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ON THE EXISTENCE OF OSCILLATORY SOLUTIONS IN THE WEISBUCH-SALOMON-ATLAN MODEL FOR THE BELOUSOV-ZHABOTINSKIJ REACTION

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The Belousov-Zhabotinskij reaction is an oscillating oxidation reaction. For the kinetics of this reaction two models have been established. The first of them has been investigated in [4] where the existence of a periodic solution in the model has been proved. As was pointed out, all solutions in this model are oscillatory. In [7] G. Weisbuch, J. Salomon and H. Atlan have proposed a new model and have examined the stability of this model. They have shown numerically that this system shows oscillations, too. In this paper the stability properties of the model are reexamined and completed, the existence and properties of oscillatory solutions are investigated and a periodic solution is found.

1. The model in question is

$$\begin{aligned} \dot{X} &= -5K_1X + K_5C\,, \\ \dot{U} &= K_1X + K_3U - K_2XU - K_4U^2\,, \\ \dot{C} &= 2K_3U - 4K_5C\,, \end{aligned} \tag{1}$$

where $K_1 - K_5$ are positive real parameters representing kinetic constants and X, U, C are concentrations, and hence, nonnegative.

The system (1) has two equilibrium points

$$a_0 = (0, 0, 0)\,, \quad a_1 = (X_0, U_0, C_0) \tag{2}$$

where

$$X_0 = \frac{11K_3^2}{10(K_2K_3 + 10K_1K_4)}\,, \quad U_0 = \frac{11K_1K_3}{K_2K_3 + 10K_1K_4}\,, \tag{3}$$

$$C_0 = \frac{11K_1K_3^2}{2K_5(K_2K_3 + 10K_1K_4)}$$

and thus

$$(4) \qquad X_0 = \frac{K_3}{10K_1} U_0 = \frac{K_5}{5K_1} C_0 \,, \quad C_0 = \frac{K_3}{2K_5} U_0 \,.$$

First a general property of the system (1) will be derived. Denote $\mathscr{P} = \{(X, U, C) : : X \geqq 0, U \geqq 0, C \geqq 0\}$ and let $\mathscr{P}^0$ be the interior of $\mathscr{P}$. Since a_0 is a non-egress point and all the other points of quarterplanes $X = 0\,(U \geqq 0, C \geqq 0)$, $U = 0\,(X \geqq 0, C \geqq 0)$, and $C = 0\,(X \geqq 0, U \geqq 0)$ are strict ingrees points of P^0 with respect to (1), we obtain on the basis of Lemma 8.1, [3], p. 53,

Theorem 1. *The system* (1) *satisfies Hypothesis V from paper* [2], *p.* 31, *i.e., for each solution* (X, U, C) *of* (1) *it holds that if there is a* t_0 *such that* $(X(t_0), U(t_0), C(t_0)) \in \mathscr{P}$, *then* $(X(t), U(t), C(t)) \in \mathscr{P}$ *for all* $t \geqq t_0$ *(from the interval of existence).*

Now the stability of critical points a_0, a_1 will be investigated. To this aim let us introduce new variables x, u, c by

$$(5) \qquad \begin{aligned} X &= X_0 + x \,, \\ U &= U_0 + u \,, \\ C &= C_0 + c \end{aligned}$$

where (X_0, U_0, C_0) can also stand for a_0. Then (1) will assume the form

$$(6) \qquad \begin{aligned} \dot{x} &= -5K_1 x + K_5 c \,, \\ \dot{u} &= (K_1 - K_2 U_0)\, x + (K_3 - K_2 X_0 - 2K_4 U_0)\, u - K_2 x u - K_4 u^2 \,, \\ \dot{c} &= 2K_3 u - 4K_5 c \,. \end{aligned}$$

The characteristic equation of the matrix of the system of the first approximation is of the form

$$(7) \qquad \lambda^3 + a\lambda^2 + b\lambda + d = 0$$

where

$$(8) \qquad \begin{aligned} a &= 5K_1 - K_3 + 4K_5 + K_2 X_0 + 2K_4 U_0 \,, \\ b &= 20K_1K_5 - 5K_1K_3 - 4K_3K_5 + (5K_1K_2 + 4K_2K_5)\, X_0 + \\ &\quad + (10K_1K_4 + 8K_4K_5)\, U_0 \,, \\ d &= -22K_1K_3K_5 + 20K_1K_2K_5X_0 + (40K_1K_4K_5 + 2K_2K_3K_5)\, U_0 \,. \end{aligned}$$

