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THE EUCLIDEAN PLANE KINEMATICS

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In his paper [1], A. Karger devoted himself to the study of the kinematic geometry in homogeneous spaces, the groups of motions of which are special 3-dimensional Lie groups. In what follows, I restrict myself to the Euclidean plane, but I am going to show a method leading to the solution of the equivalence problem for all Lie groups of motions. Besides this, I present all transitive one-parametric systems of motions in E^2.

Let E^2 denote the Euclidean plane, let G be the Lie group of direct isometries of E^2 onto itself. By a motion we will understand a mapping $g : J \to G$, $J \subseteq \mathbb{R}$ being an interval. Our first problem may be formulated as follows: Let $g, \tilde{g} : J \to G$ be two motions; we have to find conditions which ensure the equivalence of g, $\tilde{g}$, i.e., the existence of a direct isometry $\sigma : E^2 \to E^2$ such that $\sigma \circ g(t) = \tilde{g}(t)$ for each $t \in J$. We will solve this problem by constructing an invariant parameter and two "curvatures" on g; the desired condition will be then reduced to the equality of invariant parameters and curvatures. As a by-product, we will present a complete list of motions with constant curvatures.

Let $\{M, e_1, e_2\}$ be an orthonormal frame in E^2. Then a general direct isometry $g \in G$ is given by

$$(1) \qquad \begin{aligned} g(M) &= M + ae_1 + be_2 \,, \\ g(e_1) &= \cos\alpha \,.\, e_1 - \sin\alpha \,.\, e_2 \,, \\ g(e_2) &= \sin\alpha \,.\, e_1 + \cos\alpha \,.\, e_2 \,. \end{aligned}$$

Thus we may identify the group G with the subgroup of $\mathrm{GL}(3, \mathbb{R})$ of matrices of the form

$$(2) \qquad g = \begin{pmatrix} 1 & a & b \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}.$$

The Lie algebra $\mathfrak{G}$ is then, as is easy to see, identified with the Lie algebra of matrices of the form

(3) $$V = \begin{pmatrix} 0 & \mathrm{d}a/\mathrm{d}t & \mathrm{d}b/\mathrm{d}t \\ 0 & 0 & -\mathrm{d}\alpha/\mathrm{d}t \\ 0 & \mathrm{d}\alpha/\mathrm{d}t & 0 \end{pmatrix};$$

let

(4) $$V_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

be its basis. Then

(5) $$[V_1, V_2] = 0, \quad [V_1, V_3] = -V_2, \quad [V_2, V_3] = V_1.$$

Now, let $g : J \to G$, $g = g(t)$ be a motion. Then we get a mapping $v : J \to \mathfrak{G}$ given by

(6) $$v(t) = g(t)^{-1} \frac{\mathrm{d}g(t)}{\mathrm{d}t}.$$

Taking into consideration (1), the motion $g(t)$ is represented by the equations

(7) $$\alpha = \alpha(t), \quad a = a(t), \quad b = b(t); \quad t \in J.$$

It is easy to see that

(8) $$v(t) = \left(\frac{\mathrm{d}a}{\mathrm{d}t} - b\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right) V_1 + \left(\frac{\mathrm{d}b}{\mathrm{d}t} + a\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right) V_2 + \frac{\mathrm{d}\alpha}{\mathrm{d}t} V_3.$$

With the second motion $\tilde{g} : J \to G$, we associate similarly the mapping $\tilde{v}$. Let In $(\mathfrak{G})$ denote the Lie group of inner automorphisms of $\mathfrak{G}$. Then it is known that g and $\tilde{g}$ are equivalent if and only if there is a $\Gamma_0 \in \mathrm{In}(\mathfrak{G})$ such that $\tilde{v}(t) = \Gamma_0\{v(t)\}$ for each $t \in J$. Because the motion $g(t)$ is given, up to the equivalence, by $v(t)$, our problem reduces to the equivalence problem for two mappings $v, \tilde{v} : J \to \mathfrak{G}$ with respect to the group In $(\mathfrak{G})$. We are now going to solve this problem even with respect to a more general group Aut $(\mathfrak{G})$ of all automorphisms of $\mathfrak{G}$.

