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A NOTE ON NONHOMOGENEOUS INITIAL AND BOUNDARY CONDITIONS IN PARABOLIC PROBLEMS SOLVED BY THE ROTHE METHOD

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When solving parabolic problems with nonhomogeneous initial and boundary conditions by the Rothe method, some difficulties are encountered leading to rather unnatural additional conditions concerning the corresponding bilinear form and the initial and boundary functions (cf. [1], [2], [3], etc.). In the present paper we show how to remove such additional assumptions in the case of the initial conditions (Chap. 2) and how to replace them by other, rather more natural assumptions in the case of the boundary conditions (Chap. 3; see especially assumption (3.7), p. 64).

In the first chapter, we summarize briefly basic results from [1] concerning the Rothe method in the case of homogeneous initial and boundary conditions. In Chaps 2, or 3, nonhomogeneous initial, or boundary conditions are considered, respectively. In these chapters, also the cause of the above mentioned difficulties will become clear. In Chap. 4, the properties of the very weak solution will be studied, especially continuous dependence on the initial condition $u_0 \in L_2(\Omega)$ and independence of the function w characterizing the boundary conditions. In Chap. 5, application of the Ritz method (or of other direct methods) to approximate solution is considered.

CHAPTER 1. THE ROTHE METHOD IN PARABOLIC PROBLEMS. HOMOGENEOUS INITIAL AND BOUNDARY CONDITIONS

Let us give a brief survey of the work [1] concerning this subject.

In [1], the parabolic problem

$$Au + \frac{\partial u}{\partial t} = f \quad \text{in} \quad Q \equiv \Omega \times (0, T)\,, \tag{1.1}$$

$$u(x, 0) = u_0(x)\,, \tag{1.2}$$

$$u = \frac{\partial u}{\partial \nu} = \ldots = \frac{\partial^{k-1} u}{\partial \nu^{k-1}} = 0 \quad \text{on} \quad \partial\Omega \times (0, T) \tag{1.3}$$

is considered.

Assumptions. Ω is a bounded region in E_N with a Lipschitz boundary $\partial\Omega$, $u_0 \in L_2(\Omega)$, $f \in L_2(\Omega)$, the form

$$((v, u)) = \sum_{|i|,|j| \leqq k} \int_\Omega a_{ij} D^i v \, D^j u \, dx \tag{1.4}$$

(corresponding to the operator A) with bounded measurable coefficients $a_{ij}(x)$ in Ω is V-elliptic, i.e.

$$((v, v)) \geqq c \|v\|^2_{W_2^{(k)}(\Omega)} \quad \text{for every} \quad v \in V .$$

Notation. (v, u), or $\|v\|$ is the scalar product, or the norm in the space $L_2(\Omega)$, respectively,

$$(v, u)_{W_2^{(k)}(\Omega)} = \sum_{|i| \leqq k} (D^i v, D^i u) , \quad \|v\|^2_{W_2^{(k)}(\Omega)} = (v, v)_{W_2^{(k)}(\Omega)} , \tag{1.5}$$

$$V = \left\{ v;\, v \in W_2^{(k)}(\Omega),\; v = \frac{\partial v}{\partial \nu} = \ldots = \frac{\partial^{k-1} v}{\partial \nu^{k-1}} = 0 \quad \text{on } \partial\Omega \text{ in the sense of traces} \right\}$$

(with the metric of the space $W_2^{(k)}(\Omega)$), ν is the outward normal to $\partial\Omega$.

The Rothe method. Divide the interval $[0, T]$ into p subintervals of the length $h = T/p$ – denote this division by d_1 – and substitute the problem (1.1)–(1.3) by the following p boundary value problems (for $t_1 = h, t_2 = 2h, \ldots, t_p = ph = T$) – the so called *Rothe problems* – assuming, first $u_0(x) \equiv 0$ (thus considering the homogeneous initial condition):

$$((v, z_1)) + \frac{1}{h}(v, z_1) = (v, f) , \quad z_1 \in V , \tag{1.6}$$

$$((v, z_2)) + \frac{1}{h}(v, z_2 - z_1) = (v, f) , \quad z_2 \in V , \tag{1.7}$$

. . . .

$$((v, z_p)) + \frac{1}{h}(v, z_p - z_{p-1}) = (v, f) , \quad z_p \in V , \tag{1.8}$$

to be satisfied for all $v \in V$. The given assumptions ensure existence (and uniqueness) of solutions of (1.6)–(1.8). We construct, in $\bar{Q}$, a piecewise linear function in t, the so-caled *Rothe function*

$$u_1(x, t) = z_j(x) + \frac{t - t_j}{h} [z_{j+1}(x) - z_j(x)] \tag{1.9}$$

for

$$t_j \leqq t \leqq t_{j+1} , \quad j = 0, 1, \ldots, p - 1 , \quad z_0(x) = 0 .$$

Consider, further, the divisions $d_2, d_3, \ldots, d_n, \ldots$, dividing successively the interval $[0, T]$ into $2p, 4p, \ldots, 2^{n-1}p(= p_n) \ldots$ subintervals of lengths $h_2 = T/(2p)$, $h_3 = T/(4p), \ldots, h_n = T/p_n, \ldots$ and solve, for every fixed n and for $t_j^n = jh_n$ ($j =$

$= 1, 2, \ldots, p_n$), the corresponding Rothe problems similar to the problems (1.6)–(1.8). Finally, construct, for every n, the Rothe function

$$u_n(x, t) = z_j^n(x) + \frac{t - t_j^n}{h}[z_{j+1}^n(x) - z_j^n(x)] \tag{1.10}$$

for $$t_j^n \leqq t \leqq t_{j+1}^n, \quad j = 0, 1, \ldots, p_n - 1 .$$

(Here, $z_j^n(x)$ is the solution of the j-th Rothe problem corresponding to the division d_n, $z_0^n(x) = 0$; for $n = 1$, we write h, t_j and z_j instead of h_1, t_j^1, z_j^1, respectively, see (1.9).)

In this way, we get the so-called *Rothe sequence* of functions $\{u_n(x, t)\}$, defined in $\bar{Q}$. They may be considered, if needed, as abstract functions $u_n(t)$ from $[0, T]$ into V.

