ we define the "flux" $\boldsymbol{q} \,.\, \nu|_\Gamma \in H^{-1/2}(\Gamma)$ by means of the relation

$$\langle \boldsymbol{q} \,.\, \nu, w \rangle = ((\boldsymbol{q}, \operatorname{grad} v)) + (v, \operatorname{div} \boldsymbol{q})_0 \,,$$

where $v \in H^1(\Omega)$ is any function such that the trace $\gamma v = w$. We write $\boldsymbol{q} \,.\, \nu|_\Gamma \geqq 0$ if

$$\langle \boldsymbol{q} \,.\, \nu, w \rangle \geqq 0 \quad \forall w \in H_+^{1/2}(\Gamma) = \{w \in H^{1/2}(\Gamma) \,|\, w \geqq 0\} \,.$$

The main tool used for the convergence proof will be the following abstract theorem (cf. [5] – chpt. 4 and [6] – Th. 3.1).

Theorem 1.1 *Let V be a Hilbert space with the norm $\|\cdot\|$, $K \subset V$ a convex closed subset, $h \in (0, 1\rangle$ a real parameter, $K_h \subset K$ convex closed sets for any h.*

Let a differentiable functional $\mathscr{J}$ on V be given, the second differential of which satisfies the following inequalities

$$\text{(1.2)} \qquad \alpha_0 \|z\|^2 \leqq D^2 \mathscr{J}(u; z, z) \leqq C\|z\|^2 \quad \forall u \in K \,, \quad z \in V \,,$$

where α_0 and C are positive constants.

Denote u and u_h the minimizing elements of $\mathscr{J}$ over K and K_h, respectively.

Assume that $v_h \in K_h$ exist such that

$$\text{(1.3)} \qquad \|u - v_h\| \to 0 \quad \text{for} \quad h \to 0 \,.$$

Then it holds

$$\text{(1.4)} \qquad u_h \to u \quad \text{in} \quad V \quad \text{for} \quad h \to 0 \,.$$

In the approximations we use the $\alpha - \beta -$ regular family of triangulations $\{\mathscr{T}_h\}$, $0 < h \leqq 1$ (i.e. positive numbers α and β exist such that for any h the minimal angle of all triangles is not less than α and the ratio of any two sides in $\mathscr{T}_h$ is less than β).

2. THE DUAL VARIATIONAL FORMULATION

First let us recall the definition of the dual problem.

We define the set of admissible functions

$$\mathscr{U} = \{\boldsymbol{q} \in Q \mid \operatorname{div} \boldsymbol{q} + f = 0, \ \boldsymbol{q} \,.\, \nu|_\Gamma \geqq 0\}$$

and the functional (complementary energy)

$$\mathscr{S}(\boldsymbol{q}) = \tfrac{1}{2}\|\boldsymbol{q}\|^2 \,.$$

The *problem* to find $\lambda^0 \in \mathscr{U}$ such that

$$\mathscr{S}(\lambda^0) \leqq \mathscr{S}(\boldsymbol{q}) \quad \forall \boldsymbol{q} \in \mathscr{U} \tag{2.1}$$

is called *dual* to the primary problem. In order to define finite element approximations to the dual problem, we introduced an equivalent dual problem, as follows.

Let $\bar{\lambda} \in Q$ be such that

$$\operatorname{div} \bar{\lambda} + f = 0 \quad \text{in} \quad \Omega .$$

Moreover, let $\mathbf{z}^0 \in Q$ be such that

$$\operatorname{div} \mathbf{z}^0 = 0 \quad \text{in} \quad \Omega ,$$

$$\mathbf{z}^0 \,.\, \nu\big|_\Gamma = -\bar{\lambda} \,.\, \nu\big|_\Gamma - g_0 ,$$

where $g_0 = (f, 1)_0/\text{mes}\, \Gamma = \text{const.}$ (We assume $g_0 \leqq 0$, in accordance with [3].)

Note that the vector $\bar{\lambda}$ can be found by a simple integration, whereas $\mathbf{z}^0 = \{-\partial\omega/\partial x_2, \partial\omega/\partial x_1\}$, if the function $\omega \in H^2(\Omega)$ satisfies the boundary condition

$$\omega(s) = -\int_{s_0}^{s} (\bar{\lambda} \,.\, \nu + g_0)\, \mathrm{d}t \quad \forall s \in \Gamma .$$