Let $\Gamma \in \mathrm{Aut}(\mathfrak{G})$ be given by

(9) $$\begin{aligned} \Gamma(V_1) &= a_1 V_1 + a_2 V_2 + a_3 V_3, \\ \Gamma(V_2) &= b_1 V_1 + b_2 V_2 + b_3 V_3, \\ \Gamma(V_3) &= c_1 V_1 + c_2 V_2 + c_3 V_3. \end{aligned}$$

From

$$(10)\qquad [\Gamma(V_1), \Gamma(V_2)] = 0\,, \quad [\Gamma(V_1), \Gamma(V_3)] = -\Gamma(V_2)\,,$$
$$[\Gamma(V_2), \Gamma(V_3)] = \Gamma(V_1)$$

we see that Aut $(\mathfrak{G}) \subset \mathrm{GL}(3, \mathbb{R})$ may be identified with the set of all matrices of the form

$$(11)\qquad \begin{pmatrix} a_1 & a_2 & 0 \\ -a_2 & a_1 & 0 \\ c_1 & c_2 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_1 & a_2 & 0 \\ a_2 & -a_1 & 0 \\ c_1 & c_2 & 1 \end{pmatrix} \quad \text{with} \quad a_1^2 + a_2^2 \neq 0\,.$$

Of course, the Lie algebra $\mathfrak{Aut}(\mathfrak{G}) \subset \mathfrak{GL}(3, \mathbb{R})$ of Aut $(\mathfrak{G})$ is the set of all matrices of the form

$$(12)\qquad \begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ -\alpha_2 & \alpha_1 & 0 \\ \beta_1 & \beta_2 & 0 \end{pmatrix}.$$

It is known that the Lie algebra $\mathfrak{In}(\mathfrak{G})$ is the set of all homomorphisms ad $u : \mathfrak{G} \to \mathfrak{G}$, $u \in \mathfrak{G}$. For $u = u^1 V_1 + u^2 V_2 + u^3 V_3$, we have

$$(13)\qquad \mathrm{ad}(u)(x^1 V_1 + x^2 V_2 + x^3 V_3) = x^1 u^3 V_2 - x^2 u^3 V_1 - x^3(u^1 V_2 - u^2 V_1)\,,$$

i.e., we may identify $\mathfrak{In}(\mathfrak{G}) \subset \mathfrak{Aut}(\mathfrak{G})$ with the set of all matrices of the form

$$(14)\qquad \begin{pmatrix} 0 & u^3 & 0 \\ -u^3 & 0 & 0 \\ u^2 & -u^1 & 0 \end{pmatrix}.$$

From this (or by a direct calculation) we get that In $(\mathfrak{G}) \subset$ Aut $(\mathfrak{G})$ is the group of all matrices (11) with $a_1^2 + a_2^2 = 1$, i.e., its identity component is the group of all matrices of the form

$$(15)\qquad \begin{pmatrix} \cos\psi_1 & \sin\psi_1 & 0 \\ -\sin\psi_1 & \cos\psi_1 & 0 \\ \psi_2 & \psi_3 & 1 \end{pmatrix}.$$

First of all, let us study the mapping $v : J \to \mathfrak{G}$ with respect to the group In $(\mathfrak{G})$ acting on $\mathfrak{G}$. By a frame of $\mathfrak{G}$ (with respect to this group) we shall mean each triple $\{v_1, v_2, v_3\}$ such that $v_i = \Gamma_0(V_i)$ for a $\Gamma_0 \in \mathrm{In}(\mathfrak{G})$ and $i = 1, 2, 3$. If $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ are two frames, then there are φ, λ, μ such that

$$(16)\qquad \begin{aligned} w_1 &= \cos\varphi \,.\, v_1 + \sin\varphi \,.\, v_2\,, \\ w_2 &= -\sin\varphi \,.\, v_1 + \cos\varphi \,.\, v_2\,, \\ w_3 &= \lambda \,.\, v_1 + \mu \,.\, v_2 + v_3\,. \end{aligned}$$

Let $\{v_1(t), v_2(t), v_3(t)\}$, $t \in J$ be a field of frames. Then there are functions $\alpha_i^j(t)$ such that

$$\frac{\mathrm{d}v_i(t)}{\mathrm{d}t} = \alpha_i^j(t) \, . \, v_j(t) \, , \tag{17}$$

but (14) implies that (17) should have the form

$$\frac{\mathrm{d}v_1}{\mathrm{d}t} = \beta v_2 \, , \quad \frac{\mathrm{d}v_2}{\mathrm{d}t} = -\beta v_1 \, , \quad \frac{\mathrm{d}v_3}{\mathrm{d}t} = \gamma_1 v_1 + \gamma_2 v_2 \, . \tag{18}$$

Consider $v = v(t)$, $t \in J$, and suppose of course $v(t) \neq 0$ for each $t \in J$. With each t, let us associate a frame $\{v_1(t), v_2(t), v_3(t)\}$ such that $v_3(t)$ and $v(t)$ are dependent; this can always be achieved because of (15) by a suitable choice of ψ_2, ψ_3. Any other field of frames $\{w_1(t), w_2(t), w_3(t)\}$ with the same property is given by

$$\begin{aligned} w_1 &= \cos\varphi \, . \, v_1 + \sin\varphi \, . \, v_2 \, , \\ w_2 &= -\sin\varphi \, . \, v_1 + \cos\varphi \, . \, v_2 \, , \\ w_3 &= v_3 \, . \end{aligned} \tag{19}$$