Thus in the case of the critical point a_0

$$(8_0) \qquad \begin{aligned} a &= 5K_1 - K_3 + 4K_5 \,, \\ b &= 20K_1K_5 - 5K_1K_3 - 4K_3K_5 \,, \\ d &= -22K_1K_3K_5 \,, \end{aligned}$$

while for a_1 we get

$$(8_1)\quad \begin{aligned} a &= 5K_1 + K_3/10 + 4K_5 + 11K_1K_3K_4/(K_2K_3 + 10K_1K_4)\,, \\ b &= 20K_1K_5 + (1/2)\,K_1K_3 + (2/5)\,K_3K_5 + K_1K_3K_4(55K_1 + \\ &\quad + 44K_5)/(K_2K_3 + 10K_1K_4)\,, \\ d &= 22K_1K_3K_5\,. \end{aligned}$$

The next lemma brings a statement on zeros of the polynomial (7) which puts together a new result obtained with a little algebra, a stability criterion from [6], p. 58, and a result on instability from [7], p. 74.

Lemma 1. *I. Let $a > 0$, $b > 0$, $d > 0$. Then there is a real negative zero of (7) and, if $d - ab < 0$, all zeros of this polynomial have negative real part. If $d - ab \geqq 0$, then there is a pair $\beta \pm i\gamma$ of complex conjugate roots of (7) such that* $\operatorname{sgn} \beta = \operatorname{sgn}(d - ab)$.

II. If $d < 0$, then there exists a positive root of (7) and if this polynomial possesses a pair $\beta \pm i\gamma$ of complex conjugate roots, then $\operatorname{sgn} \beta = \operatorname{sgn}(d - ab)$.

Let us consider the problem of stability of a_0. With respect to (8_0), Lemma 1 implies that (7) has a positive root. Further, (1) satisfies the condition (1.8) in [1], p. 316. Hence Theorem 1.2 [1], p. 317 yields

Theorem 2. *The equilibrium point a_0 of the system (1) is unstable.*

As to the stability of the critical point a_1 of the system (1), we see from (5) that it is equivalent to the stability of the equilibrium point a_0 of (6). Here, by (8_1), we have to deal with the case $a > 0$, $b > 0$, $d > 0$ of coefficients of (7).

Denote

$$(9)\qquad K_1/K_2 = k\,, \quad K_3/K_4 = l\,.$$

When $k = l$, i.e. $K_1 = kK_2$, $K_3 = kK_4$, we get from (8_1) that

$$\begin{aligned} a &= k(5K_2 + 1{,}1K_4) + 4K_5\,, \\ b &= k^2\,6K_2K_4 + k(20K_2K_5 + 4{,}8K_4K_5)\,, \\ d &= k^2\,22K_2K_4K_5\,. \end{aligned}$$

Then

$$d - ab = k^2(22K_2K_4K_5 - 24K_2K_4K_5) + \ldots\,,$$

where all omitted terms are negative. Therefore

$$d - ab < -2K_2K_4K_5k^2 < 0$$

and from Lemma 1 by the Ljapunov Theorem, [6], p. 213 we obtain

Theorem 3. *If* $K_1/K_2 = K_3/K_4$, *then the equilibrium point* a_1 *of the system* (1) *is exponentially asymptotically stable.*

Corollary. *In this case the system* (1) *is a generalized Volterra equation.*

Proof. By Theorem 1, the system (1) satisfies Hypothesis V and has at least one asymptotically stable solution which passes through $\mathscr{P}$ and is bounded. By Definition 4, [2], p. 33 this means that (1) is a generalized Volterra equation.

