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CONTACT PROBLEM OF TWO ELASTIC BODIES – Part I

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INTRODUCTION

In tunnel construction the following problem occurs: Find displacements and stresses on the tunnel wall and the surrounding earth, particularly on the boundary between the tunnel wall and the earth. It includes the problem how to find the part of the boundary where the tunnel wall and the earth are in contact after deformation.

We formulate the simplest model in two dimensions. Our aim is a complete numerical study of the model. For this reason we have devided the paper into three parts. The first part of our paper has the following chapters:

1. Formulation of the problem
2. Finite dimensional elasticity model
3. Numerical solution of the finite dimensional model

The convergence analysis is left to the second part of our paper while the third part includes the implementation of the algorithm and numerical results.

The contact problem between an elastic and a rigid body was studied by Lions and Duvaut [1] and by Fremond [2] from the numerical point of view. Fremond's idea of an artificial bolt was generalized by Janovský [4] to the case that both bodies are elastic.

1. FORMULATION OF THE PROBLEM

We consider a displacement problem concerning a tunnel beneath the earth where the wall of the tunnel (Ω'') is subjected to forces due to the surrounding earth (Ω'). It is assumed that slipping may occur at the tunnel-earth interface, so that in fact both bodies may lose contact at certain points on the common boundary Γ. The problem (before the deformation occurs) is illustrated in two-dimensional situation (section view) in Figure 1. Symmetry about the vertical axis (x_2) is assumed so that only one half of the region is considered.

The following assumption concerning the physical situation are made:

(1) there is a plane deformation,
(2) both materials are isotropic,
(3) the displacements are small,
(4) the problem is symmetric about the x_2-axis,
(5) friction forces on Γ are neglected,
(6) there are no displacements along Γ_2,
(7) there is a loading on Γ_1,
(8) there is no loading on Γ_5,
(9) the boundaries $\partial\Omega'$ and $\partial\Omega''$ of Ω' and Ω'' are respectively Lipschitz and piecewise infinitely smooth,
(10) both Ω' and Ω'' are simply connected domains of the plane.

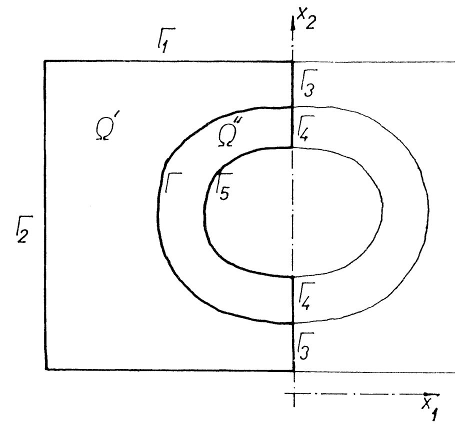

Fig. 1.

Displacements are described by a vector function $u = [u_1, u_2]$ of the variable $x = (x_1, x_2)$; $\Omega \equiv \Omega' \cup \Omega''$, where $u_i = u_i(x)$ is the displacement of the point x in the direction of the x_i-axis $(i = 1, 2)$. The restriction of any function to Ω' or Ω'' is denoted respectively by one prime or by two primes; e.g. $u/\Omega' = u'$ and $u/\Omega'' = u''$. The vector functions $F = [F_1, F_2]$ and $P = [P_1, P_2]$ describe volume and surface forces on Ω and Γ_1, respectively; let us suppose that $F_i \in L_2(\Omega)$ and $P_i \in L_2(\Gamma_1)$ for $i = 1, 2$.

Definition 1.1. *We denote*

$$V \equiv \{w = [w_1, w_2] \text{ on } \Omega;\ w_i' \in W^{1,2}(\Omega'),\ w_i'' \in W^{1,2}(\Omega'') \text{ for } i = 1, 2;$$
$$w_i' = 0 \text{ a.e. on } \Gamma_2 \text{ for } i = 1, 2;\ w_1' = 0 \text{ a.e. on } \Gamma_3;$$
$$w_1'' = 0 \text{ a.e. on } \Gamma_4\}$$

and

$$\|w\| = \left(\sum_{i=1}^{2} \|w_i'\|^2_{W^{1,2}(\Omega')} + \sum_{i=1}^{2} \|w_i''\|^2_{W^{1,2}(\Omega'')}\right)^{1/2}$$

the norm on V.

Remark 1.2. The extensions of w_i' and w_i'' to $\partial\Omega'$ and $\partial\Omega''$ respectively are defined by means of the traces of $w_i' \in W^{1,2}(\Omega')$ and $w_i'' \in W^{1,2}(\Omega'')$, $i = 1, 2$.

Remark 1.3. The space V is the Banach space of all admissible displacements which satisfy the essential boundary conditions on Γ_2, Γ_3, Γ_4 but not on Γ.

Let $v = (v_1, v_2)$ be the outward normal vector to Γ with respect to Ω'. The functions $u_v' = v_1 u_1' + v_2 u_2'$ and $u_v'' = v_1 u_1'' + v_2 u_2''$ (on Γ) are displacements of the bodies Ω' and Ω'' in the v-direction along the contact boundary Γ. With respect to assumption (3) concerning "small" displacements, it may be satisfactory to formulate the essential boundary condition along the contact boundary as follows:

$$[u]_v \equiv u_v' - u_v'' \leqq 0 \quad \text{a.e. on } \Gamma\,. \tag{1.1}$$

Definition 1.2. *We denote*

$$K \equiv \{w \in V;\ [w]_v \leqq 0 \ \textit{a.e. on } \Gamma\}\,.$$

Remark 1.3. The set K is the cone of admissible displacements with respect to the essential contact condition (1.1).

Definition 1.3. *We denote*

$$J(w) \equiv \tfrac{1}{2}A(w, w) - \sum_{i=1}^{2} \int_{\Omega} F_i w_i \,\mathrm{d}x_1\,\mathrm{d}x_2 - \sum_{i=1}^{2} \int_{\Gamma_1} P_i w_i \,\mathrm{d}\sigma$$

for all $w \in V$, where

$$A(w, v) = \sum_{i,j=1}^{2} \int_{\Omega' \cup \Omega''} \tau_{ij}(w)\, e_{ij}(v)\,\mathrm{d}x_1\,\mathrm{d}x_2 \quad \forall v, w \in V$$

and

$$e_{ij} = e_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right),$$

$$\tau_{ij} = \tau_{ij}(w) = \lambda\theta\delta_{ij} + 2\eta e_{ij}\,,$$

$$\theta = \theta(w) = \sum_{j=1}^{2} e_{jj}(w)\,,$$

δ_{ij} is the Kronecker delta, λ, η are positive Lamme's constants.

Remark 1.4. The functional $J = J(w)$ represents the amount of potential energy of both the bodies Ω' and Ω'' after a deformation $w \in V$.

The problem of finding the displacement $u = [u_1, u_2]$ can be formulated mathematically as follows.

Problem. Find $u \in K$ such that

$$(1.2) \qquad J(u) \leqq J(w) \quad \forall w \in K .$$

Remark 1.1. The problem (1.2) is equivalent to the following one: Find $u \in K$ such that

$$(1.3) \qquad D\, J(u, w - u) \geqq 0 \quad \forall w \in K ,$$

where $D\, J(u, \psi)$ is the Fréchet derivative in the direction ψ of the functional J at the point u; i.e.

$$(1.4) \qquad D\, J(u, \psi) = A(u, \psi) - \sum_{i=1}^{2} \int_{\Omega} F_i \psi_i \, dx_1 \, dx_2 - \sum_{i=1}^{2} \int_{\Gamma_1} P_i \psi_i \, d\sigma .$$

Lemma 1.1. (Inequality of Korn's type.)
Let

$$P \equiv \{w = [w_1, w_2] \text{ on } \Omega;\ w_1' = a_1' + b' x_2,\ w_2' = a_2' - b' x_1,\ w_1'' = a_1'' + b'' x_2 , \ w_2'' = a_2'' - b'' x_1 \text{ where } a_1', a_2', a_1'', a_2'', b', b'' \text{ are constants}\} .$$

If

(a) *W is a closed linear subspace of V,*
(b) *$\psi = [\psi_1, \psi_2] \in W \cap P$ iff $\psi_i \equiv 0$ on Ω for $i = 1, 2$*

then there exists a constant C such that

$$A(w, w) \geqq C \|w\|^2$$

for all $w \in W$.

Remark 1.6. We do not prove the above lemma. For detailed information see [3]. Korn's inequality states that the energy functional J is coercive of any Banach space of admissible displacements W except that one for which the displacements of both the bodies Ω' and Ω'' are rigid.