Define the new set of admissible functions

$$\mathscr{U}_0 = \{\boldsymbol{q}/\boldsymbol{q} \in Q, \operatorname{\mathbf{div}} \boldsymbol{q} = 0 \quad \text{in}\ \Omega,$$

$$(\boldsymbol{q} + \lambda^f) \,.\, \nu\big|_\Gamma \geqq 0\} , \quad \lambda^f = \bar{\lambda} + \mathbf{z}^0 ,$$

and the functional

$$J(\boldsymbol{q}) = \tfrac{1}{2}\|\boldsymbol{q}\|^2 + ((\lambda^f, \boldsymbol{q})) .$$

Then the problem to find a $\boldsymbol{q}^0 \in \mathscr{U}_0$ such that

$$J(\boldsymbol{q}^0) \leqq J(\boldsymbol{q}) \quad \forall \boldsymbol{q} \in \mathscr{U}_0 \tag{2.2}$$

is equivalent with the dual problem (2.1) and the solutions satisfy the relation $\lambda^0 = \lambda^f + \boldsymbol{q}^0$. Recall that $-\lambda^f \,.\, \nu\big|_\Gamma = g_0$ and

$$\boldsymbol{q} \in \mathscr{U}_0 \Rightarrow \langle \boldsymbol{q} \,.\, \nu, w \rangle \geqq g_0 \int_\Gamma w \, \mathrm{d}s \quad \forall w \in H_+^{1/2}(\Gamma) . \tag{2.2'}$$

We employed the spaces $\mathscr{N}_h(\Omega)$ of solenoidal finite elements (see [4]) to define

$$\mathscr{U}_0^h = \{\boldsymbol{q} \in \mathscr{N}_h(\Omega) \,|\, \boldsymbol{q} \,.\, \nu\big|_\Gamma \geqq g_0\} = \mathscr{U}_0 \cap \mathscr{N}_h(\Omega) .$$

Finally, a vector $\lambda^h = \lambda^f + \boldsymbol{q}^h$, $\boldsymbol{q}^h \in \mathscr{U}_0^h$ is called a finite element approximation to the dual problem, if

$$J(\boldsymbol{q}^h) \leqq J(\boldsymbol{q}) \quad \forall \boldsymbol{q} \in \mathscr{U}_0^h . \tag{2.3}$$

In the paper [3] the following results were proven:

(i) the dual problem (2.1) has a solution if and only if

$$(f, 1)_0 \leqq 0 \tag{2.4}$$

and if (2.4) holds, the solution is unique;

(ii) if u and λ^0 are the solutions of the primary and of the dual problem, respectively, then

$$\lambda^0 = \operatorname{grad} u \, , \tag{2.5}$$

Moreover, we obtained the following a priori error estimate:

Theorem 2.1 *Let Ω be simply connected, $(f, 1)_0 < 0$, let the solution $\boldsymbol{q}^0$ of the problem (2.2) belong to $[H^2(\Omega)]^2$ and $\boldsymbol{q}^0 . \nu \in H^2(\Gamma_m)$ for any side Γ_m, $m = 1, 2, \dots, M$, of the polygonal boundary Γ. Then for $\alpha - \beta -$ regular family of triangulations $\{\mathscr{T}_h\}$ it holds*[1])

$$\|\lambda^h - \lambda^0\| \leqq C h^{3/2} \{ \|\boldsymbol{q}^0\|_{2,\Omega} + \sum_{m=1}^{M} \|\boldsymbol{q}^0 . \nu\|_{2,\Gamma_m} \} \, . \tag{2.6}$$

Convergence of the dual finite element approximations

As the regularity of the solution required in Theorem 2.1 cannot be expected in general (cf. [7], [8], [9]), we shall study the convergence of the dual finite element approximations under less restrictive assumptions. To this end we employ Theorem 1.1.

The above theorem will be applied with:

$$V = Q_0(\Omega) = \{\boldsymbol{q} \in Q \mid \operatorname{div} \boldsymbol{q} = 0 \text{ in } \Omega\} \, ,$$

$$\|\boldsymbol{q}\| = (\sum_{i=1}^{2} \|q_i\|_0^2)^{1/2} \, ,$$

$$K = \mathscr{U}_0 \, , \quad K_h = \mathscr{U}_0^h \, , \quad \mathscr{J} = J \, , \quad u = \boldsymbol{q}^0 \, , \quad u_h = \boldsymbol{q}^h \, .$$

It is not difficult to prove that $\mathscr{U}_0$ is closed in $Q_0(\Omega)$ and that the functional J satisfies (1.2) with $\alpha_0 = c = 1$.