For $\{w_1(t), w_2(t), w_3(t)\}$, we may write equations similar to (18):

$$\frac{\mathrm{d}w_1}{\mathrm{d}t} = \tilde{\beta} w_2 \, , \quad \frac{\mathrm{d}w_2}{\mathrm{d}t} = -\tilde{\beta} w_1 \, , \quad \frac{\mathrm{d}w_3}{\mathrm{d}t} = \tilde{\gamma}_1 w_1 + \tilde{\gamma}_2 w_2 \, . \tag{20}$$

From the equations (18)–(20) we obtain

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} . \sin\varphi + \beta \, . \sin\varphi = \tilde{\beta} \, . \sin\varphi \, , \quad \frac{\mathrm{d}\varphi}{\mathrm{d}t} . \cos\varphi + \beta \, . \cos\varphi = \tilde{\beta} \, . \cos\varphi \, , \tag{21}$$

$$\tilde{\gamma}_1 \, . \cos\varphi - \tilde{\gamma}_2 \, . \sin\varphi = \gamma_1 \, , \quad \tilde{\gamma}_1 \, . \sin\varphi + \tilde{\gamma}_2 \, . \cos\varphi = \gamma_2 \, .$$

From $(21_{3,4})$ we get

$$\gamma_1^2 + \gamma_2^2 = \tilde{\gamma}_1^2 + \tilde{\gamma}_2^2 \, , \tag{22}$$

and $\gamma_1^2 + \gamma_2^2$ is thus an invariant of our system of frames. We have to distinguish two cases. Let us, first of all, consider the case

$$\gamma_1 = \gamma_2 = 0 \, . \tag{23}$$

The equations $(21_{1,2})$ reduce to

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} + \beta = \tilde{\beta} \, , \tag{24}$$

and we may choose, at least locally, $\varphi(t)$ in such a way that $\tilde{\beta} = 0$. Thus we may associate with $v(t)$ a field of frames such that

(25) $$\frac{dv_1}{dt} = 0\,, \quad \frac{dv_2}{dt} = 0\,, \quad \frac{dv_3}{dt} = 0\,,$$

i.e., $v(t)$ reduces to a fixed line and $g(t)$ may be parametrized in such a way that it becomes a one-parametric subgroup of G. Let us suppose

(26) $$\gamma_1^2 + \gamma_2^2 \neq 0\,.$$

It is easy to see that we may then choose the frames restricted by the condition

(27) $$\gamma_2 = 0\,;$$

from (21) we get

(28) $$\tilde{\beta} = \beta\,, \quad \tilde{\gamma}_1 = \gamma_1\,,$$

β and γ_1 being thus the invariants of $v(t)$. From (19) we see that the condition (27) associates with $v(t)$ just one field of frames.

Let us now determine explicitly the invariants of our initial motion (7). For

(29) $$A(t) = \frac{da}{dt} - b\frac{d\alpha}{dt}\,, \quad B(t) = \frac{db}{dt} + a\frac{d\alpha}{dt}\,, \quad C(t) = \frac{d\alpha}{dt}\,,$$

(8) takes the form

(30) $$v(t) = A(t)\,.\,V_1 + B(t)\,.\,V_2 + C(t)\,.\,V_3\,.$$

It is not necessary to study the case $C(t) = 0$; in this case $\alpha =$ constant, and our motion consists just of a group of translations. Thus we may suppose

(31) $$C(t) \neq 0 \quad \text{for} \quad t \in J\,.$$

The general field of frames associated with $v(t)$ is then

(32) $$\begin{aligned} v_1(t) &= \cos\xi(t)\,.\,V_1 + \sin\xi(t)\,.\,V_2\,,\\ v_2(t) &= -\sin\xi(t)\,.\,V_1 + \cos\xi(t)\,.\,V_2\,,\\ v_3(t) &= A(t)\,.\,C(t)^{-1}\,.\,V_1 + B(t)\,.\,C(t)^{-1}\,.\,V_2 + V_3 \end{aligned}$$

and we get

(33) $$\frac{dv_1}{dt} = \frac{d\xi}{dt}\,.\,v_2\,, \quad \frac{dv_2}{dt} = -\frac{d\xi}{dt}\,.\,v_1\,,$$

$$\frac{dv_3}{dt} = \left\{\frac{d}{dt}(AC^{-1})\,.\cos\xi + \frac{d}{dt}(BC^{-1})\,.\sin\xi\right\}v_1 + \\ + \left\{-\frac{d}{dt}(AC^{-1})\,.\sin\xi + \frac{d}{dt}(BC^{-1})\,.\cos\xi\right\}v_2\,.$$