We define a certain splitting of V into two parts such that one of them has the character of the space W, see Lemma 1.1.

Definition 1.4. *Let Γ_0 be a fixed part of Γ such that*

$$q_0 \equiv \left(\int_{\Gamma_0} v_2 \, d\sigma \right)^{-1} , \quad q_0^{-1} \neq 0 .$$

(It is evident that such Γ_0 exists.) We define two operators $T_i : V \to V$ $(i = 1, 2)$ as follows:

If $w \in V$ then

(a) $T_2 w \in V$, $(T_2 w)'_i \equiv 0$ *on* Ω' *for* $i = 1, 2$, $(T_2 w)''_i \equiv 0$ *on* Ω'',

$$(T_2 w)''_2 \equiv q_0 \int_{\Gamma_0} q_0 [w]_\nu \, d\sigma \text{ on } \Omega'',$$

(b) $T_1 w = w + T_2 w$.

Remark 1.7. It follows from Definition 1.4 that

$$T_2(T_2 w) = -T_2 w,$$
$$T_1(T_1 w) = T_1 w$$

for each $w \in V$.

Lemma 1.2. *There exists a constant C such that*

$$A(T_1 w, T_1 w) \geqq C \|T_1 w\|^2 \quad \forall w \in V.$$

Proof. It is sufficient to show that

(i) the range of the operator T_1 is closed;

(ii) if $T_1 w \in P \cap V$ then $T_1 w \equiv 0$;

the required results then follow by applying Lemma 1.1. Assertion (i) follows immediately from Remark 1.7 and from the continuity of T_1. To prove (ii), observe that definitions of V and P imply

$$(T_1 w)'_i \equiv 0 \text{ on } \Omega' \quad (i = 1, 2),$$
$$(T_1 w)''_1 \equiv 0 \text{ on } \Omega'',$$
$$(T_1 w)''_2 \equiv \alpha \text{ on } \Omega'',$$

where α is a constant.

Hence

$$w'_i \equiv 0 \text{ on } \Omega' \quad (i = 1, 2),$$
$$w''_1 \equiv 0 \text{ on } \Omega''$$

and

$$\alpha = w''_2 + q_0 \int_{\Gamma_0} [w]_\nu \, d\sigma \text{ on } \Omega''. \tag{1.5}$$

Thus w''_2 is a constant, say α_1, on Ω''. Substitution of α_1 into (1.5) together with integration of the boundary integral gives

$$\alpha = \alpha_1 - q_0 \alpha_1 \int_{\Gamma_0} \nu_2 \, d\sigma$$

and so $\alpha = 0$.

Theorem 1.1. *If*

$$\int_{\Omega''} F_2'' \, dx_1 \, dx_2 \neq 0 ,$$

then there exists at most one solution of Problem (1.2).

Proof. Suppose that this is not so and that u and $u^{(1)}$ are two such solutions. Then (1.3) implies

$$0 = A(u - u^{(1)}, u - u^{(1)}) .$$

Since

$$A(u - u^{(1)}, u - u^{(1)}) = A(T_1(u - u^{(1)}), T_1(u - u^{(1)})),$$

Lemma 1.2 gives

$$T_1(u - u^{(1)}) \equiv 0 .$$

By definition $J(u) = J(u^{(1)})$, and after some manipulations this gives that

$$(T_2 u)_2'' \int_{\Omega''} F_2'' \, dx_1 \, dx_2 = (T_2 u^{(1)})_2'' \int_{\Omega''} F_2'' \, dx_1 \, dx_2 .$$

From the hypothesis of the theorem, we conclude that

$$T_2(u - u^{(1)}) \equiv 0 .$$

Theorem 1.2. *If*

$$\int_{\Omega''} F_2'' \, dx_1 \, dx_2 \neq 0 ,$$

then there exists at least one solution of our Problem (1.2).

Proof. It is evident that the functional $J(\cdot)$ is weakly lower semi-continuous on V. According to the standard theory concerning minimalization of the functional J over a closed convex set K it is thus sufficient to prove that J is coercive on K, i.e. if a sequence $\{w^{(k)}\}_{k=1}^{\infty}$ of elements of K is given such that

$$\|w^{(k)}\| \to +\infty \quad \text{for} \quad k \to +\infty ,$$

then

$$J(w^{(k)}) \to +\infty .$$

Let us assume that

$$\int_{\Omega''} F_2'' \, dx_1 \, dx_2 > 0 .$$

We consider the "splitting" operators T_1 and T_2 (see Definition 1.4) where Γ_0 is chosen so that $q_0 < 0$; it is evident that such $\Gamma_0 \subset \Gamma$ exists. Using splitting $w^{(k)} = T_1 w^{(k)} - T_2 w^{(k)}$, we know that

$$\|w^{(k)}\| \to +\infty$$

only if either

$$\|T_1 w^{(k)}\| \to +\infty \tag{1.6}$$

or

$$\{\|T_1 w^{(k)}\|\} \text{ is bounded, } \|T_2 w^{(k)}\| \to +\infty\,. \tag{1.7}$$

In accordance with Lemma 1.2 there exists a constant C such that

$$C\|T_1 w^{(k)}\|^2 - \sum_{i=1}^{2} \int_{\Omega} F_i \,.\, (T_1 w^{(k)})_i \,\mathrm{d}x_1 \,\mathrm{d}x_2 - \sum_{i=1}^{2} \int_{\Gamma_1} P_i \,.\, (T_1 w^{(k)})_i \,\mathrm{d}\sigma + \int_{\Omega''} F_2'' \,.\, (T_2 w^{(k)})_2'' \,\mathrm{d}x_1 \,\mathrm{d}x_2 \leqq J(w^{(k)})\,.$$

But

$$(T_2 w^{(k)})_2'' = q_0 \int_{\Gamma_0} [w^{(k)}]_\nu \,\mathrm{d}\sigma \geqq 0\,,$$

due to the fact that $w^{(k)} \in K$.

Hence

$$\int_{\Omega''} F_2'' \,.\, (T_2 w^{(k)})_2'' \,\mathrm{d}x_1 \,\mathrm{d}x_2 \geqq 0$$

and

$$C\|T_1 w^{(k)}\|^2 - \sum_{i=1}^{2} \int_{\Omega} F_i \,.\, (T_1 w^{(k)})_i \,\mathrm{d}x_1 \,\mathrm{d}x_2 - \sum_{i=1}^{2} \int_{\Gamma_1} P_i \,.\, (T_1 w^{(k)})_i \,\mathrm{d}\sigma \leqq J(w^{(k)})\,. \tag{1.10}$$

If assumption (1.6) holds then (according to (1.10)) we have

$$J(w^{(k)}) \to +\infty\,.$$

If assumption (1.7) holds then the absolute value of

$$q_0 \int_{\Gamma_0} [w^{(k)}]_\nu \,\mathrm{d}\sigma$$

must converge to $+\infty$ (see Definition 1.4). In this case it follows from (1.9) that

$$\int_{\Omega''} F_2'' \,.\, (T_2 w^{(k)})_2'' \,\mathrm{d}x_1 \,\mathrm{d}x_2 \to +\infty$$

and so (due to (1.8))

$$J(w^{(k)}) \to +\infty\,.$$

2.1. Finite dimensional spaces of displacements and other auxiliary spaces

We define a partition of Ω into distinct "triangular" non-overlapping subdomains $\Omega_{i,p}$, $i = 1, \ldots, K(p)$, where the integer p is a parameter and $K(p)$ is the number of subdomains.

Definiton 2.1. *The system* $\{\Omega_{i,p}\}_{i=1}^{K(p)} \equiv \Omega^{(p)}$ *is a triangulation of* Ω *iff*

(i) $\bigcup_{i=1}^{K(p)} \bar{\Omega}_{i,p} = \bar{\Omega}$,

(ii) *either* $\Omega_{i,p} \subset \Omega'$ *or* $\Omega_{i,p} \subset \Omega''$,

(iii) $\partial\Omega_{i,p}$ *is the boundary of* $\Omega_{i,p}$. *In general, the element* $\bar{\Omega}_{i,p}$ *is a triangle. In order to match curved boundaries* Γ, Γ_5, *we admit curved sides (see Fig. 2) of the corresponding triangles;*

(iv) *triangles* $\bar{\Omega}_{i,p}$, $\bar{\Omega}_{j,p}$ *are either disjoint or have a vertex in common or have a common side.*

All vertices of triangles $\{\Omega_{i,p}\}_{i=1}^{K(p)}$ are *nodal points*. The partition $\Omega^{(p)}$ induces a partition on Γ as follows:

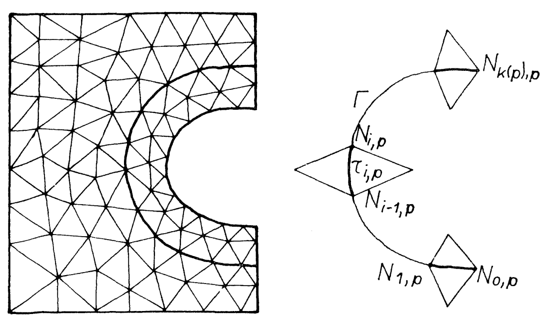

Fig. 2.