The main problem, however, is to verify (1.3). We have to find $\boldsymbol{p}^h \in \mathscr{U}_0^h$ such that

$$\lim_{h \to 0} \|\boldsymbol{q}^0 - \boldsymbol{p}^h\| = 0 \, .$$

To this end we choose the following approach. First we prove that a smooth vector $\boldsymbol{q} \in \mathscr{U}_0 \cap [C^\infty(\bar{\Omega})]^2$ exists, which is arbitrarily close to $\boldsymbol{q}^0$ in $Q_0(\Omega)$.

[1]) In the right-hand side of (2.6) even the norms can be replaced by seminorms.

Theorem 2.2 *Let Ω be a convex polygonal domain such that the sum of any two neighbouring angles is not less than π. Assume that $\lambda^f = \bar{\lambda} + \mathbf{z}^0 \in [H^1(\Omega)]^2$.*

Then for any $\eta > 0$ there exists a $\mathbf{q} \in \mathscr{U}_0 \cap [C^\infty(\bar{\Omega})]^2$ such that

$$\|\mathbf{q}^0 - \mathbf{q}\| < \eta . \tag{2.7}$$

Remark 2.1 If Ω is a convex polygon, the solution u of the primary problem belongs to $H^2(\Omega)$ (cf. Grisvard [9]).

Proof of Th. 2.2 is based on several lemmas.

Lemma 2.1 *Let the assumptions of Th. 2.2 hold. Let Ω^* be a bounded polygonal domain such that $\Omega^* \supset \bar{\Omega}$, the sides of $\partial\Omega^*$ are parallel with those of $\partial\Omega$ and denote $G = \Omega^* \dot{-} \bar{\Omega}$. Let k, $k \in \langle 1, k_0 \rangle$ and $\varkappa_0 > 0$ be real parameters; let the origin of the global coordinate system (x_1, x_2) be in Ω.*

Then there exists an extension $\mathbf{p} \in Q_0(G)$ of $\mathbf{q}^0$ such that

$$\mathbf{p} \,.\, \nu|_\Gamma = \mathbf{q}^0 \,.\, \nu|_\Gamma , \tag{2.8}$$

$$\mathbf{p}(\mathbf{z}) \,.\, \nu(\mathbf{x}) \geqq g_0 \tag{2.9}$$

holds for almost all $\mathbf{x} \in \Gamma$ and $\mathbf{z} \in G$, $|\mathbf{z} - k\mathbf{x}| \leqq k\varkappa_0$, $k \in \langle 1, k_0 \rangle$.

Proof. Let us introduce neighbourhoods G_m of the vertices of Γ such that the total "strip" G is divided by segments parallel with the sides of Γ into subdomains G_m and the trapezoidal domains F_m, $m = 1, 2, \ldots, M$ (see Fig. 1).

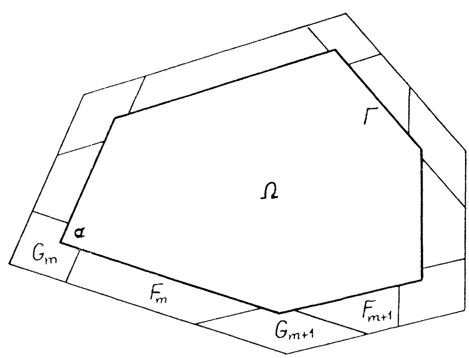

Fig. 1.

Consider an arbitrary subdomain G_m. Introduce a skew coordinate system by means of the linear mapping

$$\mathbf{y} = \mathscr{F}(\mathbf{x}) \equiv \begin{cases} y_1 = x_1 \sin \alpha - x_2 \cos \alpha , \\ y_2 = x_2 . \end{cases} \tag{2.10}$$

The system $\{y_1, y_2\}$ corresponds with a new basis $\{\mathbf{e}^1 \sin^{-1} \alpha, \mathbf{e}^2 \sin^{-1} \alpha\}$ (see Fig. 2), where $\mathbf{e}^1$, $\mathbf{e}^2$ are unit tangential vectors.

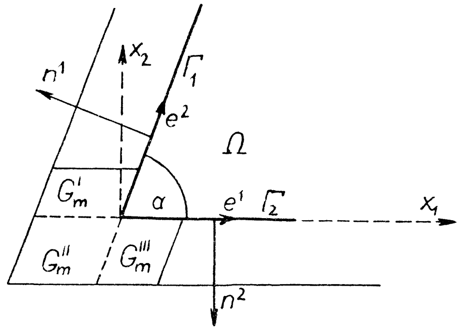

Fig. 2.

Then the segment Γ_2 is mapped onto the positive y_1-axis, the segment Γ_1 onto the positive y_2-axis and the domain Ω into the first quadrant $\{y_1 > 0, y_2 > 0\}$.