The case $K_1/K_2 \neq K_3/K_4$ is more interesting. From (8_1) it follows that

$$d - ab = H_1 - H_2 \tag{10}$$

where

$$H_1 = 16K_1K_3K_5 - 2{,}5K_1^2K_3 - 100K_1^2K_5 - 0{,}05K_1K_3^2 - \\ - 0{,}04K_3^2K_5 - 80K_1K_5^2 - 1{,}6K_3K_5^2 \tag{11}$$

and

$$H_2 = [1/(K_2K_3 + 10K_1K_4)] \, . \, [660K_1^2K_3K_4K_5 + 176K_1K_3K_4K_5^2 + \\ + 8{,}8K_1K_3^2K_4K_5 + 11K_1^2K_3^2K_4 + 275K_1^3K_3K_4 + \\ + 605K_1^3K_3^2K_4^2/(K_2K_3 + 10K_1K_4) + \\ + 484K_1^2K_3^2K_4^2K_5/(K_2K_3 + 10K_1K_4)] \, . \tag{12}$$

Note that H_1 depends only on K_1, K_3, K_5 and that for fixed K_1, K_3, K_4, K_5 the expression $H_1 - H_2$ is an increasing function of K_2. Further, the relations $\lim_{K_2 \to 0+} (H_1 - H_2) < 0$, $\lim_{K_2 \to \infty} (H_1 - H_2) = H_1$ are true. Thus, for such K_1, K_3, K_5 that $H_1 > 0$ and for an arbitrary K_4 there exists a unique $K_2 = K_2^0$ for which $H_1 - H_2 = 0$. Similarly $\lim_{K_4 \to 0+} (H_1 - H_2) = H_1$ and $\lim_{K_4 \to \infty} (H_1 - H_2) = -\infty$ imply that if $H_1 > 0$ and K_2 is arbitrary, there is at least one $K_4 = K_4^0$ such that $H_1 - H_2 = 0$.

A sufficient condition for H_1 to be positive will be now derived. Similarly as in [7], p. 75 we put

$$K_3 = \nu K_1 \, , \\ K_1 = \mu K_5 \, . \tag{13}$$

Then

$$H_1 = \mu K_5^3(-p\mu^2 + q\mu - r) \, , \tag{14}$$

where

$$p = 2{,}5\nu + 0{,}05\nu^2 \, , \\ q = 16\nu - 100 - 0{,}04\nu^2 \, , \\ r = 1{,}6\nu + 80 \, . \tag{15}$$

Let v be such that

(16) $$q > 0\,, \quad q^2 - 4pr > 0$$

and let $(0<)\ \mu_1 < \mu_2$ be the roots of the equation

(17) $$-pv^2 + qv - r = 0\,.$$

Then for all μ such that

(18) $$\mu_1 < \mu < \mu_2\,,$$

the expression $-p\mu^2 + q\mu - r$ is positive and, with respect to (14), $H_1 > 0$. Hence, the following lemma holds.

Lemma 2. *Let K_1, K_3, K_5 be such that v, μ which are determined by (13) satisfy the inequalities (16) and (18) where p, q, r are defined by (15) and $\mu_1 < \mu_2$ are zeros of (17). Then H_1 given by (11) is positive.*

Remark. A little calculation gives that the inequalities $7 < v < 393$ imply that $q > 0$. In [7], p. 75 the implication $17 < v < 152 \Rightarrow q^2 - 4pr > 0$ is stated. Hence

(19) $$17 < v < 152$$

is a sufficient condition for v to satisfy (16). For such v the corresponding μ_1, μ_2 are given in Fig. 4 in [7], p. 75. E.g., when $v = 50$ then $\mu_1 \doteq 0.3$ and $\mu_2 \doteq 2.1$.

On the basis of (10), Lemma 1 implies

Theorem 4. *Let $K_1 - K_5$ be such that H_1, H_2 which are defined by (11), (12), fulfil*

(20) $$H_1 - H_2 > 0\,.$$

Then the equilibrium point a_1 of the system (1) is conditionally stable.

Remark. When K_1, K_3, K_5 satisfy the assumption of Lemma 2 (e.g. when (19), (18) are fulfilled) and K_4 is arbitrary, then by the above argument there exists a $K_2^0 > 0$ such that for $K_2 > K_2^0$ (or $K_2 = K_2^0$ or $K_2 < K_2^0$) the hypothesis (20) (or $H_1 - H_2 = 0$ or $H_1 - H_2 < 0$, respectively) is satisfied. At the same time (K_1, K_3, K_4, K_5 being fixed) there is a K_2^1 such that $K_2 > K_2^1$ (or $K_2 = K_2^1$ or $K_2 < K_2^1$) implies $k < l$ (or $k = l$ or $k > l$, respectively). By Theorems 3 and 4, $K_2^1 \geqq K_2^0$ cannot occur. Thus $K_2^1 < K_2^0$ and the following statement is true:

$$\textit{If} \quad H_1 - H_2 > 0\,, \quad \textit{then} \quad k < l\,.$$

Moreover, there is an $\varepsilon_1 > 0$ such that $k < l$ for $K_2^0 - \varepsilon_1 < K_2 < \infty$. Since we shall consider only such K_2's, *we shall assume in the sequel that $k < l$.* In paper [7], p. 73 even $10k$ has been neglected with respect to l.