Definition 2.2. *Given an integer p, the intersection of the relevant curved triangles of the partition* $\Omega^{(p)}$ *with* Γ *defines a set of distinct nonempty arcs*

$$\{\tau_{i,p}\}_{i=1}^{k(p)} \equiv \tau^{(p)},$$

where $k(p)$ *is the number of subarcs on* Γ. *The elements of* $\tau^{(p)}$ *are ordered so that the nodes on* Γ *are*

$$\{N_{i,p}\}_{i=0}^{k(p)} \equiv N^{(p)} \quad (\textit{see Fig. 2}).$$

In order to approximate the space V and the cone K (see Definitions 1.1 and 1.2) we define a finite dimensional subspace $V^{(p)}$ and a cone $K^{(p)}$. For the purpose of Section 3 we introduce subsets $V_A^{(p)}$ and $K_A^{(p)}$ of $V^{(p)}$ and $K^{(p)}$ with a "frozen" displacement at the point $A \in \Gamma$.

Definiton 2.3. *Given an integer p, we define the space $V^{(p)} \equiv \{\psi; \psi$ is a linear function on each $\Omega_{i,p} \in \Omega^{(p)}$ and satisfies the essential boundary conditions at any nodal point $Q \in \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, $\psi \in C(\Omega)\}$ and the convex subset*

$$K^{(p)} = \{\psi; \psi \in V^{(p)} \text{ and at points } Q \in N^{(p)} \text{ it is } [\psi]_v \leqq 0\} .$$

For a given point $A \in N^{(p)}$ we define

$$V_A^{(p)} = \{\psi; \psi \in V^{(p)}, [\psi]_v = 0 \text{ at } A\}$$
$$K_A^{(p)} = \{\psi; \psi \in K^{(p)}, [\psi]_v = 0 \text{ at } A\} .$$

Some spaces involving the reactions on Γ are also needed.

Definition 2.4. *Let p be an integer and let $N^{(p)}$ be as in Definition 2.2, then*

$$\Lambda^{(p)} = \{\lambda; \lambda \text{ is a real function on } N^{(p)}, \lambda \geqq 0\} .$$

For a given point $A \in N^{(p)}$ we define

$$\Lambda_A^{(p)} = \{\lambda; \lambda \text{ is a real function on } N^{(p)}, \lambda \geqq 0 \text{ on } N^{(p)} \setminus \{A\}\} .$$

For the purpose of numerical integration on Γ, we define quadrature formulae

Definition 2.5. *Let p be an integer and let $N^{(p)}$, $\tau^{(p)}$ be as in Definition 2.2, then*

$$I^{(p)}(z) = \tfrac{1}{2} \sum_{i=1}^{k(p)} [z(N_{i,p}) + z(N_{i-1,p})] \text{ meas } \tau_{i,p}$$

for any real function z over $N^{(p)}$, where

$$\text{meas } \tau_{i,p} = \int_{\tau_{i,p}} d\sigma ,$$

the integral being taken in the Lebesgue sense. For a given $A \in N^{(p)}$ we define

$$I_A^{(p)}(z) = \tfrac{1}{2} \sum_{i=1}^{k(p)} [\tilde{z}(N_{i,p}) + \tilde{z}(N_{i-1,p})] \text{ meas } \tau_{i,p} ,$$

where

$$\tilde{z}(N) = 0 \text{ for } N = A \text{ and } \tilde{z}(N) = z(N) \text{ for } N \in N^{(p)} , \quad N \neq A .$$

The balance condition upon the reactive forces on Γ must also be satisfied, and to ensure this we define

Definition 2.6. *Let p be an integer, then*

$$\tilde{\Lambda}^{(p)} = \left\{\lambda;\ \lambda \in \Lambda^{(p)},\ \int_{\Omega''} F_2''\, dx_1\, dx_2 + I^{(p)}(\lambda \,.\, \nu_2) = 0\right\},$$

where $\nu_2 = \nu_2(x)$ is the second co-ordinate of the outside normal vector ν at the point $x \in \Gamma$.

For a given $A \in N^{(p)}$ we define

$$\tilde{\Lambda}_A^{(p)} = \left\{\lambda;\ \lambda \in \Lambda_A^{(p)},\ \int_{\Omega''} F_2''\, dx_1\, dx_2 + I^{(p)}(\lambda \,.\, \nu_2) = 0\right\}.$$

We introduce a certain subset of $N^{(p)}$ whose importance will be made clear in Section 3.

Definition 2.7. *For a given integer p we define $N_0^{(p)} = \{x;\ x \in N^{(p)}$ such that $\nu_2(x) \neq 0$, where $\nu_2(x)$ is defined in Definition 2.6$\}$.*

2.2. Discrete problems

From Definition 2.3 we define the discrete form of Problem (1.2) as follows:

Problem. For a given integer p find $u^{(p)} \in K^{(p)}$ such that

$$J(u^{(p)}) \leqq J(w) \quad \forall w \in K^{(p)}\,. \tag{2.1}$$

Assumption: In all theorems and lemmas of this paper we assume

$$\int_{\Omega''} F_2''\, dx_1\, dx_2 \neq 0\,. \tag{2.2}$$

This means that in our concrete technical problem the tunnel Ω'' is subjected to a volume force F'' such that the resultant of its vertical component F_2'' is nonzero (e.g. F_2'' is the specific weight of the tunnel which is supposed to be nonzero).

Theorem 2.1. *There exists a unique solution $u^{(p)}$ to Problem (2.1).*

Proof. First we introduce a discrete version of the operators T_1 and T_2, see Definition 1.4. Let A be a point of $N_0^{(p)}$. We define splitting operators $T_i : V^{(p)} \to V^{(p)}$ for $i = 1, 2$ as follows: If $w \in V^{(p)}$ then

(a) $T_2 w \in V^{(p)}$, $(T_2 w)_i' \equiv 0$ on Ω' for $i = 1, 2$, $(T_2 w)_1'' \equiv 0$ and $(T_2 w)_2'' \equiv (\nu_2(A))^{-1}\,[w(A)]_\nu$ on Ω'';

(b) $T_1 w = w + T_2 w$.

For this choice of T_i it can be easily checked that Remark 1.7 and Lemma 1.2 are valid if we replace V and q_0 and

$$\int_{\Gamma_0} [w]_\nu \, d\sigma$$

respectively by $V^{(p)}$ and $(\nu_2(A))^{-1}$ and $[w(A)]_\nu$.

The proof of uniqueness of $u^{(p)}$ is similar to that of Theorem 1.1. The existence of $u^{(p)}$ can be proved by adapting slightly the idea of the proof of Theorem 1.2: It is sufficient to verify that J is coercive on $K^{(p)}$. We assume (without loss of generality) that

$$\int_{\Omega''} F_2'' \, dx_1 \, dx_2 > 0$$

and choose $A \in N_0^{(p)}$ such that $\nu_2(A) < 0$. For this choice of A we consider the splitting operators T_1, T_2 defined above. If $\{w^{(k)}\}_{k=1}^{\infty}$ is a sequence of elements of $K^{(p)}$ such that $\|w^{(k)}\| \to \infty$, then (1.8) and (1.10) hold again, because $(T_2 w^{(k)})_2'' \geqq 0$. Moreover we can assume either $\|T_1 w^{(k)}\| \to \infty$ or $\|T_1 w^{(k)}\|$ is bounded and $(T_2 w^{(k)})_2'' = (\nu_2(A))^{-1} [w(A)]_\nu \to +\infty$. As an easy consequence of (1.8) and (1.10) we obtain that $J(w^{(k)}) \to +\infty$.

If one is interested not only in displacement $u^{(p)}$ but also in the surface forces along the boundary Γ, then the following reformulation of Problem (2.1) in terms of Lagrange multipliers can be used. Note that this is a minimax formulation.