For any vector $\boldsymbol{p}$ we may write

$$\boldsymbol{p} = \sum_{j=1}^{2} p^{(j)} \mathbf{e}^j \sin^{-1} \alpha . \tag{2.11}$$

Let $\boldsymbol{n}^j$ be the unit outward normal vectors. Since $\mathbf{e}^j . \boldsymbol{n}^j = -\sin \alpha$ for $j = 1, 2$, we have

$$\boldsymbol{p}^{(j)} = -\boldsymbol{p} . \boldsymbol{n}^j , \quad j = 1, 2 .\ ^1)$$

[1]) The $p^{(j)}$ are components of the contravariant vector.

According to the definition, $\boldsymbol{p} \in Q_0(G_m)$ if

$$\int_{G_m} \boldsymbol{p} . \operatorname{grad} v \, \mathrm{d}\mathbf{x} = 0 \quad \forall v \in C_0^\infty(G_m) .$$

Using the mapping $\mathscr{F}^{-1}$ inverse to $\mathscr{F}$, we obtain

$$\int_{G_m} \boldsymbol{p} . \operatorname{grad} v \, \mathrm{d}\mathbf{x} = \sin^{-1} \alpha \int_{\mathscr{F}(G_m)} \sum_{j=1}^{2} p^{(j)} \frac{\partial \tilde{v}}{\partial y_j} \, \mathrm{d}\mathbf{y} , \tag{2.12}$$

where $\tilde{v}(\mathbf{y}) = v(\mathscr{F}^{-1}(\mathbf{y}))$, $\tilde{v} \in C_0^\infty(\mathscr{F}(G_m))$.

Consequently, if the integral over $\mathscr{F}(G_m)$ in (2.12) vanishes for all $\tilde{v} \in C_0^\infty \mathscr{F}((G_m))$, then the corresponding vector

$$\boldsymbol{p} = (p_1, p_2) \in Q_0(G_m) .$$

Recall that $\boldsymbol{q}^0 = \lambda^0 - \lambda^f = \operatorname{grad} u - \lambda^f \in [H^1(\Omega)]^2$. Hence the traces $\gamma q_i^0 \in \in H^{1/2}(\Gamma)$, $i = 1, 2$. Then we have

$$\gamma q^{0(j)} = -\gamma q^0 . n^j \in H^{1/2}(\Gamma_j), \quad j = 1, 2$$

and we may set:

$$\text{(2.13)} \qquad \left.\begin{aligned} p^{(1)}(y_1, y_2) &= \gamma q^{0(1)}(y_2) \\ p^{(2)}(y_1, y_2) &= -g_0 \end{aligned}\right\} \text{ in } \mathscr{F}(G'_m),$$

(i.e., for $y_1 < 0$, $y_2 > 0$),

$$\text{(2.14)} \qquad \left.\begin{aligned} p^{(1)} &= -g_0 \\ p^{(2)} &= \gamma q^{0(2)}(y_1) \end{aligned}\right\} \text{ in } \mathscr{F}(G'''_m)$$

(i.e. for $y_1 > 0$, $y_2 < 0$) and

$$\text{(2.15)} \qquad p^{(1)} = p^{(2)} = -g_0 \quad \text{in} \quad \mathscr{F}(G''_m) \quad (\text{for } y_1 < 0,\ y_2 < 0).$$

Obviously, $(p^{(1)}, p^{(2)}) \in Q_0(\mathscr{F}(G_m))$; (2.8) and (2.9) can be verified for $\boldsymbol{z} \in G_m$, since it holds

$$\text{(2.16)} \qquad -\gamma q^{0(j)} \geqq g_0 \quad \text{on} \quad \Gamma_j, \quad j = 1, 2.$$

In fact, to prove (2.16), we write (2.2') for any $w \in H_+^{1/2}(\Gamma)$, supp $w \subset \Gamma_j$:

$$\langle \boldsymbol{q}^0 . v, w\rangle = \int_\Omega \boldsymbol{q}^0 . \operatorname{grad} v \,\mathrm{d}\boldsymbol{x} = -\int_\Omega v \operatorname{div} \boldsymbol{q}^0 \,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_j} w\boldsymbol{q}^0 . v \,\mathrm{d}s \geqq g_0 \int_{\Gamma_j} w \,\mathrm{d}s.$$

Inserting $\boldsymbol{q}_0 . v = -\gamma q^{0(j)}$ and $\operatorname{div} \boldsymbol{q}^0 = 0$ in Ω, we obtain

$$\int_{\Gamma_j} (-\gamma q^{0(j)} - g_0)\, w \,\mathrm{d}s \geqq 0$$

which implies (2.16).