According to our previous discussion, we may choose $\xi(t)$ in such a way that

$$\frac{d}{dt}(AC^{-1}) . \sin \xi - \frac{d}{dt}(BC^{-1}) . \cos \xi = 0 , \tag{34}$$

the functions

$$\frac{d\xi}{dt}, \quad \frac{d}{dt}(AC^{-1}) . \cos \xi + \frac{d}{dt}(BC^{-1}) . \sin \xi \tag{35}$$

being then the invariants of our motion. For $g(t)$ we may introduce a canonical parameter by the condition $v(t) = v_3(t)$, i.e., $C(t) = 1$; from (29) we see that $t =$ $= \alpha +$ const. From now on, let us use the canonical parameter α. Define ϱ by

$$\varrho = \left\{ \left(\frac{dA}{d\alpha} \right)^2 + \left(\frac{dB}{d\alpha} \right)^2 \right\}^{1/2} . \tag{36}$$

The case $\varrho = 0$ corresponds to (23). Therefore, let us suppose $\varrho \neq 0$. To get the invariants, we have to choose

$$\sin \xi = \varrho^{-1} . \frac{dB}{d\alpha}, \quad \cos \xi = \varrho^{-1} . \frac{dA}{d\alpha} \tag{37}$$

and we see that the invariants (35) take the final form

$$J_1 = \left\{ \left(\frac{d}{d\alpha} \left(\varrho^{-1} . \frac{dB}{d\alpha} \right) \right)^2 + \left(\frac{d}{d\alpha} \left(\varrho^{-1} . \frac{dA}{d\alpha} \right) \right)^2 \right\}^{1/2} , \tag{38}$$

$$J_2 = \left\{ \left(\frac{dA}{d\alpha} \right)^2 + \left(\frac{dB}{d\alpha} \right)^2 \right\}^{1/2} .$$

Let us turn our attention to the study of motions with constant invariants. First of all, let us study the case $\varrho = J_2 = 0$.

Introduce the functions $r, s : J \to \mathbb{C}$ by

$$r(\alpha) = a(\alpha) + i\, b(\alpha) , \quad s(\alpha) = \frac{dr(\alpha)}{d\alpha} + i\, r(\alpha) . \tag{39}$$

Then it is easy to see that

$$J_2 = \left| \frac{ds}{d\alpha} \right| . \tag{40}$$

$J_2 = 0$ implies the existence of $\varkappa_1 \in \mathbb{C}$ such that

$$\frac{dr}{d\alpha} + ir = \varkappa_1 . \tag{41}$$

The general solution of this equation is then

$$r = \varkappa_2 \, \mathrm{e}^{-i\alpha} - i\varkappa_1 \,, \quad \varkappa_2 \in \mathbb{C} \,. \tag{42}$$

In E^2, introduce the coordinates (x, y) of a point X with respect to the basis $\{M, e_1, e_2\}$ by means of $X = M + xe_1 + ye_2$. By $g \in G$ (1), this point is mapped onto the point $g(X) = g(M) + xg(e_1) + yg(e_2)$; let it have coordinates x^*, y^*. Then

$$x^* = \cos\alpha \,.\, x + \sin\alpha \,.\, y + a \,, \quad y^* = -\sin\alpha \,.\, x + \cos\alpha \,.\, y + b \,, \tag{43}$$

i.e.,

$$z^* = e^{-i\alpha} \,.\, z + r \,; \quad z = x + iy \,, \quad z^* = x^* + iy^* \,. \tag{44}$$

Thus our $g(\alpha)$ associates with the point z the point

$$z^* = e^{-i\alpha}(z + \varkappa_2) - i\varkappa_1 \,. \tag{45}$$

Suppose that the canonical parameter is chosen in such a way that $g(0)$ is the identity mapping. Then $\varkappa_2 = i\varkappa_1$ and the motion is given by $(\varkappa = \varkappa_2)$

$$z^* = e^{-i\alpha}(z + \varkappa) - \varkappa \,. \tag{46}$$

Hence

$$|z^* + \varkappa| = |z + \varkappa| \tag{47}$$

and we see that the motion considered is just the rotation around the point $-\varkappa$.