Problem. For a given integer p find a pair $\{u^{(p)}, \lambda^{(p)}\}$ such that $u^{(p)} \in V^{(p)}$ and $\lambda^{(p)} \in \Lambda^{(p)}$ and

$$(2.3) \qquad J(u^{(p)}) + I^{(p)}(\lambda^{(p)}[u^{(p)}]_\nu) \leqq J(w) + I^{(p)}(\lambda^{(p)}[w]_\nu), \quad \forall w \in V^{(p)}$$

and

$$(2.4) \qquad I^{(p)}(\eta[u^{(p)}]_\nu) \leqq I^{(p)}(\lambda^{(p)}[u^{(p)}]_\nu), \quad \forall \eta \in \Lambda^{(p)}.$$

Remark 2.1. If the pair $\{u^{(p)}, \lambda^{(p)}\}$ solves Problem (2.3)–(2.4) then the first component $u^{(p)}$ solves Problem (2.1). The component $\lambda^{(p)}$ can be interpreted as a reactive force of the body Ω' at the points of the set $N^{(p)}$. The reactive force of the body Ω'' is equal to $-\lambda^{(p)}$, of course. From the mathematical point of view, the function $\lambda^{(p)}$ is the Lagrange multiplier.

Remark 2.2. Condition (2.3) is equivalent to

$$(2.5) \qquad D\, J(u^{(p)}; \varphi) + I^{(p)}(\lambda^{(p)}[\varphi]_\nu) = 0$$

$\forall \varphi \in V^{(p)}$; the bilinear form $D\, J(\cdot,\cdot)$ is derivative of J (see 1.4).

Remark 2.3. Substituting $\psi \in V^{(p)}$, $\psi_i' \equiv 0$ for $i = 1, 2$, $\psi_1'' \equiv 0$ and $\psi_2'' \equiv 1$ into (2.5), we easily verify that $\lambda^{(p)}$ must satisfy the balance condition, i.e. $\lambda^{(p)} \in \tilde{\Lambda}^{(p)}$, see Definition 2.6.

Lemma 2.1. *The pair* $\{u^{(p)}, \lambda^{(p)}\}$ *solves Problem* (2.3)–(2.4) *iff condition* (2.5) *is satisfied and*

$$[u^{(p)}] \leqq 0\,, \quad \lambda^{(p)} \geqq 0\,, \quad [u^{(p)}]_\nu\, \lambda^{(p)} = 0 \text{ holds on } N^{(p)}\,. \tag{2.6}$$

The proof is straightforward and therefore omitted.

Theorem 2.2. *The solution to Problem* (2.3)–(2.4), *if it exists, is unique.*

Proof. Let both $\{u^{(p)}, \lambda^{(p)}\}$ and $\{w^{(p)}, \eta^{(p)}\}$ be solutions of Problem (2.3)–(2.4). Then (Remark 2.1) functions $u^{(p)}$ and $w^{(p)}$ solve Problem (2.1), which has a unique solution (Theorem 2.1), i.e. $u^{(p)} \equiv w^{(p)}$. Hence, the condition (2.5) implies that

$$I^{(p)}((\lambda^{(p)} - \eta^{(p)})\,[\varphi]_\nu) = 0 \quad \forall \varphi \in V^{(p)}\,.$$

It is evident that the function $\varphi \in V^{(p)}$ can be chosen so that $[\varphi]_\nu = \lambda^{(p)} - \eta^{(p)}$ on $N^{(p)}$ and hence $I^{(p)}((\lambda^{(p)} - \eta^{(p)})^2) = 0$. From Definition 2.5 it follows that $\lambda^{(p)} = \eta^{(p)}$ on $N^{(p)}$.

We now prove the existence of the above solution using the well-known geometric interpretation of the Hahn-Banach theorem.

Theorem 2.3. *Let both* X *and* Y *be convex subsets of a Banach space* Z; *let the interior* $\mathring{X}$ *of* X *be nonempty and* $\mathring{X} \cap Y = \emptyset$. *Then there exists a bounded linear functional* $F(\cdot)$ *over* Z *and a constant* C *such that*

$$F \not\equiv 0$$

and

$$F(z_1) \geqq C \geqq F(z_2) \quad \forall z_1 \in X,\, z_2 \in Y\,.$$

Proof see [6], Theorem 3.1.3, page 51.

Definition 2.8. *Let* $u^{(p)}$ *be the solution to Problem* (2.1). *Let the symbol* $\Xi^{(p)}$ *denote the space of all real functions over the set* $N^{(p)}$. *Two subsets* $S^{(p)}$ *and* $T^{(p)}$ *of the Cartesian product* $\mathbb{R}_1 \times \Xi^{(p)}$ *are defined as follows: For* $(p, q) \in \mathbb{R}_1 \times \Xi^{(p)}$

(i) $(p, q) \in S^{(p)}$ *iff there exist* $w \in V^{(p)}$ *and* $s_0 \in \mathbb{R}_1$ *and* $s \in \Xi^{(p)}$ *such that*

$$s_0 \geqq 0\,, \quad s \leqq 0$$
$$p = J(w) - J(u^{(p)}) + s_0$$
$$q = [w]_\nu - s$$

(ii) $(p, q) \in T^{(p)}$ *iff there exist* $t_0 \in \mathbb{R}_1$ *and* $t \in \Xi^{(p)}$ *such that*

$$t_0 > 0\,, \quad t \leqq 0$$
$$p = -t_0$$
$$q = t\,.$$

Lemma 2.2. *It holds that*

(i) *both* $S^{(p)}$ *and* $T^{(p)}$ *are convex subsets of* $\mathbb{R}_1 \times \Xi^{(p)}$,

(ii) $S^{(p)} \cap T^{(p)} = \emptyset$,

(iii) *the set* $S^{(p)}$ *contains at least one interior point.*

Proof. The assertion (i) is trivial. Let us suppose the contrary of the assertion (ii), i.e. there exist real numbers s_0, t_0 and functions $s, t \in \Xi^{(p)}$ and $w_0 \in V^{(p)}$ such that

$$s_0 \geqq 0\,, \quad t_0 > 0$$
$$s \leqq 0\,, \quad t \leqq 0 \text{ on } N^{(p)}$$
$$J(w_0) - J(u^{(p)}) + s_0 = -t_0$$
$$[w]_\nu - s = t\,.$$

However, then $J(w_0) < J(u^{(p)})$ and $[w_0]_\nu \leqq 0$ on $N^{(p)}$, i.e. $u^{(p)}$ is not the solution to Problem (2.1) which is a contradiction. Finally we show that the point (p_0, q_0), where $p_0 = -J(u^{(p)}) + 1$ and $q_0 = 1$ is an interior point of the set $S^{(p)}$. Choosing $w \equiv 0$, $s \equiv -1$ and $s_0 \equiv 1$ we verify that $(p_0, q_0) \in S^{(p)}$ – see Definition 2.8. But for any $\{p, q\} \in \mathbb{R}_1 \times \Xi^{(p)}$, where $p_0 - 1 < p < p_0 + 1$, $q_0 - 1 < q < q_0 + 1$ (i.e. $\{p, q\}$ is an arbitrary element in the "ball" of radius 1 with the centre at $\{p_0, q_0\}$) we can find $s_0 > 0$, $s < 0$, $w \equiv 0$ such that

$$p = -J(u^{(p)}) + s_0 + J(w)\,,$$
$$q = -s + [w]_\nu$$

and hence $\{p, q\} \in S^{(p)}$.

Remark 2.4. A functional $F(\cdot,\cdot)$ is linear and bounded on $\mathbb{R}_1 \times \Xi^{(p)}$ iff there exists a pair $(\alpha_0; \alpha) \in \mathbb{R}_1 \times \Xi^{(p)}$ such that

$$F(p, q) = \alpha_0 p + I(\alpha q) \quad \forall (p, q) \in \mathbb{R}_1 \times \Xi^{(p)}\,.$$

Thus $F \equiv 0$ iff $\alpha_0 = 0$ and $\alpha \equiv 0$ on $N^{(p)}$.

Theorem 2.4. *There exists at least one solution of Problem* (2.3)–(2.4).

Proof. We use Theorem 2.3 where X and Y and Z are to be replaced by $S^{(p)}$ and $T^{(p)}$ and $\mathbb{R}_1 \times \Xi^{(p)}$, respectively. In this case it follows from Lemma 2.2 that the assumptions of Theorem 2.3 are satisfied. Using Remark 2.4, we interpret the statements of Theorem 2.3 as follows:

There exist constants α_0, C and a function $\alpha \in \Xi^{(p)}$ such that

$$\alpha_0^2 + I^{(p)}(\alpha^2) \neq 0 \tag{2.7}$$

and

$$\alpha_0(J(w) - J(u^{(p)}) + s_0) + I^{(p)}(\alpha([w]_\nu - s)) \geqq C \geqq \alpha_0(-t_0) + I^{(p)}(\alpha t) \tag{2.8}$$

$\forall w \in V^{(p)}$, $s_0 \in \mathbb{R}_1$, $t_0 \in \mathbb{R}_1$, $s \in \Xi^{(p)}$, $t \in \Xi^{(p)}$ such that $s_0 \geqq 0$, $s \leqq 0$, $t \leqq 0$, $t_0 > 0$.