Next let us consider the trapezoidal domain F_m, $m = 1, \ldots, M$. If F_m is a parallelogram, we reduce it to a straight segment $\bar{G}_m \cap \bar{G}_{m+1}$ by extending G_m and G_{m+1}. We set in (2.14), (2.15) and (2.13):

$$\text{(2.17)} \qquad \begin{aligned} \bar{p}^{(1)} &= 0 \quad \text{in} \quad \mathscr{F}(G''_m \cup G'''_m) \\ \bar{p}^{(2)} &= 0 \quad \text{in} \quad \mathscr{F}(G'_{m+1} \cup G''_{m+1}), \end{aligned}$$

($\bar{p}^{(j)}$ are components of the vector $\bar{\boldsymbol{p}}$ in the local system of G_{m+1}). Thus we obtain the continuity of fluxes $\boldsymbol{p} . v$ on $\bar{G}_m \cap \bar{G}_{m+1}$.

Let us consider a general shape of the trapezoid F_m. Introduce a new coordinate system (y_1, y_2) by means of the mapping

$$\text{(2.18)} \qquad \boldsymbol{x} = T\boldsymbol{y} \equiv \begin{cases} x_1 = y_1 y_2, \\ x_2 = y_2, \end{cases}$$

where the origin of the local Cartesian coordinates (x_1, x_2) is situated at the intersection of the straight lines AB and CD (see Fig. 3). Then the trapezoid

$$F_m = \{\mathbf{x} \in \mathbb{R}^2 |\ a < x_1/x_2 < b,\ 0 < c < x_2 < d\}$$

is the image $T\mathscr{R}_m$ of a rectangle

$$\mathscr{R}_m = \{\mathbf{y} \in \mathbb{R}^2 |\ a < y_1 < b,\ c < y_2 < d\}\ .$$

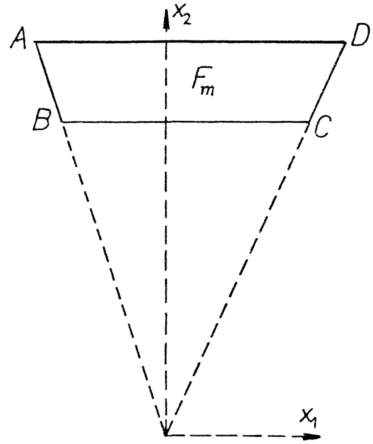

Fig. 3.

Then we have

$$\int_{F_m} \boldsymbol{p} \cdot \operatorname{grad} v \,\mathrm{d}\mathbf{x} = \int_{\mathscr{R}_m} \sum_{i=1}^{2} \tilde{p}^i \frac{\partial \tilde{v}}{\partial y_i}\, y_2 \,\mathrm{d}y = -\int_{\mathscr{R}_m} \tilde{v}\left(y_2 \frac{\partial \tilde{p}^1}{\partial y_2} + y_2 \frac{\partial \tilde{p}^2}{\partial y_2} + \tilde{p}^2\right) \mathrm{d}\mathbf{y}$$

where $v(T(\mathbf{y})) \equiv \tilde{v}(\mathbf{y}) \in C_0^\infty(\mathscr{R}_m)$, provided $v \in C_0^\infty(F_m)$, and

$$\tilde{p}^i = \sum_{k=1}^{2} p_k \frac{\partial y_i}{\partial x_k}\,, \quad p_i = \sum_{k=1}^{2} \tilde{p}^k \frac{\partial x_i}{\partial y_k}\,.$$

It is obvious that

$$(2.19) \qquad y_2 \operatorname{div} \tilde{\boldsymbol{p}} + p^2 = 0 \quad \text{in} \quad \mathscr{R}_m \Rightarrow \boldsymbol{p} \in Q_0(F_m)\,.$$

Moreover, if

$$(2.20) \qquad -p_2 = \boldsymbol{p} \cdot \nu|_{AD} = -\tilde{p}^2 \geqq g_0 \quad \text{in} \quad \mathscr{R}_m$$

and if $\boldsymbol{p}$ is defined by means of (2.13)–(2.15) in G_m, G_{m+1}, then (2.9) holds for $\mathbf{z} \in G_m \cup F_m \cup G_{m+1}$.