Apriori estimates. Denote

$$Z_j(x) = \frac{z_j(x) - z_{j-1}(x)}{h}, \quad j = 1, \ldots, p \tag{1.11}$$

("derivative with respect to t" at the time $t = t_j$). Especially,

$$Z_1(x) = \frac{z_1(x)}{h}, \tag{1.12}$$

because $u_0(x) \equiv 0$ according to the assumption. Putting $v = z_1$ in (1.6), we get

$$\|z_1\| \leqq h\|f\| \tag{1.13}$$

in consequence of $((z_1, z_1)) \geqq 0$. Thus

$$\|Z_1\| \leqq \|f\| . \tag{1.14}$$

Subtracting (1.6) from (1.7) and putting $v = z_2 - z_1$, we get

$$\|Z_2\| \leqq \|Z_1\| \leqq \|f\| \tag{1.15}$$

and in a similar way (for details see in [1])

$$\|Z_j\| \leqq \|f\| . \tag{1.16}$$

Denoting, similarly,

$$Z_j^n = \frac{z_j^n - z_{j-1}^n}{h_n} \tag{1.17}$$

we get, using the same procedure,

$$\|Z_j^n\| \leqq \|f\| \tag{1.18}$$

which means the uniform boundedness of $\|Z_j^n\|$ (thus not depending on the divison d_n). From (1.18) the uniform boundedness of $\|z_j^n\|$ and $\|z_j^n\|_V$ immediately follows (for details see in [1]; for $z \in V$ we write briefly $\|z\|_V$ instead of $\|z\|_{W_2^{(k)}(\Omega)}$). Denote

$$U_n(x, t) = Z_{j+1}^n(x) \quad \text{for} \quad t_j^n \leqq t \leqq t_{j+1}^n, \quad j = 0, 1, \ldots, p_n - 1 . \tag{1.19}$$

We shall also write $U_n(t)$, considering the function (1.19) as an abstract function from $[0, T]$ into $L_2(\Omega)$.

Convergence of the Rothe sequence $\{u_n(t)\}$. Denote, briefly,

$$I = [0, T] .$$

Let $L_2(I, V)$, or $L_2(I, L_2(\Omega))$ be Hilbert spaces of square integrable (in the Bochner sense) abstract functions from $[0, T]$ into V, or $L_2(\Omega)$, respectively. In consequence of uniform boundedness of $\|z_j^n\|_V$ and $\|Z_j^n\|$, the functions $u_n(t)$ and $U_n(t)$ are uniformly bounded in $L_2(I, V)$, or $L_2(I, L_2(\Omega))$, respectively. Then it is possible to find subsequences

$$\{u_{j_n}(t)\} , \quad \text{or} \quad \{U_{j_n}(t)\} , \tag{1.20}$$

converging weakly to some functions

$$u(t) \in L_2(I, V) , \quad \text{or} \quad U(t) \in L_2(I, L_2(\Omega)) , \tag{1.21}$$

respectively. In [1] it is shown that:

$$u(t) \in C(I, L_2(\Omega)) \tag{1.22}$$

($u(t)$ is even absolutely continuous),

$$U(t) = u'(t) \quad \text{in} \quad L_2(I, L_2(\Omega)) , \tag{1.23}$$

$$u(0) = 0 \quad \text{in} \quad C(I, L_2(\Omega)) , \tag{1.24}$$

the integral identity

$$\int_0^T ((v(t), u(t)))\,dt + \int_0^T (v(t), u'(t))\,dt = \int_0^T (v(t), f)\,dt \tag{1.25}$$

holds for every $v(t) \in L_2(I, V)$.

Definition 1.1. The function $u(t)$ is called the *weak solution of the problem* (1.1)–(1.3) *with* $u_0 = 0$.

In [1], uniqueness of this solution is proved, yielding, in the usual manner, weak convergence of the whole sequence $\{u_n(t)\}$ to the function $u(t)$ in $L_2(I, V)$. Moreover, it is shown that $\{u_n(x, t)\}$ converges *strongly* to $u(x, t)$ in $L_2(Q)$.

CHAPTER 2. NONHOMOGENEOUS INITIAL CONDITIONS

Let us turn to the problem (1.1)–(1.3) with $u_0(x) \neq 0$, $u_0 \in L_2(\Omega)$. Using the Rothe method, (1.6)–(1.8) turn into

$$((v, z_1)) + \frac{1}{h}(v, z_1 - u_0) = (v, f) , \quad z_1 \in V , \tag{2.1}$$

$$((v, z_2)) + \frac{1}{h}(v, z_2 - z_1) = (v, f) , \quad z_2 \in V , \tag{2.2}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$((v, z_p)) + \frac{1}{h}(v, z_p - z_{p-1}) = (v, f) , \quad z_p \in V \tag{2.3}$$

($v \in V$). It is easily seen that the procedure from Chap. 1, leading to the basic apriori estimates (1.16), (1.18), cannot be applied here, because if $u_0 \neq 0$ in $L_2(\Omega)$, (1.14) is no more valid. If

$$u_0 \in W_2^{(2k)} \cap V, \tag{2.4}$$

it seems natural to use the substitution $u = u_0 + z$ and to convert, in this way, the problem (1.1)–(1.3) into a similar problem with $z_0 = 0$ in $L_2(\Omega)$ and with the right-hand side $f - Au_0$ instead of f. It follows that some additional assumptions are to be imposed upon the operator A, or upon the corresponding bilinear form. In [1], it is required that

$$Ay \in L_2(\Omega), \quad ((v, u_0)) = (v, Au_0) \tag{2.5}$$

holds for every $y \in W_2^{(2k)}(\Omega) \cap V$, $v \in V$ and u_0 satisfying (2.4). (Cf. rather similar assumptions in [2], etc.)

In this way, one comes in [1] to the weak solution (according to Def. 1.1) $z(t)$ with $z_0 = 0$. The function $z(t) + u_0$ is then the so-called *weak solution of the problem* (1.1)–(1.3). Showing then the continuous dependence of this weak solution on the initial conditions, one removes in [1] the assumption (2.4): Let $u_0 \in L_2(\Omega)$ and

$$u_i \to u_0 \quad \text{in} \quad L_2(\Omega), \tag{2.6}$$

u_i satisfying (2.4); then the corresponding weak solution $u_i(t)$ converge, in $L_2(I, L_2(\Omega))$, to a uniquely determined function $u(t)$ which is called, in [1], the *generalized solution of* (1.1)–(1.3).