2. In order to get more information in the case when (20) is fulfilled, consider the vector field given by the system (1) in the part of the phase space which has a real meaning in the Belousov - Zhabotinskij model, i.e. for $X \geqq 0$, $U \geqq 0$, $C \geqq 0$. From the first and the third equation of (1) we conclude that

a) $\dot{X} = 0$ *is true in the quarterplane* $X = (K_5/5K_1)\, C$, $0 \leqq C < \infty$, $0 \leqq U < \infty$, *while* $\dot{X} < 0$ $(\dot{X} > 0)$ *holds for* $X > (K_5/5K_1)\, C$, $(X < (K_5/5K_1)\, C)$, $0 \leqq C < \infty$, $0 \leqq U < \infty$ (*and* $0 \leqq X < \infty$).

b) $\dot{C} = 0$ *is valid in the quarterplane* $C = (K_3/2K_5)\, U$, $0 \leqq U < \infty$, $0 \leqq X < \infty$, *while* $\dot{C} < 0$ $(\dot{C} > 0)$ *holds for* $C > (K_3/2K_5)\, U (C < (K_3/2K_5)\, U)$, $0 \leqq U < \infty$, $0 \leqq X < \infty$ (*and* $0 \leqq C < \infty$).

The second equation in (1) can be written in the form

$$\dot{U} = U(K_3 - K_4 U) + X(K_1 - K_2 U)$$

and the condition $\dot{U} = 0$ leads to the quadratic equation

$$- K_4 U^2 + (K_3 - K_2 X)\, U + K_1 X = 0\,,$$

which, for each $X \geqq 0$, possesses two real roots, a positive one $U_p(X)$ and a non-positive one $U_n(X)$. Here

$$U_p(X) = (K_3 - K_2 X + \sqrt{((K_3 - K_2 X)^2 + 4K_1 K_4 X)})/(2K_4)\,(X \geqq 0)\,. \tag{21}$$

Further,

$$U_p'(X) = (-K_2/(2K_4))\left[1 + \frac{K_3 - K_2 X - 2K_1 K_4/K_2}{\sqrt{((K_3 - K_2 X)^2 + 4K_1 K_4 X)}}\right].$$

Since

$$\left(\frac{K_3 - K_2 X - 2K_1 K_4/K_2}{\sqrt{((K_3 - K_2 X)^2 + 4K_1 K_4 X)}}\right)^2 < 1\,,$$

$$U_p'(X) < 0 \quad (X \geqq 0)\,,$$

while

$$U_p''(X) = [-2K_1(K_1 K_4 - K_3 K_2)]/[(K_3 - K_2 X)^2 + 4K_1 K_4 X]^{3/2} > 0 \quad (X \geqq 0)\,.$$

The inequality $U_p'(X) < 0$, together with $U_p(0) = l$, gives $l - U_p(X) > 0$ for $0 < X < \infty$. As U_p satisfies the equality $U(l - U)/K_2 + X(k - U)/K_4 = 0$ in which the first term is positive, we also have $k - U_p(X) < 0$ $(0 < X < \infty)$ and thus, the inequalities

$$k < U_p(X) < l \quad \text{for} \quad 0 < X < \infty \tag{22}$$

are true. The results can be put together in the following statement:

c) $\dot{U} = 0$ *on the surface* $U = U_p(X)$, *where* U_p *is determined by* (21), $0 \leqq X < \infty$,

$0 \leqq C < \infty$, *while* $\dot{U} < 0$ $(\dot{U} > 0)$ *for* $U > U_p(X)$ $(U < U_p(X))$, $0 \leqq X < \infty$, $0 \leqq C < \infty$ (*and* $0 \leqq U < \infty$, $X^2 + U^2 > 0$). *The function* U_p *is decreasing, strictly convex in* $[0, \infty)$ *and satisfies the inequalities* (22).