Let us now consider the case

$$J_2 = \text{constant} \neq 0 \,. \tag{48}$$

It is not difficult to see that (48) implies

$$J_1 = J_2^{-1} \left|\frac{\mathrm{d}^2 s}{\mathrm{d}\alpha^2}\right| \,. \tag{49}$$

First of all, let

$$J_1 = 0 \,, \tag{50}$$

i.e.,

$$s = \varrho_1 \alpha + \varrho_2 \,; \quad \varrho_1, \varrho_2 \in \mathbb{C} \,. \tag{51}$$

The general solution of the differential equation

$$\frac{\mathrm{d}r}{\mathrm{d}\alpha} + ir = \varrho_1 \alpha + \varrho_2 \tag{52}$$

is

$$r = \varrho_3 \, e^{-i\alpha} - i\varrho_1 \alpha + \varrho_1 - i\varrho_2 \,; \quad \varrho_3 \in \mathbb{C} \,; \tag{53}$$

and our motion is given by (44), i.e.,

$$z^* = e^{-i\alpha}(z + \varrho_3) - i\varrho_1\alpha + \varrho_1 - i\varrho_2 . \tag{54}$$

Let us suppose $g(0)$ to be identity. Then $\varrho_3 + \varrho_1 - i\varrho_2 = 0$ and (54) reduces to

$$z^* = e^{-i\alpha}(z + \varrho) - i\varrho_1\alpha - \varrho\,; \quad \varrho = \varrho_3 \in \mathbb{C}\,; \tag{55}$$

of course, $J_2 = |\varrho_1|$. Let us calculate the centroids of (55). The fixed centroid is the set of points $z_0(\alpha)$ such that $(\mathrm{d}z^*/\mathrm{d}\alpha) = 0$, i.e., it is given by $z_0 = -\varrho_1 e^{i\alpha} - \varrho$. Because of $|z_0 + \varrho| = |\varrho_1|$, we see that it is a circle with the center $-\varrho$ and the radius J_2. The moving centroid is then given by $z^* = e^{-i\alpha}(z_0 + \varrho) - i\varrho_1\alpha - \varrho = -i\varrho_1\alpha - \varrho_1 - \varrho$, and it is the straight line with the equation

$$\bar{\varrho}_1 z + \varrho_1 \bar{z} + \varrho\bar{\varrho}_1 + \varrho_1\bar{\varrho} + 2\varrho_1\bar{\varrho}_1 = 0 . \tag{56}$$

Thus our motion is produced by rolling a straight line upon a circle of radius J_2.

Let us turn our attention to the case

$$J_1 = \text{constant} \neq 0\,, \quad J_2 = \text{constant} \neq 0 . \tag{57}$$

The function $s(\alpha)$ should then be the solution of the differential equations (see (49))

$$\left|\frac{\mathrm{d}s}{\mathrm{d}\alpha}\right| = J_2\,, \quad \left|\frac{\mathrm{d}^2 s}{\mathrm{d}\alpha^2}\right| = J_1 J_2 . \tag{58}$$

From (58_1) we get

$$\frac{\mathrm{d}s}{\mathrm{d}\alpha} \cdot \frac{\mathrm{d}^2\bar{s}}{\mathrm{d}\alpha^2} + \frac{\mathrm{d}\bar{s}}{\mathrm{d}\alpha} \cdot \frac{\mathrm{d}^2 s}{\mathrm{d}\alpha^2} = 0 . \tag{59}$$

Multiplying this by

$$\frac{\mathrm{d}s}{\mathrm{d}\alpha} \cdot \frac{\mathrm{d}^2 s}{\mathrm{d}\alpha^2}$$

and inserting from (58), we have

$$J_2^2 \left(\frac{\mathrm{d}^2 s}{\mathrm{d}\alpha^2}\right)^2 + J_1^2 J_2^2 \left(\frac{\mathrm{d}s}{\mathrm{d}\alpha}\right)^2 = 0\,, \tag{60}$$

i.e.,

$$\frac{\mathrm{d}^2 s}{\mathrm{d}\alpha^2} = -\varepsilon i J_1 \cdot \frac{\mathrm{d}s}{\mathrm{d}\alpha}\,; \quad \varepsilon = \pm 1 . \tag{61}$$

The general solution of this equation is

$$s = \varepsilon i \varkappa_1 J_1^{-1} e^{-\varepsilon i J_1 \alpha} + \varkappa_2\,, \quad \varkappa_1, \varkappa_2 \in \mathbb{C} . \tag{62}$$

From (58_1) we obtain

(63) $$|\varkappa_1| = J_2 .$$

Now, the general solution of $(\mathrm{d}r/\mathrm{d}\alpha) + ir = s$ (see (39)) is

(64) $$r = \varepsilon\varkappa_1 J_1^{-1}(1 - \varepsilon J_1)^{-1} e^{-\varepsilon i J_1 \alpha} + \varkappa_3 e^{-i\alpha} - i\varkappa_2 ; \quad \varkappa_3 \in \mathbb{C}$$

in the case

(65) $$\varepsilon J_1 \neq 1 ;$$

the case $\varepsilon J_1 = 1$ will be considered later on. The motion $g(\alpha)$ is then (see (44))