In the present chapter, we show how to remove the additional assumptions (2.5). The form $((v, u))$ being V-elliptic, a set M exists (see [4], pp. 131, 132), dense in V, and consequently in $L_2(\Omega)$, with the following property: If $s \in M$, then there exists precisely one $g \in L_2(\Omega)$ such that

$$((v, s)) = (v, g) \quad \text{holds for every } v \in V. \tag{2.7}$$

Thus let $s \in M$. Replacing, in (2.1), u_0 by s, we get

$$((v, z_1)) + \frac{1}{h}(v, z_1 - s) = (v, f), \quad z_1 \in V. \tag{2.8}$$

Putting $z_1 = s + \tilde{z}_1$ and using (2.7),

$$((v, s)) = (v, g), \tag{2.9}$$

(2.8) becomes

$$((v, \tilde{z}_1)) + \frac{1}{h}(v, \tilde{z}_1) = (v, f - g), \quad \tilde{z}_1 \in V. \tag{2.10}$$

Similarly, putting in (2.2) $z_2 = s + \tilde{z}_2$, etc., we get

$$((v, \tilde{z}_2)) + \frac{1}{h}(v, \tilde{z}_2 - \tilde{z}_1) = (v, f - g), \quad \tilde{z}_2 \in V, \tag{2.11}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$((v, \tilde{z}_p)) + \frac{1}{h}(v, \tilde{z}_p - \tilde{z}_{p-1}) = (v, f - g), \quad \tilde{z}_p \in V. \tag{2.12}$$

But (2.10)–(2.12) are Rothe problems of the form (1.6)–(1.8) with f replaced by $f - g$. Thus the sequence $\{\tilde{u}_n\}$ of corresponding Rothe functions (1.10) with z_j^n replaced by $\tilde{z}_j^n$ converges weakly to a function $\tilde{u}(t)$ having the properties (1.21)–(1.24) and satisfying the integral identity

$$(2.13) \qquad \int_0^T ((v(t), \tilde{u}(t)))\,\mathrm{d}t + \int_0^T (v(t), \tilde{u}'(t))\,\mathrm{d}t = \int_0^T (v(t), f - g)\,\mathrm{d}t$$

(for every $v(t) \in L_2(I, V)$). The function

$$(2.14) \qquad u(t) = \tilde{u}(t) + s$$

has then similar properties, with

$$(2.15) \qquad u(0) = s \quad \text{in} \quad C(I, L_2(\Omega)),$$

and satisfies the integral identity

$$(2.16) \qquad \int_0^T ((v(t), u(t)))\,\mathrm{d}t + \int_0^T (v(t), u'(t))\,\mathrm{d}t = \int_0^T (v(t), f)\,\mathrm{d}t$$

for every $v(t) \in L_2(I, V)$. (Note that for every such $v(t)$ we have $(v(t), g) = ((v(t), s))$ for almost all $t \in I$ and that $\tilde{u}'(t) = u'(t)$.)

Definition 2.1. The function $u(t)$ is called the *weak solution of the problem* (1.1)–(1.3) *with* $u_0 = s \in M$.

Uniqueness of this solution follows in the same way as in [1].

We show that this weak solution depends continuously (in $L_2(I, L_2(\Omega))$) on $s \in M$ (from $L_2(\Omega)$. Thus let $\hat{s} \in M$. In the same way as before we get the weak solution $\hat{u}(t)$ of the problem (1.1)–(1.3) with $u_0 = \hat{s}$. The Rothe problems (2.8), (2.2), (2.3) become

$$(2.17) \qquad ((v, \hat{z}_1)) + \frac{1}{h}(v, \hat{z}_1 - \hat{s}) = (v, f), \quad \hat{z}_1 \in V,$$

$$(2.18) \qquad ((v, \hat{z}_2)) + \frac{1}{h}(v, \hat{z}_2 - \hat{z}_1) = (v, f), \quad \hat{z}_2 \in V,$$

. . . .

$$(2.19) \qquad ((v, \hat{z}_p)) + \frac{1}{h}(v, \hat{z}_p - \hat{z}_{p-1}) = (v, f), \quad \hat{z}_p \in V.$$

Subtracting (2.8) from (2.17), (2.2) from (2.18), etc., and writing $\hat{z}_j - z_j = \bar{z}_j$, we get

$$(2.20) \qquad ((v, \bar{z}_1)) + \frac{1}{h}(v, \bar{z}_1 - (\hat{s} - s)) = 0, \quad \bar{z}_1 \in V,$$

$$(2.21) \qquad ((v, \bar{z}_2)) + \frac{1}{h}(v, \bar{z}_2 - \bar{z}_1) = 0, \quad \bar{z}_2 \in V,$$

. . . .

$$(2.22) \qquad ((v, \bar{z}_p)) + \frac{1}{h}(v, \bar{z}_p - \bar{z}_{p-1}) = 0, \quad \bar{z}_p \in V,$$

wherefrom (putting $v = \bar{z}_1$ in (2.20), $v = \bar{z}_2$ in (2.21), etc.)

$$\|\bar{z}_1\| \leqq \|\hat{s} - s\| , \tag{2.23}$$

$$\|\bar{z}_2\| \leqq \|\bar{z}_1\| \leqq \|\hat{s} - s\| , \tag{2.24}$$

and, in general,

$$\|\bar{z}_j\| \leqq \|\hat{s} - s\| . \tag{2.25}$$

Analogously, we get, for the division d_n,

$$\|\hat{z}_j^n - z_j^n\| \leqq \|\bar{z}_j^n\| \leqq \|\hat{s} - s\| \tag{2.26}$$

and, in view of the form of the Rothe functions,

$$\|\hat{u}_n(x, t) - u_n(x, t)\| \leqq \|\hat{s} - s\| \tag{2.27}$$

for every $t \in [0, T]$. Taking the square and integrating between 0 and T, we find

$$\|\hat{u}_n(t) - u_n(t)\|_{L_2(I, L_2(\Omega))} \leqq \sqrt{(T)}\|\hat{s} - s\|_{L_2(\Omega)} . \tag{2.28}$$

Having in mind that $u(t)$, or $\hat{u}(t)$ are weak limits, in $L_2(I, V)$, and, consequently, in $L_2(I, L_2(\Omega))$, of the sequences $u_n(t)$, or $\hat{u}_n(t)$, respectively, we finally get

$$\|\hat{u}(t) - u(t)\|_{L_2(I, L_2(\Omega))} \leqq \sqrt{(T)}\|\hat{s} - s\|_{L_2(\Omega)} , \tag{2.29}$$

which expresses the required continuous dependence of the weak solution on the initial condition from M.