The statements c), b) and a) together with (2), (3), (4) imply the following

Lemma 3. *Let the numbers* U_i, C_i *and* X_i, $i = 1, 2$ *be such that*

$$0 < U_1 \leqq k, \quad l < U_2,$$
$$0 < C_1 < (K_3/2K_5)\, U_1, \quad (K_3/2K_5)\, U_2 < C_2,$$
$$0 < X_1 < (K_5/5K_1)\, C_1, \quad (K_5/5K_1)\, C_2 < X_2.$$

Let $\mathscr{R} = \{(X, U, C) : X_1 \leqq X \leqq X_2,\ U_1 \leqq U \leqq U_2,\ C_1 \leqq C \leqq C_2\}$ *and let* $\mathscr{R}^0$ *be its interior.*

Then the following statements are true:

1. *The orbit of each solution to* (1) *passing through a point of* $\mathscr{R}$ *enters* $\mathscr{R}^0$ *and remains in* $\mathscr{R}^0$.

2. *The system* (1) *has a unique equilibrium point in* $\mathscr{R}^0$, *namely* a_1.

Lemma 3 completes the statement of Theorem 1 and implies that each solution of (1) the orbit of which passes through a point of $\mathscr{R}$ is defined on an interval $[t_0, \infty)$ for some t_0 (depending on that solution).

Further, for the solutions of (1) the following alternative holds:

Lemma 4. *Let* (X, U, C) *be a solution of the system* (1), *the orbit of which goes through a point from* $\mathscr{R}$ *and which is different from the equilibrium point. Then either each of its components is ultimately strictly monotone, i.e. strictly monotone in an interval* $[t_1, \infty)$, $t_0 < t_1$, *or each of its components is oscillating in the sense that its derivative changes its sign infinitely many times in each subinterval* $[t_1, \infty)$ *of* $[t_0, \infty)$.

Proof. First, by the properties of the vector field determined by (1), it follows that if a component of the solution (X, U, C) is monotone in an interval, then it is strictly monotone in that interval. Therefore it suffices to prove the following three implications:

1. If U is ultimately strictly monotone (briefly u.s.m.), then C is u.s.m., too.
2. If C is u.s.m., then so is X.
3. If X is u.s.m., then U is also u.s.m..

Let U be u.s.m., let U be increasing in $[t_1, \infty)$. Then from the third equation of (1) it follows that $C'(t) < 0$ $(C'(t) > 0)$ at all points t where $C(t) > (K_3/2K_5)\, U(t)$ $(C(t) < (K_3/2K_5)\, U(t))$. Since $(K_3/2K_5)\, U$ is increasing, there exist no points t_2, t_3 with $t_1 < t_2 < t_3$ such that $C(t_i) = (K_3/2K_5).\ U(t_i)$, $i = 2, 3$, and $C'(t) < 0$ in (t_2, t_3). Hence C' must be in a neighbourhood of ∞ either everywhere negative or everywhere nonnegative. If C were only nondecreasing, it would contradict the assumption that U is increasing. Thus in both cases the implication 1 follows. Similarly we can proceed when U is decreasing or when proving the implication 2.

The implication 3 will be proved only in the case when X is increasing in $[t_1, \infty)$. Consider the curve $(X(t), U(t))$ in the phase plane X, U. By the statement c) which precedes Lemma 3, it follows that if $U(t) > U_p[X(t)]$ $(U(t) < U_p[X(t)])$, then $\dot{U}(t) < 0$ $(\dot{U}(t) > 0)$. Since $U_p[X(t)]$ is decreasing, there exist no points t_2, t_3 with $t_1 < t_2 < t_3$ such that $U(t_i) = U_p[X(t_i)]$, $i = 2, 3$ and $\dot{U}(t) > 0$ in (t_2, t_3). This gives the implication 3 in a similar way as in the previous case.