The following substitutions (a)–(d) into (2.8) are made:

(a) $t_0 > 0$ fixed, $s_0 = 0$, $w = u^{(p)}$, $s = [w]_v$, $t \equiv 0$, i.e. $0 \geqq -\alpha_0 t_0$ and hence

(2.9) $$\alpha_0 \geqq 0\,;$$

(b) $t_0 > 0$ fixed, $s_0 = 0$, $w = u^{(p)}$, $t \equiv 0$, $s = [w]_v + \tilde{s}$ where $\tilde{s} \in \Xi^{(p)}$, $\tilde{s} \leqq 0$, i.e. $-I^{(p)}(\alpha\,.\,\tilde{s}) \geqq -\alpha_0 t_0$ $\forall \tilde{s}$ and hence

(2.10) $$\alpha \geqq 0\,;$$

(c) $s_0 = 0$, $s \equiv 0$, $t \equiv 0$, $w = u^{(p)}$, i.e. $-\alpha_0 t_0 \leqq I^{(p)}(\alpha\,.\,[u^{(p)}]_v) \leqq 0$ $\forall t_0 > 0$ where the last inequality follows from (2.10) and the fact that $[u^{(p)}]_v \leqq 0$. As a consequence we have

(2.11) $$I^{(p)}(\alpha\,.\,[u^{(p)}]_v) = 0\,,$$

(d) $s_0 = 0$, $s = [u^{(p)}]_v$, $t \equiv 0$, i.e. $\alpha_0(J(w) - J(u^{(p)})) + I^{(p)}(\alpha[w]_v - \alpha[u]_v) \geqq -\alpha_0 t_0$ $\forall t_0 > 0$, $w \in V^{(p)}$.

Hence we can write

(2.12) $$\alpha_0\, J(u^{(p)}) + I^{(p)}(\alpha[u^{(p)}]_v) \leqq \alpha_0\, J(w) + I^{(p)}(\alpha[w]_v) \quad \forall w \in V^{(p)}\,.$$

A stronger condition than (2.9), namely $\alpha_0 > 0$, is required. Thus suppose that $\alpha_0 = 0$. Then (2.12) and (2.11) imply

$$0 = I^{(p)}(\alpha[u^{(p)}]_v) \leqq I^{(p)}(\alpha[w]_v)$$

$\forall w \in V^{(p)}$. Choosing $w \in V^{(p)}$ such that $[w]_v = -\alpha$ on $N^{(p)}$, we immediately have $I^{(p)}(\alpha^2) = 0$, and hence it is $\alpha_0^2 + I^{(p)}(\alpha^2) = 0$. But the last statement contradicts (2.7) and hence $\alpha_0 > 0$.

We now verify that the pair $\{u^{(p)}, \alpha/\alpha_0\}$ solves Problem (2.3)–(2.4). The function α/α_0 belongs to $\Lambda^{(p)}$ because of (2.10) and $\alpha_0 > 0$. Then the condition (2.3) follows from (2.12) and the condition (2.4) is a consequence of (2.11) and the fact that $[u^{(p)}]_v \leqq 0$ on $N^{(p)}$ (see Definition 2.3 for the cone $K^{(p)}$).

The following assertion will be very useful in Section 3:

Lemma 2.3. *If a pair* $\{u^{(p)}, \lambda^{(p)}\}$ *solves Problem* (2.3)–(2.4) *then there exists at least one point* $A \in N_0^{(p)}$ *(see Definition* 2.7*) such that* $[u^{(p)}]_v = 0$ *at* A *(i.e. the contact occurs at the point* A*).*

Proof. Let us suppose the contrary, i.e. $[u^{(p)}]_v < 0$ on $N_0^{(p)}$. Then we define $w^{(p)}$ such that

$$(w^{(p)})'_i = (u^{(p)})'_i \text{ on } \Omega'\,, \quad i = 1, 2\,,$$
$$(w^{(p)})''_1 = (u^{(p)})''_1 \text{ on } \Omega''\,,$$
$$(w^{(p)})''_2 = (u^{(p)})''_2 + \alpha \text{ on } \Omega''\,,$$

where α is a constant. If we express $[w^{(p)}]_v$ then we get

$$[w^{(p)}]_v = [u^{(p)}]_v - \alpha v_2 \text{ on } N^{(p)} .$$

At the points $X \in N^{(p)} \setminus N_0^{(p)}$ we have $[w^{(p)}]_v = [u^{(p)}]_v$ because of $v_2(x) = 0$, see Definition 2.7. Using the assumption that $[u^{(p)}]_v < 0$ and the fact that $N_0^{(p)}$ is a finite set, we can choose $\alpha \neq 0$ sufficiently small so that $[w^{(p)}]_v < 0$ on $N_0^{(p)}$ and hence

$$[w^{(p)}]_v \leqq 0 \text{ on } N^{(p)} .$$

We now prove that the pair $\{w^{(p)}, \lambda^{(p)}\}$ solves Problem (2.3)–(2.4).

In fact we verify the equivalent formulation by (2.5) and (2.6) – see Lemma 2.1. Because $D\, J(w^{(p)}, \varphi) = D\, J(u^{(p)}, \varphi) \forall \varphi \in V^{(p)}$, the condition (2.5) is satisfied. We know from the previous conclusion that $[w^{(p)}]_v \leqq 0$ and $\lambda^{(p)} \geqq 0$ on $N^{(p)}$. It remains to show that $\lambda^{(p)}[w^{(p)}]_v = 0$ on $N^{(p)}$ (see (2.6)).

According to Lemma 2.1, $\lambda^{(p)}[u^{(p)}]_v = 0$ on $N^{(p)}$. Using the assumption $[u^{(p)}]_v < 0$ on $N_0^{(p)}$, we know that $\lambda^{(p)} = 0$ on $N_0^{(p)}$. Hence $\lambda^{(p)}[w^{(p)}]_v = 0$ on $N_0^{(p)}$ and because $[w^{(p)}]_v = [u^{(p)}]_v$ on $N^{(p)} \setminus N_0^{(p)}$, we have $\lambda^{(p)}[w^{(p)}]_v = 0$ on $N^{(p)} \setminus N_0^{(p)}$.

3. NUMERICAL SOLUTION OF THE FINITE DIMENSIONAL MODEL

3.1. Introduction

The method generally recommended for the solution of Problem (2.3)–(2.4) is the Uzawa method; see [5]. Thus, let $\varrho > 0$ be a real number and let $\lambda^{(p,1)} \in \Lambda^{(p)}$ be given. Then the quoted method involves the repeated application of the following two steps:

(i) Find $u^{(p,k)} \in V^{(p)}$ such that

$$D\, J(u^{(p,k)}, \varphi) + I^{(p)}(\lambda^{(p,k)}[\varphi]_v) = 0 \quad \forall \varphi \in V^{(p)} .$$

(ii) Find $\lambda^{(p,k+1)} \in \Lambda^{(p)}$ such that

$$\lambda^{(p,k+1)} = P(\lambda^{(p,k)} + \varrho[u^{(p,k)}]_v) ,$$

where the operator $P : \Xi^{(p)} \to \Lambda^{(p)}$ is defined as follows:

$$\text{if } \mu \in \Xi^{(p)} \text{ then } \kappa = P\mu \text{ iff}$$

$\kappa \in \Lambda^{(p)}$ and $I^{(p)}((\mu - \kappa)(\omega - \kappa)) \leqq 0$ for all $\omega \in \Lambda^{(p)}$ i.e. κ is the projection of μ into $\Lambda^{(p)}$.