Now, let $u_0 \in L_2(\Omega)$ and let $\{s_i\}$ be a sequence of M such that

$$s_i \to u_0 \quad \text{in} \quad L_2(\Omega) . \tag{2.30}$$

In consequence of (2.29), the sequence of corresponding solutions $u_i(t)$ is a Cauchy sequence in $L_2(I, L_2(\Omega))$, thus converging, in $L_2(I, L_2(\Omega))$ to a function $u(t)$ (uniquely determined by the function u_0 because of the just proved continuous dependence).

Definition 2.2. The function $u(t)$ is called the *very weak solution of the problem* (1.1)–(1.3) *with* $u_0 \in L_2(\Omega)$.

In this way, we came to the concept of a solution of (1.1)–(1.3) *without additional assumptions* (2.5).

From the above constructions it is easily seen that this is the only difference between the very weak solution introduced by Def. 2.2 and the generalized solution introduced in [1]. Except this, these concepts are identical.

In [1], some properties of the generalized solution are derived. Because the proofs are based on the continuous dependence only, without using (2.5), they remain unchanged for the case of our very weak solution. Thus we are not going to reproduce these proofs here, and only summarize these properties in the following theorems:

Theorem 2.1. *For the very weak solution $u(t)$ from Def.* 2.2 *we have*

$$u(t) \in C(I, L_2(\Omega)) . \tag{2.31}$$

Further,

$$u(0) = u_0 \quad in \quad C(I, L_2(\Omega)) . \tag{2.32}$$

Theorem 2.2. (*Continuous dependence on the initial condition.*) *For $u(t)$ from Def. 2.2, the inequality (2.29) is preserved:*

If $u_1(t)$, or $u_2(t)$ is the very weak solution of the problem (1.1)–(1.3) with $u_{01} \in L_2(\Omega)$, or $u_{02} \in L_2(\Omega)$, respectively, we have

$$\|u_2(t) - u_1(t)\|_{L_2(I, L_2(\Omega))} \leqq \sqrt{(T)}\|u_{02} - u_{01}\|_{L_2(\Omega)} . \tag{2.33}$$

Moreover,

$$\|u_2(t) - u_1(t)\|_{C(I, L_2(\Omega))} \leqq \|u_{02} - u_{01}\|_{L_2(\Omega)} . \tag{2.34}$$

Theorem 2.3. *The very weak solution $u(t)$ from Def. 2.2 is the weak limit (in $L_2(I, L_2(\Omega))$) of the Rothe sequence $\{u_n(t)\}$, where z_j* (or z_j^n) in (1.9) (or (1.10)) *are the solutions of the problems* (2.1)–(2.3) (or of similar problems corresponding to the division d_n). *Moreover, we have*

$$\lim_{n \to \infty} u_n(x, t) = u(x, t) \tag{2.35}$$

strongly in $L_2(Q)$.

Remark 2.1. Obviously, the weak solution introduced by Def. 2.1 is a special case, for $u_0 \in M$, of the very weak solution introduced by Def. 2.2. The weak solution by Def. 1.1 is a special case, for $u_0 = 0$, of the weak solution by Def. 2.1.

CHAPTER 3. NONHOMOGENEOUS INITIAL AND BOUNDARY CONDITIONS

Consider, now, the problem

$$Au + \frac{\partial u}{\partial t} = f \quad \text{in} \quad Q = \Omega \times (0, T) , \tag{3.1}$$

$$u(x, 0) = u_0(x) , \tag{3.2}$$

$$u - w \in V , \tag{3.3}$$

where $u_0 \in L_2(\Omega)$, and $w \in W_2^{(k)}(\Omega)$ is a given function characterizing the nonhomogeneous boundary conditions. (For $w = 0$ we have the problem (1.1)–(1.3).)

The corresponding Rothe problems are

$$((v, u_1)) + \frac{1}{h}(v, u_1 - u_0) = (v, f) , \quad u_1 - w \in V , \tag{3.4}$$

$$((v, u_2)) + \frac{1}{h}(v, u_2 - u_1) = (v, f) , \quad u_2 - w \in V , \tag{3.5}$$

. . . .

$$((v, u_p)) + \frac{1}{h}(v, u_p - u_{p-1}) = (v, f) , \quad u_p - w \in V . \tag{3.6}$$

In Chap. 2 we have seen that if $u_0 \neq 0$ it is not possible to use the procedure from Chap. 1, because the apriori estimates (1.16), (1.18) do no more hold. The less can this procedure be applied in the present case. Difficulties arise here even if $u_0 = 0$, because we cannot put $v = u_1$ into (3.4) to get an estimate analogous to (1.13), since we have not $u_1 \in V$. If we put $u_1 = w + z_1$ to reach $z_1 \in V$, we get a term $-((v, w))$ on the right hand side which makes it again impossible to obtain estimates of the type (1.16), (1.18). It seems to be natural to put $u = w + z$ immediately in (3.1)–(3.3) and to convert this problem, in this way, into a problem with homogeneous boundary conditions. But such an approach requires additional assumptions on the operator A, such as $Aw \in L_2(\Omega)$ (see [2], [3], etc.). In this paper, we choose an other approach, and *will assume – throughout the paper – that the form* $((v, u))$ *is such that*

$$(3.7) \qquad (((v, u))) = ((v, u)) + (v, u)$$

is an equivalent scalar product in $W_2^{(k)}(\Omega)$. (In details, (3.7) is a scalar product on the elements of the space $W_2^{(k)}(\Omega)$, and generates a norm equivalent with the norm of this space.)

This assumption requires, first of all, the symmetry of the form $((v, u))$. Nevertheless, it seems to be more natural than the assumption $Aw \in L_2(\Omega)$ and, at the same time, more suitable for applications. A trivial example is as follows: $A = -\Delta$, $V = \mathring{W}_2^{(1)}(\Omega)$.