Remarks. 1. If each component of a solution (X, U, C) of (1) the orbit of which goes through a point from $\mathcal{R}$ is u.s.m., then, in virtue of Lemma 3,

$$\lim_{t\to\infty} X(t) = X_0\,, \quad \lim_{t\to\infty} U(t) = U_0\,, \quad \lim_{t\to\infty} C(t) = C_0\,.$$

2. Each of the quarterplanes mentioned in the statements a), b), as well as the surface $U = U_p(X)$ from the statement c), define a decomposition of the parallelepiped $\mathcal{R}$ into two subsets whereby the derivative $\dot{X}$, $\dot{U}$, $\dot{C}$ of each component of a solution to (1) has a constant in each subset of the same decomposition and these signs are mutually different in these two subsets. Hence, if each component of a solution to (1) is oscillating, then its semiorbit goes from one subset of the mentioned three decompositions into the other infinitely many times.

Lemma 5. *Let (X, U, C) be a solution of the system (1) the orbit of which goes through a point from $\mathcal{R}$, is different from the equilibrium point a_1 and is oscillatory in the sense of Lemma 4. Then it is oscillating around the equilibrium point a_1, i.e. the functions $x = X - X_0$, $u = U - U_0$, $c = C - C_0$ change their sign in each interval $[t_1, \infty)$ infinitely many times.*

Proof. The vector function (x, u, c) is a non-trivial solution of (6). It is easy to see that none of its components can be identically 0 on any interval.

Suppose now that $u(t) \geqq 0$ for $t \in [t_1, \infty)$. Then the third equation of the system (6) implies that for an arbitrary but fixed $t_2 < t_1$ we have

$$c(t) = c(t_2) \exp\left[-4K_5(t - t_2)\right] + \int_{t_2}^{t} \exp\left[-4K_5(t - s)\right] . 2K_3\, u(s)\, \mathrm{d}s\,.$$

This implies that either $c(t) < 0$ in $[t_1, \infty)$ or there is a $t_2 > t_1$ such that $c(t) > 0$ for $t > t_2$.

i) If $c(t) < 0$ in $[t_1, \infty)$, then $\dot{c}(t) > 0$ in the same interval.

ii) If $c(t) > 0$ in (t_2, ∞), then similarly as in the previous case but from the first equation of (6) we conclude that either $x(t) < 0$ in (t_2, ∞) and then $\dot{x}(t) > 0$ for the same t or $x(t) > 0$ in (t_3, ∞), $t_2 < t_3$.

iii) The case $x(t) > 0$, $u(t) \geqq 0$ in (t_3, ∞), with respect to the inequalities $K_1 - K_2U_0 < 0$, $K_3 - K_2X_0 - 2K_4U_0 < 0$ leads *to* $\dot{u}(t) < 0$ in (t_3, ∞). Hence, if $u(t) \geqq 0$ in $[t_1, \infty)$, then by Lemma 4 the solution (x, u, c) is u.s.m. A similar implication also holds when $u(t) \leqq 0$ in $[t_1, \infty)$.

Therefore u changes its sign in each interval $[t_1, \infty)$ infinitely many times.

Let now $U = U_p(X)$ be the function determined by (21). From (5) we get $U_0 + u = U_p(X_0 + x)$, hence $u = U_p(X_0 + x) - U_0 = u_p(x)$. As $U_p(X_0) = U_0$, $u_p(0) = 0$ holds, and by the statement c) the function u_p is decreasing and strictly convex. Consider the curve (x, u). By Lemma 4, it intersects infinitely many times the graph of u_p. If all intersection points for sufficiently great t lay in the halfplane $u > 0$, then, again using the statement c), we should get that $u(t) > 0$ for all t sufficiently great. Similarly we can exclude the case that almost all intersection points lie in the halfplane $u < 0$. Therefore there exist infinitely many intersection points of the curve (x, u) with the graph of u_p in the halfplane $u > 0$ and infinitely many of them in the halfplane $u < 0$. As u_p is decreasing and $u_p(0) = 0$, the intersection points in the halfplane $u > 0$ $(u < 0)$ also lie in the halfplane $x < 0$ $(x > 0)$. So we have proved that x changes its sign infinitely many times in each interval $[t_1, \infty)$.

If c did not possess the same property, we should obtain from the first equation of (6) by a similar argument as above that x is ultimately positive or ultimately negative, which gives a contradiction with the previous statement. This completes the proof of the lemma.