Let us define

$$(2.21)\qquad \tilde{p}^2 = \gamma\tilde{q}^{02}(y_1) + \frac{\mathscr{A}}{b-a}\left(\frac{d}{y_2} - 1\right)$$

$$(2.22)\qquad \tilde{p}^1 = -g_0\frac{\sqrt{(1+a^2)}}{y_2} + \int_a^{y_1}\left[-\frac{\partial\tilde{p}^2}{\partial y_2} - \frac{1}{y_2}\,\tilde{p}^2(t, y_2)\right]\mathrm{d}t\,,$$

$$\gamma\tilde{q}^{02}(y_1) = \gamma q_2^0(x_1)\,,\quad x_1 = y_1 d\,,$$

where

$$\mathscr{A} = g_0(\sqrt{(1+a^2)} + \sqrt{(1+b^2)}) + \int_a^b \gamma\tilde{q}^{02}(y_1)\,\mathrm{d}y_1 = \text{const.}$$

Then the condition (2.19), i.e.

$$y_2\left(\frac{\partial\tilde{p}^1}{\partial y_1} + \frac{\partial\tilde{p}^2}{\partial y_2}\right) + \tilde{p}^2 = 0 \quad \text{in} \quad \mathscr{R}_m\,,$$

can easily be verified.

Moreover, since $\gamma\tilde{q}^{02} \leqq -g_0$, we have

$$\mathscr{A} \leqq g_0(\sqrt{(1+a^2)} + \sqrt{(1+b^2)} - (b-a)) < 0\,.$$

Therefore

$$(2.23)\qquad \tilde{p}^2 \leqq \gamma\, q_2^0(y_1) \leqq -g_0\,,\quad \mathbf{y} \in \mathscr{R}_m$$

and (2.20) is satisfied.

Finally, let us calculate the values of the boundary flux $\boldsymbol{p}\,.\,\nu$ on AB and CD, respectively.

For $y_1 = a$ we have

$$(2.24)\quad \tilde{p}^{(1)} = -\frac{g_0}{y_2}\sqrt{(1+a^2)} \Rightarrow -\boldsymbol{p}\,.\,\boldsymbol{n}^1 = p^{(1)} = \tilde{p}^{(1)}\frac{y_2}{\sqrt{(1+a^2)}} = -g_0\,,$$

where $\boldsymbol{n}^1$ is the unit normal to AB (outward with respect to F_m).

For $y_1 = b$ we obtain

$$(2.25)\quad p^{(1)} = \frac{g_0}{y_2}\sqrt{(1+b^2)} \Rightarrow -\boldsymbol{p}\,.\,\bar{\boldsymbol{n}}^2 = \bar{p}^{(2)} = -\tilde{p}^{(1)}\frac{y_2}{\sqrt{(1+b^2)}} = -g_0\,,$$

where $\bar{\boldsymbol{n}}^2$ is the unit normal to CD (outward to F_m) and $\bar{p}^2$ the component of $\bar{\boldsymbol{p}}$ in the local system of G_{m+1}.

Comparing (2.14) with (2.24) and (2.13) (for G_{m+1}) with (2.25), respectively, we conclude that the flux $\boldsymbol{p}\,.\,\nu$ is continuous on the lines AB and CD.

In this way the vector field $\boldsymbol{p}$ can be constructed, satisfying all conditions of lemma 2.1. Q.E.D.

Next let us define the following extension of $\boldsymbol{q}^0$:

(2.26)
$$E\boldsymbol{q}^0 = \begin{cases} \boldsymbol{q}^0 & \text{in } \Omega \\ \boldsymbol{p} & \text{in } G . \end{cases}$$

Obviously, $E\boldsymbol{q}^0 \in Q_0(\Omega^*)$.

Let $k = 1 + \varepsilon$, $0 \leqq \varepsilon < k_0 - 1$ and define

(2.27)
$$\boldsymbol{q}^\varepsilon(\mathbf{y}) = E\boldsymbol{q}^0(k\mathbf{y})$$

for $\mathbf{y} \in k^{-1}\Omega^*$.

Regularizing $\boldsymbol{q}^\varepsilon$, we obtain

(2.28)
$$\mathbf{q}(\mathbf{x}) = R_\varkappa\, \boldsymbol{q}^\varepsilon(\mathbf{x}) = \int_{|\mathbf{x}-\mathbf{y}|<\varkappa} \omega_\varkappa(\mathbf{x} - \mathbf{y})\, \boldsymbol{q}^\varepsilon(\mathbf{y})\, \mathrm{d}\mathbf{y} .$$

Lemma 2.2 *Let the assumptions of Theorem* 2.2 *hold. Let* $\mathbf{q}$ *be defined by* (2.26) *to* (2.28) *and* $\boldsymbol{p}$ *be the extension from Lemma* 2.1.