Let us denote by $\overline{W}_2^{(k)}(\Omega)$ the space the elements of which are the elements of the space $W_2^{(k)}(\Omega)$ and in which the scalar product is given by (3.7). The space $\overline{W}_2^{(k)}(\Omega)$ is complete and the space V, provided with the same scalar product (3.7), is its subspace. Thus, the function w from (3.3) can be uniquely decomposed, in $\overline{W}_2^{(k)}(\Omega)$, into the sum

$$(3.8) \qquad w = w_1 + w_2\,, \quad ^{1)}\; w_1 \in V, \quad w_2 \perp V.$$

Especialy, we have

$$(3.9) \qquad ((v, w_2)) = -(v, w_2) \quad \text{for all} \quad v \in V,$$

because

$$(3.10) \qquad (((v, w_2))) = ((v, w_2)) + (v, w_2) = 0 \quad \text{for all} \quad v \in V.$$

Thus let (3.7) be an equivalent scalar product in $W_2^{(k)}(\Omega)$. Put, in (3.4)–(3.6),

$$u_j = \tilde{u}_j + \hat{u}_j$$

and substitute the p problems (3.4)–(3.6) by $2p$ problems

$$(3.11) \qquad ((v, \tilde{u}_1)) + \frac{1}{h}(v, \tilde{u}_1 - w_2) = (v, f)\,, \quad \tilde{u}_1 - w \in V,$$

$$(3.12) \qquad ((v, \tilde{u}_2)) + \frac{1}{h}(v, \tilde{u}_2 - \tilde{u}_1) = (v, f)\,, \quad \tilde{u}_2 - w \in V,$$

. . . .

$$(3.13) \qquad ((v, \tilde{u}_p)) + \frac{1}{h}(v, \tilde{u}_p - \tilde{u}_{p-1}) = (v, f)\,, \quad \tilde{u}_p - w \in V$$

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[1]) Let us note here that a proper decomposition of w can be obtained *without* the additional assumption (3.7). See a new monography prepared by K. Rektorys on the considered method.

and

$$((v, \hat{u}_1)) + \frac{1}{h}(v, \hat{u}_1 - (u_0 - w_2)) = 0\,, \quad \hat{u}_1 \in V\,, \tag{3.14}$$

$$((v, \hat{u}_2)) + \frac{1}{h}(v, \hat{u}_2 - \hat{u}_1) = 0\,, \quad \hat{u}_2 \in V\,, \tag{3.15}$$

. . . .

$$((v, \hat{u}_p)) + \frac{1}{h}(v, \hat{u}_p - \hat{u}_{p-1}) = 0\,, \quad \hat{u}_p \in V \tag{3.16}$$

(to be satisfied for all $v \in V$).

Obviously, the problems (3.11)–(3.13), or (3.14)–(3.16) correspond to be problems

$$A\tilde{u} + \frac{\partial \tilde{u}}{\partial t} = f \quad \text{in} \quad Q = \Omega \times (0, T)\,, \tag{3.17}$$

$$\tilde{u}(x, 0) = w_2\,, \tag{3.18}$$

$$\tilde{u} - w \in V\,, \tag{3.19}$$

or

$$A\hat{u} + \frac{\partial \hat{u}}{\partial t} = 0 \quad \text{in} \quad Q = \Omega \times (0, T)\,, \tag{3.20}$$

$$\hat{u}(x, 0) = u_0 - w_2\,, \tag{3.21}$$

$$\hat{u} \in V\,, \tag{3.22}$$

respectively.

Remark 3.1. Let us note here that the function w_2 plays an auxiliary role here, in theoretical considerations only. As concerns numerical methods (see Chap. 5), the decomposition $w = w_1 + w_2$ is not to be carried out.

Let us put, in (3.11)–(3.13),

$$\tilde{u}_j = \tilde{z}_j + w_2\,, \quad j = 1, 2, \ldots, p\,. \tag{3.23}$$

We get (note that $w - w_2 \in V$)

$$((v, \tilde{z}_1 + w_2)) + \frac{1}{h}(v, \tilde{z}_1) = (v, f)\,, \quad \tilde{z}_1 \in V\,, \tag{3.24}$$

etc., or substituting $-(v, w_2)$ for $((v, w_2))$ according to (3.9),

$$((v, \tilde{z}_1)) + \frac{1}{h}(v, \tilde{z}_1) = (v, f + w_2)\,, \quad \tilde{z}_1 \in V\,, \tag{3.25}$$

$$((v, \tilde{z}_2)) + \frac{1}{h}(v, \tilde{z}_2 - \tilde{z}_1) = (v, f + w_2)\,, \quad \tilde{z}_2 \in V\,, \tag{3.26}$$

. . . .

$$((v, \tilde{z}_p)) + \frac{1}{h}(v, \tilde{z}_p - \tilde{z}_{p-1}) = (v, f + w_2)\,, \quad \tilde{z}_p \in V \tag{3.27}$$

(for all $v \in V$). But these are Rothe problems of the form (1.6)–(1.8) with $f + w_2$ instead of f. Consequently, the corresponding Rothe sequence converges weakly in $L_2(I, V)$ to a function $\tilde{z}(t)$ which is the weak solution, according to Def. 1.1, of the problem

$$A\tilde{z} + \frac{\partial \tilde{z}}{\partial t} = f + w_2 \quad \text{in} \quad Q = \Omega \times (0, T)\,, \tag{3.28}$$

$$\tilde{z}(x, 0) = 0\,, \tag{3.29}$$

$$\tilde{z} \in V\,. \tag{3.30}$$

Especially, $\tilde{z}(t)$ satisfies

$$\tilde{z}(0) = 0 \quad \text{in} \quad C(I, L_2(\Omega)) \tag{3.31}$$

and

$$\int_0^T ((v(t), \tilde{z}(t)))\,\mathrm{d}t + \int_0^T (v(t), \tilde{z}'(t))\,\mathrm{d}t = \int_0^T (v(t), f + w_2)\,\mathrm{d}t \tag{3.32}$$

for all $v(t) \in L_2(I, V)$. The function

$$\tilde{u}(t) = \hat{z}(t) + w_2 \tag{3.33}$$

will then have similar properties (see (1.21)–(1.23); especially we shall have $\tilde{u}(t) - w_2 \in L_2(I, V)$, $\tilde{u}(t) \in L_2(I, L_2(\Omega))$, $\tilde{u}(t) \in C(I, L_2(\Omega))$) and will satisfy

$$\tilde{u}(0) = w_2 \quad \text{in} \quad C(I, L_2(\Omega)) \tag{3.34}$$

and

$$\int_0^T ((v(t), \tilde{u}(t)))\,\mathrm{d}t + \int_0^T (v(t), \tilde{u}'(t))\,\mathrm{d}t = \int_0^T (v(t), f)\,\mathrm{d}t \tag{3.35}$$

for all $v(t) \in L_2(I, V)$. (Note that $(v(t), w_2) = -((v(t), w_2))$ for almost all $t \in [0, T]$ and that $\tilde{z}'(t) = \tilde{u}'(t)$.)