With help of the above lemmas we prove

Theorem 5. *Let the assumption of Theorem* 4 *be fulfilled. Then with the exception of exactly two positive semiorbits, the positive semiorbit of each solution of* (1) *which is different from the equilibrium point* a_1 *and which goes through a point of* $\mathcal{R}$ *is oscillating in the sense given above and cannot tend to* a_1 *as* $t \to \infty$. *The exceptional positive semiorbits belong to the solutions of* (1) *each component of which is u.s.m. and that tend to* a_1 *(as* $t \to \infty$*).*

Proof. Using the relation between the solutions of the systems (1) and (6) we shall prove that there exist exactly two positive semiorbits of the solutions of (6) each component of which is u.s.m.. By Remark 1, these semiorbits tend to 0 as $t \to \infty$. The positive semiorbits of all the other solutions of (6) corresponding to the solutions of (1) mentioned in the theorem are oscillatory.

Hence, consider the system (6) under the assumption of Theorem 4. By Lemma 1 it follows that the matrix

$$A = \begin{pmatrix} -5K_1, & 0, & K_5 \\ K_1 - K_2U_0, & K_3 - K_2X_0 - 2K_4U_0, & 0 \\ 0, & 2K_3, & -4K_5 \end{pmatrix}$$

of the first approximation to (6) has a real eigenvalue $\alpha < 0$ and a pair of complex conjugate eigenvalues $\beta \pm i\gamma$, whereby $\beta > 0$. Further, the vector function $f(z) = (0, -K_2xu - K_4u^2, 0)$ of the vector variable $z = (x, u, c) = (x_1. x_2, x_3)$ satisfies the condition $\partial f/\partial z = 0$ at $z = 0$. Thus the system (6) satisfies all assumptions of Theorem 4.1, [1], p. 330 and, moreover, there exists a real nonsingular 3×3 constant matrix Q such that (6) can be brought by the transformation $y = Qz$ to the form which already fulfils the assumptions of Theorem 8.1, [5], p. 248. By these two theorems there exists a one-dimensional manifold S in the space z, the equations of which are ([1], p. 330)

(23)
$$x_i = q_{i1}y_1 + q_{i2}\,\psi_2(y_1) + q_{i3}\,\psi_3(y_1) = x_i(y_1) \quad y_1 \in [-\tilde{\delta}, \tilde{\delta}]$$

where ψ_2, ψ_3 are real analytic functions defined in $[-\tilde{\delta}, \tilde{\delta}]$,

(24)
$$\psi_i(0) = 0\,, \quad \psi_i'(0) = 0$$

and the matrix $(q_{ik}) = Q^{-1}$, $i, k = 1, 2, 3$, is the inverse of Q. With respect to (24), $x_i(0) = 0$ and $x_i'(0) = q_{i1}$, $i = 1, 2, 3$. Since Q^{-1} is nonsingular, at least one of the numbers q_{i1}, $i = 1, 2, 3$, is different from zero. Suppose $q_{11} \neq 0$. (In the other cases we should proceed similarly.) Then we can take $\tilde{\delta}$ so small that $x_1(y_1)$ is one-to-one in $[-\tilde{\delta}, \tilde{\delta}]$.

The manifold S enjoys the following property ([5], p. 248). There exist two positive numbers $\delta_0 \leqq \delta$ such that the following statements hold:

1. If the initial point of a positive semiorbit of a solution $z = (x, u, c)$ of (6) is from S and $|x| \leqq \delta_0$, then the whole semiorbit lies on S and $\lim_{t\to\infty} z(t) = 0$.

2. If a positive semiorbit of a solution z of (6) lies in the δ-neighbourhood of the point 0, e.g. if $\lim_{t\to\infty} z(t) = 0$, then it must lie on S.