Then $\mathbf{q} \in \mathscr{U}_0 \cap C^\infty[(\bar{\Omega})]^2$ *and* (2.7) *holds for* ε *and* $\varkappa = \varkappa(\varepsilon)$ *sufficiently small.*

Proof. 1° Let

$$\Omega_d = \{\mathbf{x} \mid \operatorname{dist}(\mathbf{x}, \bar{\Omega}) < d\} , \quad \bar{\Omega}_d \subset \Omega^* .$$

Then $\boldsymbol{q}^\varepsilon \in Q_0(\Omega_d)$ for ε sufficiently small. In fact, let $v \in H_0^1(\Omega_d)$ and define $\tilde{v}(\mathbf{y}) = v(\mathbf{y}/k)$. Since $k\bar{\Omega}_d \subset \Omega^*$ for sufficiently small ε and $\tilde{v} \in H_0^1(k\Omega_d)$, we can extend $\tilde{v}$ by zero to obtain $P\tilde{v} \in H_0^1(\Omega^*)$. Then

$$\int_{\Omega_d} \boldsymbol{q}^\varepsilon \,.\, \operatorname{grad} v \,\mathrm{d}\mathbf{x} = \int_{\Omega_d} E\boldsymbol{q}^0(k\mathbf{x}) \,.\, \operatorname{grad} v(\mathbf{x})\, \mathrm{d}\mathbf{x} =$$
$$= \frac{1}{k} \int_{k\Omega_d} Eq_i^0(\mathbf{y}) \frac{\partial \tilde{v}}{\partial y_i}\, \mathrm{d}\mathbf{y} = \frac{1}{k} \int_{\Omega^*} E\boldsymbol{q}^0 \,.\, \operatorname{grad} P\tilde{v}\, \mathrm{d}\mathbf{y} = 0$$

follows, using $E\boldsymbol{q}^0 \in Q_0(\Omega^*)$. Hence $\boldsymbol{q}^\varepsilon \in Q_0(\Omega_d)$.

Let us calculate

$$\frac{\partial \mathbf{q}_i}{\partial x_i}(\mathbf{x}) = - \int_{|\mathbf{x}-\mathbf{y}|<\varkappa} \frac{\partial \omega_\varkappa(\mathbf{x} - \mathbf{y})}{\partial y_i}\, q_i^\varepsilon(\mathbf{y})\, \mathrm{d}\mathbf{y} , \quad \mathbf{x} \in \Omega .$$

If $\varkappa < d$, $\omega_\varkappa(\mathbf{x} - \cdot) \in C_0^\infty(\Omega_d)$ and $\operatorname{div} \mathbf{q}(\mathbf{x}) = 0$.

2° Since it holds

$$g_0 = \int_{|\mathbf{x}-\mathbf{y}|<\varkappa} \omega_\varkappa(\mathbf{x} - \mathbf{y})\, g_0\, \mathrm{d}y \quad \forall \mathbf{x} \in \Gamma ,$$

we may write

$$v_i(\mathbf{x})\, q_i(\mathbf{x}) - g_0 = \int_{|\mathbf{x}-\mathbf{y}|<\varkappa} \omega_\varkappa(\mathbf{x} - \mathbf{y}) \left[v_i(\mathbf{x})\, q_i^\varepsilon(\mathbf{y}) - g_0\right] \mathrm{d}\mathbf{y} .$$

It is obvious that

$$\Gamma \subset k^{-1}G \quad \forall k \equiv 1 + \varepsilon > 1 .$$

Consequently, if $\varkappa < \operatorname{dist}(\Gamma, k^{-1}\Gamma)$ and $\varkappa < \varkappa_0$, then

$$|\mathbf{y} - \mathbf{x}| < \varkappa \Rightarrow k\mathbf{y} \in G , \quad |k\mathbf{y} - k\mathbf{x}| < k\varkappa < k\varkappa_0 ,$$

and using Lemma 2.1, (2.9) yields that

$$\nu_i(\mathbf{x})\, q_i^\varepsilon(\mathbf{y}) = \nu_i(\mathbf{x})\, p_i(k\mathbf{y}) \geqq g_0 .$$

Since $\omega_\varkappa \geqq 0$, we obtain $\nu \,.\, \mathbf{q}(\mathbf{x}) - g_0 \geqq 0\ \forall \mathbf{x} \in \Gamma$, i.e.