Definition 3.1. The function $\tilde{u}(t)$ is called the *weak solution of the problem* (3.17)–(3.19).

The problems (3.14)–(3.16) are Rothe problems of the type (2.1)–(2.3) with $u_0 - w_2$, or 0 instead of u_0, or f, respectively. According to Chap. 2, the corresponding Rothe sequence converges weakly in $L_2(I, L_2(\Omega))$ to the very weak solution $\hat{u}_2(t)$ of the problem (3.20)–(3.22).

Definition 3.2. The function

$$u(t) = \tilde{u}(t) + \hat{u}(t) \tag{3.36}$$

is called the *very weak solution of the problem* (3.1)–(3.3).

This concept – intuitively clear, because the problem (3.1)–(3.3) is the "sum" of problems (3.17)–(3.19) and (3.20)–(3.22) – deserves a more detailed discussion. Especially, uniqueness of this very weak solution – inclusive its independence of the possible choice of the functions w (characterizing the same boundary conditions) and thus of the corresponding functions w_2 — is to be shown. To clarify these questions and to derive some properties of this very weak solution is the purpose of the next chapter.

CHAPTER 4. THE DEFINITION 3.2 ESTABLISHED. SOME PROPERTIES OF THE VERY WEAK SOLUTION OF THE PROBLEM (3.1)–(3.3)

Let w (and thus w_2) be fixed.

Let, first, $u_0 - w_2 \in M$ (on the set M see the text related to (2.7), p. 60). Then the very weak solution $\hat{u}(t)$ of the problem (3.20)–(3.22) turns into the weak solution according to Def. 2.1. Thus, it has the properties (1.21)–(1.23) and satisfies

$$\hat{u}(0) = u_0 - w_2 \quad \text{in} \quad C(I, L_2(\Omega)) \tag{4.1}$$

and

$$\int_0^T ((v_\cdot(t), \hat{u}(t)))\,\mathrm{d}t + \int_0^T (v(t), \hat{u}'(t))\,\mathrm{d}t = 0 \tag{4.2}$$

for every $v(t) \in L_2(I, V)$. Consequently, the function

$$u(t) = \tilde{u}(t) + \hat{u}(t) \tag{4.3}$$

will satisfy

$$u(t) - w_2 \in L_2(I, V)\,, \tag{4.4}$$

$$u'(t) \in L_2(I, L_2(\Omega))\,, \tag{4.5}$$

$$u(t) \in C(I, L_2(\Omega)) \text{ (even absolutely continuous)}\,, \tag{4.6}$$

$$u(0) = u_0 \quad \text{in} \quad C(I, L_2(\Omega))\,, \tag{4.7}$$

$$\int_0^T ((v(t), u(t)))\,\mathrm{d}t + \int_0^T (v(t), u'(t))\,\mathrm{d}t = \int_0^T (v(t), f)\,\mathrm{d}t \tag{4.8}$$

for all $v(t) \in L_2(I, V)$. The properties (4.4) and (4.7) correspond to the conditions (3.3) and (3.2), respectively, the equation (3.1) is fulfilled in the sense (4.8). Thus, in the case $u_0 - w_2 \in M$, the term (very weak) *solution* of the problem (3.1)–(3.3) for this function is justified. Such a solution is *unique*: Let us have two functions with properties (4.4)–(4.8). Then their difference satisfies (1.21)–(1.25) with $f = 0$ and, consequently, is equal to zero (see [1], Theorem 1). Moreover, *the solution* (4.3) *depends continuously on the initial conditions*: Let $u_{01} \in L_2(\Omega)$, $u_{02} \in L_2(\Omega)$ be such functions that

$$u_{01} - w_2 \in M\,, \quad u_{02} - w_2 \in M \tag{4.9}$$

(w is always kept fixed). Let $u_1(t)$, $u_2(t)$ be corresponding solutions. Then their difference

$$u(t) = u_2(t) - u_1(t) \tag{4.10}$$

is the weak solution of the problem (2.1)–(2.3) (p. 59) with $f = 0$ and with $u_{02} -$

$- u_{01} \in M$ instead of $u_0 \in M$. Thus we may apply (2.29) and get

$$\|u_2(t) - u_1(t)\|_{L_2(I,L_2(\Omega))} \leqq \sqrt{(T)}\, \|u_{02} - u_{01}\|_{L_2(\Omega)}\,. \tag{4.11}$$

Let, now, $u_0 - w_2 \notin M$. Then we can find such a sequence of functions $s_i \in M$ that

$$s_i \to u_0 - w_2 \quad \text{in} \quad L_2(\Omega) \tag{4.12}$$

(i.e.

$$s_i + w_2 \to u_0 \quad \text{in} \quad L_2(\Omega))\,. \tag{4.13}$$

Corresponding solutions $u_i(t)$ of the problem (3.1)–(3.3) with u_0 replaced by $s_i + w_2$ then satisfy (4.4)–(4.8) with $u_i(0) = s_i + w_2$ in $C(I, L_2(\Omega))$. In consequence of (4.11), the sequence $\{u_i(t)\}$ is a Cauchy sequence in $L_2(I, L_2(\Omega))$, and thus converges, in $L_2(I, L_2(\Omega))$, to a function $u(t) \in L_2(I, L_2(\Omega))$. Because of the above mentioned continuous dependence, this function is uniquely determined by the function u_0 (it does not depend on the choice of the sequence $\{s_i \in M\}$ with the property (4.12)). From the construction of this function and of the function $\hat{u}$ from Chap. 3 it immediately follows that $u(t)$ coincides with the very weak solution of the problem (3.1)–(3.3) introduced by Def. 3.2.

Thus, w being fixed, uniqueness of the very weak solution of the problem (3.1)–(3.3) is shown.