However, it may happen that the problem (i) is not solvable. The necessary and sufficient condition for a solution to exist is that $\lambda^{(p,k)} \in \tilde{\Lambda}^{(p)}$, i.e. $\lambda^{(p,k)} \in \Lambda^{(p)}$ and

$$\int_{\Omega''} F_2'' \, dx_1 \, dx_2 + I^{(p)}(\lambda^{(p,k)} v_2) = 0 .$$

Hence the initial multiplier $\lambda^{(p,1)}$ must belong to $\tilde{\Lambda}^{(p)}$ and the condition (ii) must be replaced by the projection of the gradient direction $\lambda^{(p,k)} + \varrho[u^{(p,k)}]_v$ into $\tilde{\Lambda}^{(p)}$ as follows:

(iii) Find $\lambda^{(p,k+1)} \in \tilde{\Lambda}^{(p)}$ such that

$$\lambda^{(p,k+1)} = \tilde{P}(\lambda^{(p,k)} + \varrho[u^{(p,k)}]_v),$$

where $\tilde{P} : \Xi^{(p)} \to \tilde{\Lambda}^{(p)}$ is such that

$$I^{(p)}((\mu - \tilde{P}\mu)(\omega - \tilde{P}\mu)) \leqq 0 \quad \forall \omega \in \tilde{\Lambda}^{(p)},$$

i.e.

$$I^{(p)}(\mu - \tilde{P}\mu)^2 \leqq I^{(p)}(\mu - \omega)^2 \quad \forall \omega \in \tilde{\Lambda}^{(p)}.$$

While condition (ii) can be easily implemented, the implementation of condition (iii) presents certain practical difficulties. These are due to the constraint

$$\int_{\Omega''} F_2'' \, dx_1 \, dx_2 + I^{(p)}(\lambda v_2) = 0.$$

We try to avoid this difficulty by constructing an auxiliary problem.

3.2. Auxiliary problem; interpretation

Problem. Let $A \in N^{(p)}$ be given. Find $u^{(p)} \in K_A^{(p)}$ such that

(3.1) $$J(u^{(p)}) \leqq J(w) \quad \forall w \in K_A^{(p)}.$$

Lemma 3.1. *There exists a unique solution $u^{(p)}$ to Problem* (3.1).

Proof. The proof of uniqueness and existence is similar to that of Theorem 2.1. We reformulate Problem (3.1) in terms of Lagrange Multipliers as follows:

Problem. Let $A \in N^{(p)}$ be given. Find $u^{(p)} \in V_A^{(p)}$ and $\lambda^{(p)} \in \Lambda_A^{(p)}$ such that

(3.2) $$J(u^{(p)}) + I^{(p)}(\lambda^{(p)}[u^{(p)}]_v) \leqq J(w) + I^{(p)}(\lambda^{(p)}[w]_v) \quad \forall w \in V^{(p)}$$

and

(3.3) $$I^{(p)}(\mu[u^{(p)}]_v) \leqq I^{(p)}(\lambda^{(p)}[u^{(p)}]_v) \quad \forall \mu \in \Lambda_A^{(p)}.$$

Remark 3.1. The condition (3.2) is equivalent to the following one:

(3.2a) $$D\, J(u^{(p)}; \varphi) + I^{(p)}(\lambda[\varphi]_v) = 0 \quad \forall \varphi \in V^{(p)}.$$

Lemma 3.2. *A pair $\{u^{(p)}, \lambda^{(p)}\}$ solves Problem* (3.2)–(3.3) *iff it holds* (3.2a) *and* $[u^{(p)}]_v \leqq 0$, $[u^{(p)}]_v \lambda^{(p)} = 0$ *on* $N^{(p)}$ *and* $\lambda^{(p)} \geqq 0$ *on* $N^{(p)} \setminus \{A\}$ *and* $[u^{(p)}]_v = 0$ *at* A.

Proof is evident.

Remark 3.2. As a consequence of (3.2a) we obtain a necessary condition upon $\lambda^{(p)}$,

$$\int_{\Omega''} F_2'' \, dx_1 \, dx_2 + I^{(p)}(\lambda^{(p)} v_2) = 0 ,$$

i.e. the balance condition.

Remark 3.3. The condition $[u]_v = 0$ at A is the condition which would be satisfied if an artificial "bold" were placed at the point A. The multipliers λ play the role of reaction forces of Ω' along Γ. The bold being placed at A has an effect iff the corresponding reaction λ is negative. In the non-negative case, the bold is placed on the part of Γ where the solution of the former Problem (2.1) has a contact.

Theorem 3.1. *The solution to Problem* (3.2)–(3.3), *if it exists, is unique.*

Proof. Let both $\{u^{(p)}, \lambda^{(p)}\}$ and $\{w^{(p)}, \mu^{(p)}\}$ be solutions of Problem (3.2)–(3.3). Then obviously the functions $u^{(p)}$ and $w^{(p)}$ solve Problem (3.1), which has a unique solution (Lemma 3.1). Hence, from the condition (3.2a) we derive that

$$I^{(p)}((\lambda^{(p)} - \mu^{(p)}) [\varphi]_v) = 0 \quad \forall \varphi \in V^{(p)} .$$

It is evident that the function $\varphi \in V^{(p)}$ can be chosen so that $[\varphi]_v = \lambda^{(p)} - \mu^{(p)}$ on $N^{(p)}$ and hence $I^{(p)}(\lambda^{(p)} - \mu^{(p)})^2 = 0$, i.e. $\lambda^{(p)} = \mu^{(p)}$ on $N^{(p)}$.

Definition 3.1. *Let $u^{(p)}$ be the solution to Problem* (3.1). *Two subsets $S^{(p)}$ and $T^{(p)}$ of the Cartesian product $\mathbb{R}_1 \times \Xi^{(p)}$ are defined as follows:*

For $(p, q) \in \mathbb{R}_1 \times \Xi^{(p)}$,

(i) $(p, q) \in S^{(p)}$ *iff there exist $w \in V^{(p)}$ and $s_0 \in \mathbb{R}_1$ and $s \in \Xi^{(p)}$ such that*

$$\begin{aligned} s_0 &\geqq 0 , \quad s \leqq 0 \\ s &= 0 \ \text{at} \ A \\ p &= J(w) - J(u^{(p)}) + s_0 \\ q &= [w]_v - s , \end{aligned}$$

(ii) $(p, q) \in T^{(p)}$ *iff there exist $t_0 \in \mathbb{R}_1$ and $t \in \Xi^{(p)}$ such that*

$$\begin{aligned} t_0 &> 0 , \quad t \leqq 0 \\ t &= 0 \ \text{at} \ A \\ p &= -t_0 \\ q &= t . \end{aligned}$$

Lemma 3.3. *If $A \in N_0^{(p)}$ then*

(i) *both $S^{(p)}$ and $T^{(p)}$ are convex subsets of $\mathbb{R}_1 \times \Xi^{(p)}$,*

(ii) $S^{(p)} \cap T^{(p)} = \emptyset$,

(iii) *the set $S^{(p)}$ contains at least one interior point.*

Proof. The assertion (i) is trivial. Let us suppose the contrary of the assertion (ii), i.e. there exist real numbers s_0, t_0 and functions $s, t \in \Xi^{(p)}$ and $w_0 \in V^{(p)}$ such that

$$\begin{aligned} s_0 &\geqq 0\,, \quad t_0 > 0\,, \\ s &\leqq 0\,, \quad t \leqq 0 \text{ on } N^{(p)}\,, \\ s &= t = 0 \text{ at } A\,, \\ J(w_0) &- J(u^{(p)}) + s_0 = -t_0\,, \\ [w_0]_\nu &- s = t\,. \end{aligned}$$

But then $J(w_0) < J(u^{(p)})$ and $[w_0]_\nu \leqq 0$ on $N^{(p)}$ and $[w_0]_\nu = 0$ at A, i.e. $u^{(p)}$ is not a solution to Problem (3.1) which contradicts Lemma 3.1.

Proof of the assertion (iii): Let us define $\{p_0, q_0\} \in \mathbb{R}_1 \times \Xi^{(p)}$ such that

$$\begin{aligned} p_0 &= -J(u^{(p)}) + 1\,, \\ q_0 &= 1 \text{ on } N^{(p)} \setminus \{A\}\,, \\ q_0 &= 0 \text{ at } A\,. \end{aligned}$$

Choosing $w \equiv 0$, $s_0 = 1$, $s = -1$ on $N^{(p)} \setminus \{A\}$, $s = 0$ at A, we verify that $\{p_0, q_0\} \in$ $\in S^{(p)}$ – see Definition 3.1.

We now prove that if

$$\begin{aligned} &\{p, q\} \in \mathbb{R}_1 \times \Xi^{(p)}\,, \\ &p_0 - \varepsilon < p < p_0 + \varepsilon\,, \\ &q_0 - \varepsilon < q < q_0 + \varepsilon\,, \end{aligned}$$

where $\varepsilon > 0$ is a sufficiently small real number, then $\{p, q\} \in S^{(p)}$, i.e. $\{p_0, q_0\}$ is an interior point.