(66) $$z^* = (z + \varkappa_3) e^{-i\alpha} + \varepsilon\varkappa_1 J_1^{-1}(1 - \varepsilon J_1)^{-1} e^{-\varepsilon i J_1 \alpha} - i\varkappa_2 .$$

The condition $g(0) =$ identity implies $\varkappa_3 - i\varkappa_2 + \varepsilon\varkappa_1 J_1^{-1}(1 - \varepsilon J_1)^{-1} = 0$, and we get

(67) $$z^* = (z + \varkappa_3) e^{-i\alpha} + \varepsilon\varkappa_1 J_1^{-1}(1 - \varepsilon J_1)^{-1} (e^{-\varepsilon i J_1 \alpha} - 1) - \varkappa_3 .$$

It is easy to see that the fixed centroid is given by

(68) $$z + \varkappa_3 = -\varkappa_1(1 - \varepsilon J_1)^{-1} e^{i(1 - \varepsilon J_1)\alpha} ,$$

and it is a circle C_F, its center S_F and its radius R_F being given by

(69) $$S_F = -\varkappa_3 , \quad R_F = J_2 |1 - \varepsilon J_1|^{-1} .$$

The moving centroid is then

(70) $$z + \varepsilon\varkappa_1 J_1^{-1}(1 - \varepsilon J_1)^{-1} + \varkappa_3 = \varepsilon\varkappa_1 J_1^{-1} e^{-\varepsilon i J_1 \alpha} ;$$

it is a circle C_M with

(71) $$S_M = -\varkappa_3 - \varepsilon\varkappa_1 J_1^{-1}(1 - \varepsilon J_1)^{-1} , \quad R_M = J_1^{-1} J_2 .$$

It remains to consider the case $\varepsilon J_1 = 1$. Because of $J_1 > 0$, we have

(72) $$J_1 = 1 , \quad \varepsilon = 1 ,$$

and (62) reduces to

(73) $$s = i\varkappa_1 e^{-i\alpha} + \varkappa_2 , \quad \varkappa_1, \varkappa_2 \in \mathbb{C} .$$

Hence

(74) $$r = (i\varkappa_1 \alpha + \varkappa_3) e^{-i\alpha} - i\varkappa_2$$

and

(75) $$z^* = (z + i\varkappa_1 \alpha + \varkappa_3) e^{-i\alpha} - i\varkappa_2 .$$

The condition $g(0)$ = identity implies $\varkappa_3 - i\varkappa_2 = 0$, i.e.,

$$z^* = (z + i\varkappa_1\alpha + \varkappa_3)\, e^{-i\alpha} - \varkappa_3 \,. \tag{76}$$

The fixed centroid is then

$$z = -i\varkappa_1\alpha + \varkappa_1 - \varkappa_3 \,, \tag{77}$$

i.e., a straight line with the equation (see (63))

$$\bar{\varkappa}_1 z + \varkappa_1 \bar{z} - 2J_2^2 + \varkappa_1\bar{\varkappa}_3 + \bar{\varkappa}_1\varkappa_3 = 0 \,. \tag{78}$$

The moving centroid is

$$z = \varkappa_1\, e^{-i\alpha} - \varkappa_3 \,, \tag{79}$$

and it is a circle with $S_M = -\varkappa_3$, $R_M = J_2$.

The summary of our results is contained in the following

Theorem 1. *Let $g(\alpha)$ be a motion in E^2 with constant invariants J_1, J_2. Then it is the so-called planetary motion, i.e., it is produced by rolling a circle C_M upon a fixed circle C_F. In the case $J_2 = 0$ both circles degenerate to one point, and $g(\alpha)$ is just the rotation around this point. In the case $J_1 = 0$, $J_2 \neq 0$, C_M becomes a straight line; in the case $J_1 = \varepsilon = 1$, $J_2 \neq 0$, C_F is a straight line.*

Finally, let us study the invariants of our motion with respect to the group Aut $(\mathfrak{G})$. By a frame we call now each triple $\{v_i\}$ such that $v_i = \Gamma(V_i)$, $\Gamma \in$ Aut $(\mathfrak{G})$. With $v(t)$ let us associate a frame $\{v_1(t), v_2(t), v_3(t)\}$ such that $v_3(t)$ and $v(t)$ are dependent. Then (see (12))

$$\frac{dv_1}{dt} = \alpha_1 v_1 + \alpha_2 v_2 \,, \quad \frac{dv_2}{dt} = -\alpha_2 v_1 + \alpha_1 v_2 \,, \quad \frac{dv_3}{dt} = \beta_1 v_1 + \beta_2 v_2 \,, \tag{80}$$

and the possible changes of the frames are given by

$$w_1 = a_1 v_1 + a_2 v_2 \,, \quad w_2 = -a_2 v_1 + a_1 v_2 \,, \quad w_3 = v_3 \,; \quad a_1^2 + a_2^2 \neq 0 \,. \tag{81}$$