To give the full establishment of Def. 3.2, we show that $u(t)$ does not depend on the choice of the function w (in the sence of Theorem 4.4). But first of all we present Theorems 4.1–4.3 which are analogues of theorems 2.1–2.3 for the very weak solution of the problem (2.1)–(2.3) (Def. 2.2). We get them immediately from these theorems, having in mind that the function (3.36) is the sum of the very weak solution $\hat{u}(t)$ of the problem (3.20)–(3.22) (for which thus these theorems are valid) and of the function $\tilde{u}(t)$ which is itself the sum of the weak (and thus very weak) solution $\tilde{z}(t)$ of a similar problem and of the "constant" function w_2.

Theorem 4.1. *For the very weak solution $u(t)$ from Def.* 3.2 *we have*

$$u(t) \in C(I, L_2(\Omega))\,. \tag{4.14}$$

Further,

$$u(0) = u_0 \quad \textit{in} \quad C(I, L_2(\Omega))\,. \tag{4.15}$$

Theorem 4.2. (*Continuous dependence on initial conditions.*) *For the very weak solutions $u_1(t)$, $u_2(t)$ of the problem* (3.1)–(3.3), *with initial conditions u_{01}, u_{02}, respectively, we have*

$$\|u_2(t) - u_1(t)\|_{L_2(I,L_2(\Omega))} \leqq \sqrt{(T)}\, \|u_{02} - u_{01}\|_{L_2(\Omega)}\,. \tag{4.16}$$

Moreover,

$$\|u_2(t) - u_1(t)\|_{C(I,L_2(\Omega))} \leqq \|u_2 - u_{01}\|\,. \tag{4.17}$$

Theorem 4.3. *The very weak solution $u(t)$ from Def.* 3.2 *is the weak limit, in $L_2(I, L_2(\Omega))$, of the Rothe sequence $\{u_n(t)\}$, where z_j* (or z_j^n) *in* (1.9) (or (1.10)) *are* *solutions of the Rothe problems* (3.4)–(3.6) (or of similar problems corresponding to the division d_n).

Moreover, we have

$$\lim_{n\to\infty} u_n(x, t) = u(x, t) \tag{4.18}$$

strongly in $L_2(Q)$.

Now, we formulate the announced Theorem 4.4.

Theorem 4.4. *The very weak solution from Def.* 3.2 *does not depend on the function* w *characterizing the boundary conditions. In detail: Let* $\bar{w} \in W_2^{(k)}(\Omega)$ *be another function such that* $\bar{w} - w \in V$. *Then for the very weak solutions* $u(t)$, *or* $\bar{u}(t)$ *of the problem* (3.1)–(3.3) *with the boundary functions* w, *or* $\bar{w}$, *respectively, we have*

$$u(t) = \bar{u}(t) \quad in \quad L_2(I, L_2(\Omega)) . \tag{4.19}$$

The proof is very simple: The solutions $u(t)$, $\bar{u}(t)$ are weak limits, in $L_2(I, L_2(\Omega))$, of the Rothe sequences $u_n(t)$, $\bar{u}_n(t)$, respectively. But these sequences are identical. Indeed, if $w - \bar{w} \in V$, then, as well known, the solutions of (3.4)–(3.6) (or of similar problems for the division d_n) are the same, independently of the choice of the function w or $\bar{w}$.

CHAPTER 5. APPLICATION OF THE RITZ METHOD. (VARIATIONAL-DIFFERENCE METHODS)

To approximate solving of problems of the type (3.4)–(3.6), direct variational methods can be used. We shall consider the Ritz method here, although other methods with similar properties can be investigated as well. As before, we assume V-ellipticity of the form $((v, u))$ as well as that (3.7) is fulfilled (implying symmetry of the form $((v, u))$).

Thus, consider the Rothe problems (3.4)–(3.6) which will be writen here in the form

$$((v, z_1)) + \frac{1}{h}(v, z_1 - z_0) = (v, f) , \quad z_1 - w \in V , \tag{5.1}$$

$$((v, z_2)) + \frac{1}{h}(v, z_2 - z_1) = (v, f) , \quad z_2 - w \in V , \tag{5.2}$$

. . . .

$$((v, z_p)) + \frac{1}{h}(v, z_p - z_{p-1}) = (v, f) , \quad z_p - w \in V , \tag{5.3}$$

(to be fulfilled for all $v \in V$; $z_0 = u_0$). Let $v_1, \ldots, v_i \ldots$ be a base in V. Solving the problem (5.1) by the Ritz method, choose a positive integer r_1^1 and denote by

$$z_{1,r_1^1}(x) = w + \sum_{i=1}^{r_1^1} a_{1,i} v_i(x) \tag{5.4}$$

the Ritz approximation of the solution z_1.[1]) Thus, $a_{1,i}$ are correspodning Ritz coefficients. (We should have denoted them by $a_{1,i}^{r_1^1}$ more precisely.) Putting z_{1,r_1^1} instead of z_1 into (5.2) and choosing r_2^1 ($r_2^1 = r_1^1$ is not excluded), we get similarly the Ritz approximation z_{2,r_1^2} of the solution z_2 of the problem (5.2) (with z_{1,r_1^1} substituted for z_1). Going on in this way, we come to the Ritz approximation z_{p,r_p^1} of the solution z_p of the problem (5.3) with z_{p-1,r_{p-1}^1} substituted for z_{p-1}.

Let us construct, similarly as in (1.9), the Ritz-Rothe function (corresponding to the division d_1)

$$u_{1,r_1^1,\dots,r_p^1}(x,t) = z_{j,r_j^1}(x) + \frac{t-t_j}{h}\left[z_{j+1,r_{j+1}^1}(x) - z_{j,r_j^1}(x)\right] \tag{5.5}$$

for

$$t_j \leqq t \leqq t_{j+1}\,, \quad j = 0, \dots, p-1\,, \quad z_{1,r_0^1} = 0\,.$$

Let us turn to the division d_n. We have

$$((v, z_1^n)) + \frac{1}{h_n}(v, z_1^n - z_0^n) = (v, f)\,, \quad z_1^n - w \in V\,, \tag{5.6}$$

$$((v, z_2^n)) + \frac{1}{h_n}(v, z_2^n - z_1^n) = (v, f)\,, \quad z_2^n - w \in V\,, \tag{5.7}$$

. . . .

$$((v, z_{p_n}^n)) + \frac{1}{h_n}(v, z_{p_n}^n - z_{p_n-1}^n) = (v, f)\,, \quad z_{p_n}^n - w \in V \tag{5.8}$$

(for all $v \in V$; $z_0^n = u_0$).