Let $\{p, q\}$ satisfy the assumptions above. We define $w \in V^{(p)}$ in the following way: $(w)'_i \equiv 0$ on Ω' $(i = 1, 2)$, $w''_1 \equiv 0$, $w''_2 \equiv -q(A)\,(\nu_2(A))^{-1}$ on Ω''. Then we can estimate $|[w]_\nu| \leqq \varepsilon(|\nu_2(A)|)^{-1}$ on $N^{(p)}$ and $|J(w)| \leqq C\varepsilon(|\nu_2(A)|)^{-1}$, where

$$C \equiv \int_{\Omega''} F''_2 \,\mathrm{d}x_1 \,\mathrm{d}x_2\,.$$

Let us now define $s \in \Xi^{(p)}$ and $s_0 \in \mathbb{R}_1$ so that

$$q = [w]_\nu - s\,, \quad p = J(w) - J(u^{(p)}) + s_0\,.$$

Then it must be

$$s = 0 \text{ at } A\,, \quad s \leqq -1 + \varepsilon + \varepsilon(|\nu_2(A)|)^{-1} \text{ on } N^{(p)} \setminus A$$

and

$$s_0 \geqq 1 - \varepsilon - C\varepsilon(|\nu_2(A)|)^{-1}\,, \quad \text{i.e.}$$

for $\varepsilon \geqq 0$ sufficiently small it is $s \leqq 0$ and $s_0 \geqq 0$. Thus we conclude that $\{p, q\} \in S^{(p)}$.

Theorem 3.2. *There exists at least one solution of Problem* (3.2)–(3.3).

Proof. We use Theorem 2.3 where X and Y and Z are to be replaced by $S^{(p)}$ and $T^{(p)}$ and $\mathbb{R}_1 \times \Xi^{(p)}$, respectively; see Definition 3.1. In this case it follows from Lemma 3.3 that the assumptions of Theorem 2.3 are satisfied. Hence (see Remark 2.4) there exist real α_0 and $\alpha \in \Xi^{(p)}$ such that

$$\alpha_0^2 + I^{(p)}(\alpha^2) \neq 0 \tag{3.4}$$

and

$$\alpha_0(J(w) - J(u^{(p)}) + s_0) + I^{(p)}(\alpha([w]_v - s)) \geqq \beta_0 \geqq \alpha_0(-t_0) + I^{(p)}(\alpha t) \tag{3.5}$$

$\forall w \in V^{(p)}$, $s_0 \in \mathbb{R}_1$, $t_0 \in \mathbb{R}_1$, $s \in \Xi^{(p)}$, $t \in \Xi^{(p)}$ such that $t_0 > 0$, $s_0 \geqq 0$, $s \leqq 0$, $t \leqq 0$, $s = t = 0$ at A.

The following substitutions (a)–(d) into (3.4) are made:

(a) $t_0 > 0$ fixed, $s_0 = 0$, $w = u^{(p)}$, $s = [w]_v$, $t \equiv 0$, i.e. $0 \geqq -\alpha_0 t_0$ and hence

$$\alpha_0 \geqq 0\,. \tag{3.6}$$

(b) $t_0 > 0$ fixed, $s_0 = 0$, $w = u^{(p)}$, $t \equiv 0$, $s = [w]_v + \tilde{s}$ where $\tilde{s} \in \Xi^{(p)}$, $\tilde{s} \leqq 0$, $\tilde{s} = 0$ at A,

i.e.

$$-I^{(p)}(\alpha\,.\,\tilde{s}) \geqq -\alpha_0 t_0 \quad \forall \tilde{s}$$

and hence

$$\alpha \geqq 0 \text{ on } N^{(p)} \setminus \{A\} \tag{3.7}$$

(no condition on the value α at the point A).

(c) $s_0 = 0$, $s \equiv 0$, $t \equiv 0$, $w = u^{(p)}$,

i.e.

$$-\alpha_0 t_0 \leqq I^{(p)}(\alpha\,.\,[u^{(p)}]_v) \leqq 0 \quad \forall t_0 > 0\,,$$

where the last inequality follows from (3.7) and the fact that $[u^{(p)}]_v \leqq 0$ and $[u^{(p)}]_v = 0$ at A. As a consequence we have

$$I^{(p)}(\alpha[u^{(p)}]_v) = 0\,. \tag{3.8}$$

(d) $s_0 = 0$, $s = [u^{(p)}]_v$, $t \equiv 0$,

i.e.

$$\alpha_0(J(w) - J(u^{(p)})) + I^{(p)}(\alpha[w]_v - \alpha[u]_v) \geqq -\alpha_0 t_0$$

$\forall t_0 > 0$, $w \in V^{(p)}$. Hence we can conclude that

$$\alpha_0\, J(u^{(p)}) + I^{(p)}(\alpha[u^{(p)}]_v) \leqq \alpha_0\, J(w) + I^{(p)}(\alpha[w]_v) \quad \forall w \in V^{(p)}\,. \tag{3.9}$$

We now prove that $\alpha_0 > 0$ (compare with (3.6)). Let us suppose that $\alpha_0 = 0$. Then (3.9) and (3.8) imply that

$$0 = I^{(p)}(\alpha[u^{(p)}]_v) \leqq I^{(p)}(\alpha[w]_v)$$

$\forall w \in V^{(p)}$. Choosing $w \in V^{(p)}$ such that $[w]_v = -\alpha$ on $N^{(p)}$, we obtain $I^{(p)}(\alpha^2) = 0$, i.e.

$$\alpha_0^2 + I^{(p)}(\alpha^2) = 0 ,$$

which contradicts (3.4).

Using (3.7)–(3.9) and the fact that $\alpha_0 > 0$, it can be easily verified that the pair $\{u^{(p)}, \alpha/\alpha_0\}$ solves Problem (3.2)–(3.3).

Theorem 3.3. *Let a point $A \in N_0^{(p)}$ be given. If the pair $\{u^{(p)}, \lambda^{(p)}\}$ solves Problem (3.2)–(3.3) then $\{u^{(p)}, \lambda^{(p)}\}$ solves Problem (2.3)–(2.4) and $u^{(p)}$ solves Problem (2.1) iff*

$$\lambda^{(p)} \geqq 0 \text{ at the point } A .$$

Moreover, there exists at least one point $A \in N_0^{(p)}$ such that $\lambda^{(p)} \geqq 0$ at A, where $\{u^{(p)}, \lambda^{(p)}\}$ solves the corresponding Problem (3.2)–(3.3).

Proof. The first assertion follows from the definitions of both Problem (3.2)–(3.3) and Problem (2.3)–(2.4). The second is a consequence of Lemma 2.3, Theorem 2.2 and Theorem 2.4.

3.3. Algorithm

In this section we describe an algorithm (the "p, A-Algorithm") for the solution of Problem (3.2)–(3.3) for any given point $A \in N_0^{(p)}$.

p, A-Algorithm. Let $A \in N_0^{(p)}$ and $\lambda^{(p,1)} \in \tilde{\Lambda}_A^{(p)}$ be given. Then the sequences

$$\{u^{(p,k)}\}_{k=1}^{\infty} \quad \text{and} \quad \{\lambda^{(p,k)}\}_{k=1}^{\infty}$$

are determined by the following iterations:

STEP 1

Find $u^{(p,k)} \in V_A^{(p)}$ such that

$$D\, J(u^{(p,k)}, \psi) + I^{(p)}(\lambda^{(p,k)}[\psi]_v) = 0 \quad \forall \psi \in V^{(p)} . \tag{3.10}$$

STEP 2

Find $\lambda^{(p,k+1)} \in \Lambda_A^{(p)}$ such that

$$\lambda^{(p,k+1)} = P_A(\lambda^{(p,k)} + \varrho[u^{(p,k)}]_v) \text{ on } N^{(p)} \setminus \{A\} \tag{3.11}$$

and (the "balance condition")

$$\int_{\Omega''} F_2'' \,\mathrm{d}x_1\, \mathrm{d}x_2 + I^{(p)}(\lambda^{(p,k+1)} v_2) = 0 \tag{3.12}$$

(i.e. $\lambda^{(p,k+1)} \in \tilde{\Lambda}_A^{(p)}$), where ϱ is a fixed positive constant and $P_A : \mathbb{R}_1 \to \mathbb{R}_1^+$ such that

$$P_A \chi = \chi \quad \text{iff} \quad \chi \geqq 0\,,$$
$$P_A \chi = 0 \quad \text{iff} \quad \chi \leqq 0$$

for any real χ.

Remark 3.4. The condition (3.12) defines the value of $\lambda^{(p,k+1)}$ at the point A. This is the reason why we restrict the application of the algorithm to those points $A \in N^{(p)}$ where $v_2 \neq 0$ (i.e. the condition $A \in N_0^{(p)}$).