For

$$\frac{dw_1}{dt} = \tilde{\alpha}_1 w_1 + \tilde{\alpha}_2 w_2 \,, \quad \frac{dw_2}{dt} = -\tilde{\alpha}_2 w_1 + \tilde{\alpha}_1 w_2 \,, \quad \frac{dw_3}{dt} = \tilde{\beta}_1 w_1 + \tilde{\beta}_2 w_2 \tag{82}$$

we get

$$\frac{da_1}{dt} + a_1\alpha_1 - a_2\alpha_2 = a_1\tilde{\alpha}_1 - a_2\tilde{\alpha}_2 \,, \quad \frac{da_2}{dt} + a_2\alpha_1 + a_1\alpha_2 = a_2\tilde{\alpha}_1 + a_1\tilde{\alpha}_2 \,, \tag{83}$$

$$\beta_1 = a_1\tilde{\beta}_1 - a_2\tilde{\beta}_2 \,, \quad \beta_2 = a_2\tilde{\beta}_1 + a_1\tilde{\beta}_2 \,.$$

Hence

(84) $$\beta_1^2 + \beta_2^2 = (a_1^2 + a_2^2)(\tilde{\beta}_1^2 + \tilde{\beta}_2^2)$$

and we have to distinguish two cases. In the first case

(85) $$\beta_1 = \beta_2 = 0$$

and $v(t)$ is situated in a fixed straight line; we have seen that this leads to a rotation. In the general case $\beta_1^2 + \beta_2^2 \neq 0$ and we may achieve

(86) $$\beta_1 = 1\,, \quad \beta_2 = 0\,.$$

These conditions determine the frames $\{v_i(t)\}$ uniquely, and $\alpha_1(t)$, $\alpha_2(t)$ are the invariants of our motion. Let us suppose (30) and

(87) $$v_1(t) = a_1 V_1 + a_2 V_2\,, \quad v_2(t) = -a_2 V_1 + a_1 V_2\,,$$
$$v_3(t) = AC^{-1} . V_1 + BC^{-1} . V_2 + V_3\,.$$

Then

(88) $$\frac{dv_1}{dt} = (a_1^2 + a_2^2)^{-1}\left\{\left(a_1 \frac{da_1}{dt} + a_2 \frac{da_2}{dt}\right) v_1 + \left(a_1 \frac{da_2}{dt} - a_2 \frac{da_1}{dt}\right) v_2\right\},$$
$$\frac{dv_2}{dt} = (a_1^2 + a_2^2)^{-1}\left\{\left(a_2 \frac{da_1}{dt} - a_1 \frac{da_2}{dt}\right) v_1 + \left(a_1 \frac{da_1}{dt} + a_2 \frac{da_2}{dt}\right) v_2\right\},$$
$$\frac{dv_3}{dt} = (a_1^2 + a_2^2)^{-1}\left\{a_1 \frac{d(AC^{-1})}{dt} + a_2 \frac{d(BC^{-1})}{dt}\right\} v_1 +$$
$$+ (a_1^2 + a_2^2)^{-1}\left\{-a_2 \frac{d(AC^{-1})}{dt} + a_1 \frac{d(BC^{-1})}{dt}\right\} v_2\,.$$

In the general case

(89) $$\left(\frac{d(AC^{-1})}{dt}\right)^2 + \left(\frac{d(BC^{-1})}{dt}\right)^2 \neq 0\,;$$

of course, we are going to study just this case. The conditions (86) determine a_1, a_2 and it is easy to see that the invariants of our motion are given by

(90) $$\alpha_1 = \frac{d}{dt} \log \left\{\left(\frac{d(AC^{-1})}{dt}\right)^2 + \left(\frac{d(BC^{-1})}{dt}\right)^2\right\}^{1/2},$$
$$\alpha_2 = \left\{\left(\frac{d(AC^{-1})}{dt}\right)^2 + \left(\frac{d(BC^{-1})}{dt}\right)^2\right\}^{-1} .$$
$$.\left\{\frac{d(AC^{-1})}{dt} . \frac{d^2(BC^{-1})}{dt^2} - \frac{d(BC^{-1})}{dt} . \frac{d^2(AC^{-1})}{dt^2}\right\}.$$