Solve, again these problems by the Ritz method, choosing positive integers $r_1^n, \dots$ $\dots, r_{p_n}^n$ and substituting z_1^n in (5.7) by the Ritz approximation z_{1,r_1^n} (cf. (5.4)), etc., and construct the corresponding Ritz-Rothe function

$$u_{n,r_1^n,\dots,r_{p_n}^n}(x,t)\,. \tag{5.9}$$

A question arises, of course, how "close" is this function to the very weak solution $u(t) = u(x,t)$ of the problem (3.1)–(3.3).

We show that to an arbitrary $\varepsilon > 0$ it is possible to find such n and such positive integers $r_1^n, \dots, r_{p_n}^n$ that

$$\|u(x,t) - u_{n,r_1^n,\dots,r_{p_n}^n}(x,t)\|_{L_2(Q)} < \varepsilon\,. \tag{5.10}$$

Having proved it, we say that the *Ritz-Rothe method for the problem* (3.1)–(3.3) *is convergent.*

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[1]) In what follows, theoretical considerations concerning the convergence of variational-difference methods take place. Practically, it is not necessary, of course, to use the "classical" Ritz procedure (5.4), where the function w is to be known.

To the proof we use the idea from [1]: First, in consequence of (4.18), to $\varepsilon/2$ it is possible to find such an m that for all $n > m$ we have

$$\|u(x, t) - u_n(x, t)\|_{L_2(Q)} < \frac{\varepsilon}{2} \tag{5.11}$$

where $u_n(x, t)$ is the Rothe function from Theorem 4.3. Thus it remains to prove that to $\varepsilon/2$ such positive integers $r_1^n, \ldots, r_{p_n}^n$ can be found that

$$\|u_n(x, t) - u_{n,r_1^n,\ldots,r_{p_n}^n}\|_{L_2(\Omega)} < \frac{\varepsilon}{2} \tag{5.12}$$

Because of the form of the functions $u_n(x, t)$, $u_{n,r_1^n,\ldots,r_{p_n}^n}$ (they are piecewise linear in t) it is sufficient to prove that

$$\|z_j^n(x) - z_{n,r_j^n}(x)\|_{L_2(\Omega)} < \frac{\varepsilon}{2\sqrt{T}} \tag{5.13}$$

for every $j = 1, \ldots, p_n$.

Let r_1^n be sufficiently large in order that

$$\|z_1^n(x) - z_{1,r_1^n}(x)\|_{L_2(\Omega)} < \delta\,. \tag{5.14}$$

(This can be always reached, even in V.) Put z_{1,r_1^n} instead of z_1^n into (5.7) and denote the solution of this problem by $\bar{z}_2^n$. Thus, $\bar{z}_2^n$ solves the problem

$$((v, \bar{z}_2^n)) + \frac{1}{h_n}(v, \bar{z}_2^n - z_{1,r_1^n}) = (v, f)\,, \quad \bar{z}_2^n - w \in V,\ v \in V\,. \tag{5.15}$$

Substract (5.15) from (5.7) and put $v = z_2^n - \bar{z}_2^n$. (This is possible, because $z_2^n - \bar{z}_2^n \in V$.) We get

$$(z_2^n - \bar{z}_2^n, z_2^n - \bar{z}_2^n - (z_1 - z_{1,r_1^n})) \leqq 0$$

and, consequently,

$$\|z_2^n - \bar{z}_2^n\|_{L_2(\Omega)} \leqq \|z_1^n - z_{1,r_1^n}\|_{L_2(\Omega)} < \delta\,. \tag{5.16}$$

Let r_2^n be sufficiently large, so that

$$\|\bar{z}_2^n - z_{2,r_2^n}\|_{L_2(\Omega)} < \delta\,, \tag{5.17}$$

replace z_2^n in the third of the problems (5.6), (5.7) by z_{2,r_2^n} and denote the solution of this problem by $\bar{z}_3^n$. Because of (5.16), (5.17) we have

$$\|z_2^n - z_{2,r_2^n}\|_{L_2(\Omega)} < 2\delta\,,$$

so that we get, similarly as in (5.16),

$$\|z_3^n - \bar{z}_3^n\|_{L_2(\Omega)} < 2\delta\,.$$

Going on in this way, we get, choosing $r_3^n, r_4^n, \ldots$ sufficiently large,

$$\|z_j^n - z_{n,r_j^n}\|_{L_2(\Omega)} < j\delta\,, \quad j = 1, \ldots, p_n\,.$$

To fulfill (5.13) it is thus sufficient to choose

$$\delta < \frac{\varepsilon}{2p_n \sqrt{T}}.$$

In this way, (5.10), and thus the required convergence theorem is proved:

Theorem 5.1. *The Ritz-Rothe method for the problem* (3.1)–(3.3) *is convergent.*

Remark 5.1. It is easy to see that the function $w_2(x)$ from the decomposition (3.8), which plays a significant role in theoretical considerations, does not appear in the numerical process (when applying the Ritz-Rothe method) at all.

Remark 5.2. The ideas of this paper can be well utilized in considering more general problems.

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Souhrn

POZNÁMKA K NEHOMOGENNÍM POČÁTEČNÍM A OKRAJOVÝM PODMÍNKÁM V PARABOLICKÝCH PROBLÉMECH, ŘEŠENÝCH ROTHEHO METODOU

KAREL REKTORYS a MARIE LUDVÍKOVÁ

Při řešení parabolických problémů Rotheho metodou (viz K. Rektorys [1]) činí po teoretické stránce určité obtíže nehomogenní počáteční a okrajové podmínky. Tyto potíže se řeší zpravidla tím, že se vysloví některé dodatečné předpoklady, týkající se příslušné bilineární formy a počátečních i okrajových funkcí (srov. [1], [2], [3] atd.).

V tomto článku je ukázáno, jak lze tyto dodatečné předpoklady odstranit (v případě počátečních podmínek – kap. 2), resp. nahradit jednoduššími a přirozenějšími předpoklady (v případě okrajových podmínek – kap. 3; viz zejména předpoklad (3.7), str. 64).

V závěru článku se zkoumá použití Ritzovy metody (resp. příbuzných přímých metod) k přibližnému řešení vzniklých eliptických problémů.

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