$$\mathbf{q} \in \mathscr{U}_0 \cap [C^\infty(\bar{\Omega})]^2 .$$

3° It remains to prove (2.7). There exists a sequence $\boldsymbol{q}^n \in [C_0^\infty(\Omega)]^2$ such that $\boldsymbol{q}^n \to \mathbf{q}^0$ in $[L_2(\Omega)]^2$. Let $E_0 q_i^n$ be an extension of q_i^n, $(i = 1, 2)$ by zero function outside Ω, $(q^n)_i^\varepsilon(\mathbf{y}) = E_0 q_i^n(k\mathbf{y})$. It holds

$$\|q_i^n - (q^n)_i^\varepsilon\|_{0,\Omega} \leqq \varepsilon\, C_1(q_i^n) , \tag{2.29}$$

$$\|q_i^\varepsilon - (q^n)_i^\varepsilon\|_{0,\Omega} \leqq \|q_i^n - q_i^0\|_{0,\Omega} + \|Eq_i^0\|_{0,k\Omega - \Omega} . \tag{2.30}$$

For the proofs – see Lemma 4.1 and 4.5 in [10], respectively.

Using (2.29), (2.30) and $E\boldsymbol{q}^0 = \boldsymbol{p}$ from Lemma 2.1, we obtain

$$\begin{aligned} \|\boldsymbol{q}^\varepsilon - \mathbf{q}^0\| &\leqq \|\boldsymbol{q}^\varepsilon - (\boldsymbol{q}^n)^\varepsilon\| + \|(\boldsymbol{q}^n)^\varepsilon - \boldsymbol{q}^n\| + \|\boldsymbol{q}^n - \mathbf{q}^0\| \leqq \\ &\leqq 3\|\boldsymbol{q}^n - \mathbf{q}^0\| + \varepsilon\, C_2(\boldsymbol{q}^n) + \|\boldsymbol{p}\|_{k\Omega - \Omega} . \end{aligned} \tag{2.31}$$

For a given $\eta > 0$, we can find $\boldsymbol{q}^n$ and ε (depending on $\boldsymbol{q}^n$) such that each of the three terms in the right-hand side of (2.31) is less than $\eta/4$.

Finally, we choose $\varkappa$ (depending on $\boldsymbol{q}^\varepsilon$) sufficiently small and such that

$$\|R_\varkappa \boldsymbol{q}^\varepsilon - \mathbf{q}^\varepsilon\| < \eta/4 .$$

Using (2.28) and (2.31), we obtain

$$\|\mathbf{q} - \mathbf{q}^0\| \leqq \|R_\varkappa \boldsymbol{q}^\varepsilon - \mathbf{q}^\varepsilon\| + \|\mathbf{q}^\varepsilon - \mathbf{q}^0\| < \eta \qquad \text{Q.E.D.}$$

The Theorem 2.2 is a consequence of Lemma 2.1 and 2.2.

Theorem 2.3 *Let Ω satisfy the assumptions of Theorem 2.2. Assume that $\lambda^f \in [H^1(\Omega)]^2$ and $(f, 1)_0 \leqq 0$.*

Then for any $\alpha - \beta$-regular family of triangulations $\{\mathscr{T}_h\}$, the dual finite element approximations converge in $[L_2(\Omega)]^2$, i.e.

$$\|\lambda^h - \lambda^0\| \to 0 \quad \text{for} \quad h \to 0 .$$

Proof. It suffices to verify the assumption (1.3) of Theorem 1.1. Using Theorem 2.2, we obtain $\mathfrak{q} \in \mathscr{U}_0 \cap [C^\infty(\bar{\Omega})]^2$ satisfying (2.7) for any $\eta > 0$.

In the second step, we apply Lemma 4.2 of [3] to $\mathfrak{q}$ and Lemma 5.3 of [1] to construct $\mathbf{W}^h \in \mathscr{U}_0^h$ such that (cf. the proof of Theorem 4.1 in [3])

$$\|\mathfrak{q} - \mathbf{W}^h\| \leqq C(\mathfrak{q})\, h^{3/2} .$$

Finally, we may write

$$\|\boldsymbol{q}^0 - \mathbf{W}^h\| \leqq \|\boldsymbol{q}^0 - \mathfrak{q}\| + \|\mathfrak{q} - \mathbf{W}^h\| \leqq \eta + C(\mathfrak{q})\, h^{3/2} .$$

Thus (1.3) is satisfied by $v_h = \mathbf{W}^h$.

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Souhrn

KONVERGENCE DUÁLNÍCH APROXIMACÍ METODOU KONEČNÝCH PRVKŮ PRO JEDNOSTRANNÉ OKRAJOVÉ ÚLOHY

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V článcích [1], [2], [3] byly odvozeny apriorní odhady chyb duálních aproximací za předpokladu, že řešení je dostatečně hladké. V této práci se dokazuje konvergence metody bez zvláštního předpokladu regularity řešení pro jistou podtřídu konvexních polygonálních oblastí.

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