Let the canonical parameter be, as above, introduced by the condition $v_3(t) = v(t)$; we see that α + constant is the set of canonical parameters. Using (39), a simple calculation yields

$$\alpha_1 = \frac{d}{d\alpha} \log \left|\frac{ds}{d\alpha}\right|, \quad \alpha_2 = \frac{1}{2} i \left|\frac{ds}{d\alpha}\right|^{-2} \left(\frac{ds}{d\alpha} \cdot \frac{d^2\bar{s}}{d\alpha^2} - \frac{d\bar{s}}{d\alpha} \cdot \frac{d^2 s}{d\alpha^2}\right). \tag{91}$$

Let us determine the motions with constant invariants. From (91_1) we have

$$\frac{ds}{d\alpha} \cdot \frac{d\bar{s}}{d\alpha} = c_1 e^{2\alpha_1 \alpha}; \quad c_1 > 0 \tag{92}$$

and

$$\frac{ds}{d\alpha} \cdot \frac{d^2\bar{s}}{d\alpha^2} + \frac{d\bar{s}}{d\alpha} \cdot \frac{d^2 s}{d\alpha^2} = 2c_1\alpha_1 e^{2\alpha_1 \alpha}. \tag{93}$$

(91_2) implies

$$\frac{ds}{d\alpha} \cdot \frac{d^2\bar{s}}{d\alpha^2} - \frac{d\bar{s}}{d\alpha} \cdot \frac{d^2 s}{d\alpha^2} = -2ic_1\alpha_2 e^{2\alpha_1 \alpha} \tag{94}$$

and we get

$$\frac{d^2 s}{d\alpha^2} = (\alpha_1 + i\alpha_2) \frac{ds}{d\alpha}, \tag{95}$$

i.e., in the case $\alpha_1^2 + \alpha_2^2 \neq 0$,

$$s = c_2 e^{(\alpha_1 + i\alpha_2)\alpha} + c_3; \quad c_2, c_3 \in \mathbb{C}; \tag{96}$$

of course,

$$c_1 = |c_2|^2 (\alpha_1^2 + \alpha_2^2). \tag{97}$$

Thus we have to solve the equation

$$\frac{dr}{d\alpha} + ir = c_2 e^{(\alpha_1 + i\alpha_2)\alpha} + c_3. \tag{98}$$

Suppose

$$\alpha_1 + i(\alpha_2 + 1) \neq 0. \tag{99}$$

The general solution of (98) is then

$$r = c_2(\alpha_1 + i\alpha_2 + i)^{-1} e^{(\alpha_1 + i\alpha_2)\alpha} + c_4 e^{-i\alpha} - ic_3; \quad c_4 \in \mathbb{C}. \tag{100}$$

Our motion normalized by the condition $g(0)$ = identity is then

$$z^* = (z + c_4) e^{-i\alpha} + c_2(\alpha_1 + i\alpha_2 + i)^{-1} \left(e^{(\alpha_1 + i\alpha_2)\alpha} - 1\right) - c_4. \tag{101}$$

In the case

$$(102) \qquad \alpha_1 = 0\,, \quad \alpha_2 = -1\,,$$

the general solution of (98) is

$$(103) \qquad r = (c_2\alpha + c_4)\, e^{-i\alpha} - ic_3$$

and the corresponding motion is

$$(104) \qquad z^* = (z + c_2\alpha + c_4)\, e^{-i\alpha} - c_4\,.$$

It remains to deal with the case $\alpha_1 = \alpha_2 = 0$. Then

$$(105) \qquad s = c_2'\alpha + c_3'\,, \quad c_2', c_3' \in \mathbb{C}\,;$$

of course,

$$(106) \qquad |c_2'|^2 = c_1\,.$$

From (105),

$$(107) \qquad r = c_4'\, e^{-i\alpha} - ic_2'\alpha + c_2' - ic_3'\,; \quad c_4' \in \mathbb{C}\,.$$

The corresponding motion is then

$$(108) \qquad z^* = (z + c_4')\, e^{-i\alpha} - ic_2'\alpha - c_4'\,.$$

Theorem 2. *The motions with constant invariants* α_1, α_2 *are given either by* (101) *or* (104) *or* (108).

Bibliography

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Souhrn

KINEMATIKA V EUKLIDOVĚ ROVINĚ

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A. Karger [1] studoval kinematickou geometrii v homogenním prostoru, jehož grupy pohybů jsou jisté speciální Lieovy grupy. Předložený článek se omezuje na Euklidovu rovinu, ale podává metodu, vedoucí k řešení problému ekvivalence pro všechny Lieovy grupy pohybů. Kromě toho jsou uvedeny všechny transitivní jednoparametrické soustavy pohybů v E^2.

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