Theorem 3.4. *If $A \in N_0^{(p)}$ and $\varrho > 0$ are "small enough"*) then*

$$u^{(p,k)} \to u^{(p)} \quad in \quad V_A^{(p)}\,,$$
$$\lambda^{(p,k)} \to \lambda^{(p)} \quad in \quad \tilde{\Lambda}_A^{(p)}$$

for $k \to +\infty$ and for any choice of the initial function $\lambda^{(p,1)} \in \tilde{\Lambda}_A^{(p)}$. Moreover, the pair $\{u^{(p)}, \lambda^{(p)}\}$ solves Problem (3.2)–(3.3).

The proof of Theorem 3.4 will be preceded by two Lemmas.

Lemma 3.4. *If $A \in N_0^{(p)}$ then there exists a constant C_1 such that*

$$A(w, w) \geqq C_1 \|w\|^2 \quad \forall w \in V_A^{(p)}\,.$$

Proof. The space $V_A^{(p)}$ is finite-dimensional. Hence it is sufficient to verify that

$$\text{if } w \in V_A^{(p)} \text{ and } A(w, w) = 0$$
$$\text{then } w \equiv 0\,.$$

Let us suppose that $A(w, w) = 0$, $w \in V_A^{(p)}$. Then $w_i' \equiv 0$ on Ω' $(i = 1, 2)$, $w_1'' \equiv 0$ on Ω'', $w_2'' \equiv \beta$ = a constant on Ω''. But $[w]_\nu = -\beta v_2 = 0$ at the point A. According to the assumption $A \in N_0^{(p)}$ it is $v_2 \neq 0$ at A and hence necessarily $\beta = 0$, i.e. $w \equiv 0$.

Lemma 3.5. *There exists a constant C_2 such that*

$$I^{(p)}([w]_\nu^2) \leqq C_2 \|w\|^2 \quad \forall w \in V_A^{(p)}\,.$$

Proof. The space $V_A^{(p)}$ is finite-dimensional. The assertion above is an easy consequence of the fact that $\|w\| = 0$ implies $I([w]_\nu^2) = 0$.

Proof of Theorem 3.4. We remember that

$$\lambda^{(p,k)} = P_A(\lambda^{(p,k-1)} + \varrho[u^{(p,k-1)}]_\nu)$$

on $N^{(p)} \setminus \{A\}$. Moreover, it is easy to verify that

$$\lambda^{(p)} = P_A(\lambda^{(p)} + \varrho[u^{(p)}]_\nu)$$

*) See (3.15) in the proof of Theorem 3.4 and Remark 3.6.

on $N^{(p)} \setminus \{A\}$; see Lemma 3.2. Subtracting both equations and taking into account the definition of the operator P_A, we can derive the estimate

$$|\lambda^{(p)} - \lambda^{(p,k)}|^2 \leqq |\lambda^{(p)} - \lambda^{(p,k-1)} + \varrho([u^{(p)}]_v - [u^{(p,k-1)}]_v)|^2 =$$
$$= |\lambda^{(p)} - \lambda^{(p,k-1)}|^2 + \varrho^2 |[u^{(p)}]_v - [u^{(p,k-1)}]_v|^2 +$$
$$+ 2\varrho(\lambda^{(p)} - \lambda^{(p,k-1)})([u^{(p)}]_v - [u^{(p,k-1)}]_v),$$

which holds at any point $X \in N_0^{(p)} \setminus \{A\}$. Hence

$$(3.13) \qquad I_A^{(p)}[(\lambda^{(p)} - \lambda^{(p,k)})^2] \leqq I_A^{(p)}[(\lambda^{(p)} - \lambda^{(p,k-1)})^2] +$$
$$+ \varrho^2 I_A^{(p)}[([u^{(p)}]_v - [u^{(p,k-1)}]_v)^2] +$$
$$+ 2\varrho I_A^{(p)}((\lambda^{(p)} - \lambda^{(p,k-1)})([u^{(p)}]_v - [u^{(p,k-1)}]_v)) =$$
$$= I_A^{(p)}[(\lambda^{(p)} - \lambda^{(p,k-1)})^2] + \varrho^2 I^{(p)}[([u^{(p)}]_v - [u^{(p,k-1)}]_v)^2] +$$
$$+ 2\varrho I^{(p)}((\lambda^{(p)} - \lambda^{(p,k-1)})([u^{(p)}]_v - [u^{(p,k-1)}]_v)).$$

See Definition 2.5 for the definition of $I_A^{(p)}$ and notice that the last equality is due to $[u^{(p)}]_v = [u^{(p,k-1)}]_v = 0$ at A. With respect to (3.2a) and (3.10) and Lemma 3.4 we can obtain

$$(3.14) \qquad I^{(p)}((\lambda^{(p)} - \lambda^{(p,k-1)})([u^{(p)}]_v - [u^{(p,k-1)}]_v)) =$$
$$= -A(u^{(p)} - u^{(p,k-1)}, \; u^{(p)} - u^{(p,k-1)}) \leqq -C_1 \|u^{(p)} - u^{(p,k-1)}\|^2 .$$

Now (3.13) and (3.14) together with Lemma 3.5 yield

$$I_A^{(p)}[(\lambda^{(p)} - \lambda^{(p,k)})^2] \leqq I_A^{(p)}[(\lambda^{(p)} - \lambda^{(p,k-1)})^2] +$$
$$+ (\varrho^2 C_2 - 2\varrho C_1) \|u^{(p)} - u^{(p,k-1)}\|^2 .$$

If

$$(3.15) \qquad \varrho \in (0, 2C_1 C_2^{-1}),$$

where C_1 and C_2 are the constants from Lemmas 3.4 and 3.5, then $\varrho^2 C_2 - 2\varrho C_1 < 0$, i.e. the sequence

$$\{I_A^{(p)}[(\lambda^{(p)} - \lambda^{(p,k)})^2]\}_{k=1}^{\infty}$$

is decreasing and convergent. Hence

$$(2\varrho C_1 - \varrho^2 C_2) \|u^{(p)} - u^{(p,k)}\|^2 \leqq$$
$$\leqq I_A^{(p)}[(\lambda^{(p)} - \lambda^{(p,k-1)})^2] - I_A^{(p)}[(\lambda^{(p)} - \lambda^{(p,k)})^2] \to 0$$

for $k \to +\infty$, i.e.

$$u^{(p,k)} \to u^{(p)} \quad \text{for} \quad k \to +\infty .$$

Substracting (3.2a) and (3.10) we obtain the identity

$$D\, J(u^{(p,k)}, \psi) - D\, J(u^{(p)}, \psi) = -I^{(p)}((\lambda^{(p,k)} - \lambda^{(p)})[\psi]_v) \quad \forall \psi \in V^{(p)} .$$

For any nodal point $P \in N^{(p)}$ fixed we can choose $\psi \in V^{(p)}$ such that $[\psi]_\nu = 0$ on $N^{(p)} \setminus P$, $[\psi]_\nu = 1$ at P. The above identity implies that the value $|\lambda^{(p,k)} - \lambda^{(p)}|$ at the point P is bounded by the norm of $u^{(p,k)} - u^{(p)}$. Thus $\lambda^{(p,k)} \to \lambda^{(p)}$.

Remark 3.5. Since the constants C_1 and C_2 (see the above proof) are not available in practice, the choice of ϱ (see (3.15)) is based on intuitive and experimental arguments. We shall discuss this question in the third part of our paper.

CONCLUSION OF CHAPTER 3

Theorems 3.3 and 3.4 justify the following global strategy of computation:

(i) Guess $A \in N_0^{(p)}$.

(ii) Solve the relevant p, A-Algorithm, i.e. find the solution $\{u^{(p)}, \lambda^{(p)}\}$ to Problem (3.2)–(3.3).

(iii) If $\lambda^{(p)} < 0$ at A then try a new guess of $A \in N_0^{(p)}$, i.e. go to (i); at least one of the choices of $A \in N_0^{(p)}$ leads to the following step:

(iv) If $\lambda^{(p)} \geqq 0$ then $\{u^{(p)}, \lambda^{(p)}\}$ solve Problem (2.3)–(2.4), i.e. $u^{(p)}$ solves Problem (2.1).

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Souhrn

KONTAKTNÍ PROBLÉM DVOU PRUŽNÝCH TĚLES — Část I

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Cílem článku je studium kontaktního problému dvou pružných těles. Problém je přímo aplikovatelný na výpočet posuvů a napjatosti horninového kontinua a obezdívky tunelu, která je horninou obklopena. V této prvé části je problém variačně formulován ve spojité i diskrétní verzi.

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