Let T be the set of all solutions z of the system (6) each component of which is u.s.m. and for which $\lim_{t\to\infty} z(t) = 0$. Let $z_1, z_2 \in T$, where $z_1 = (x_1^*, u_1, c_1)$, $z_2 = (x_2^*, u_2, c_2)$. Then there is an interval $[t_1, \infty)$ in which both functions x_1^*, x_2^* are one-to-one and possess a constant sign. Let both functions x_1^*, x_2^* be positive in this interval. If we denote by t_i the inverse function to x_i^*, $i = 1, 2$, t_i maps the interval $(0, x_i^*(t_1)]$ onto $[t_1, \infty)$ and a positive semiorbit corresponding to the solution z_i is given by the relations

(25)
$$u = u_i[t_i(x)]\,, \quad c = c_i[t_i(x)]\,, \quad x \in (0, x_i^*(t_1)]\,, \quad i = 1, 2\,.$$

On the other hand, since $\lim_{t\to\infty} z_i(t) = 0$, this semiorbit must lie on S (taking t_1 sufficiently large). Hence, in virtue of (23) we have

(26)
$$x = x_1(y_1)\,, \quad u = x_2(y_1)\,, \quad c = x_3(y_1)\,, \quad y_1 \in [-\tilde{\delta}, \tilde{\delta}]\,.$$

Now we choose an $x \in (0, x_1^*(t_1)] \cap (0, x_2^*(t_1)]$. As x_1 is injective, there exists exactly one y_1 such that $x = x_1(y_1)$. Comparing (25) with (26) we get at the chosen point

$$u_1[t_1(x)] = x_2(y_1) = u_2[t_2(x)], \quad c_1[t_1(x)] = x_3(y_1) = c_2[t_2(x)]$$

and hence both semiorbits pass through the same point which implies that they are identical (for all $x \in (0, x_1^*(t_1)] \cap (0, x_2^*(t_1)]$). The set T can have at most two semiorbits (one for $x > 0$ and another for $x < 0$).

Further, we prove that no oscillatory solution z of (6) can satisfy $\lim_{t \to \infty} z(t) = 0$. Otherwise a positive semiorbit of this solution would lie on S and since z is oscillating, two points t_1, t_2 would exist with $t_1 < t_2$, such that $z(t_1) = z(t_2)$. This would imply that z is periodic which contradicts the fact that $\lim_{t \to \infty} z(t) = 0$. There is even no positive semiorbit of an oscillatory solution z of (6) lying entirely in a δ-neighbourhood of 0 where $\delta > 0$ is sufficiently small. Hence only positive semiorbits of the solutions of (6) each component of which is u.s.m. may lie on S. From the property 1 of S it follows that such semiorbits do exist. This completes the proof of Theorem 5.

Further properties of oscillatory solutions of (1) are given by

Theorem 6. *Let the assumptions of Theorem 4 be fulfilled. Le (X, U, C) be a solution of the system (1) the orbit of which goes through a point from $\mathcal{R}$, is different from the equilibrium point a_1 and is oscillatory in the sense of Lemma 4 and Lemma 5, respectively. Then the following statement holds for the functions*

$$x = X - X_0, \quad u = U - U_0, \quad c = C - C_0$$

in each interval (t_1, ∞) where t_1 is a zero-point of one of them: Between two successive zeros of x and u and c, there exists exactly one zero of u and c and x respectively. All zeros of the functions x, u, c in (t_1, ∞) are simple.

Proof. The function (x, u, c) is a solution of (6) which is defined in a neighbourhood of ∞, say in (t_0, ∞) where $t_0 < t_1$. Let $t_2, t_3 \in R$ be such that $t_1 \leqq t_3 \leqq t_2$, $t_1 < t_2$. Directly from (6) we get the statement:

1. *None of the functions x, u, c possesses a zero of multiplicity 3 (and hence all zeros of these functions are isolated and cannot have a limit point in (t_0, ∞)), no pair of these functions possess a double zero at the same point. Further, $u(t_2) = \dot{u}(t_2) = 0$ implies $x(t_2) = 0$, $x(t_2) = \dot{x}(t_2) = 0$ gives $c(t_2) = 0$ and $c(t_2) = \dot{c}(t_2) = 0$ yields the equality $u(t_2) = 0$.*

The following relations between x, u, c can be derived from (6):

$$(27) \quad x(t) = x(t_3) \exp[-5K_1(t - t_3)] + \int_{t_3}^{t} \exp[-5K_1(t - s)] K_5 c(s) \, ds,$$

$$(28) \quad c(t) = c(t_3) \exp[-4K_5(t - t_3)] + \int_{t_3}^{t} \exp[-4K_5(t - s)] 2K_3 u(s) \, ds,$$

$$(29)\quad u(t) = u(t_3)\exp\left[(K_3 - K_2X_0 - 2K_4U_0)(t - t_3) - K_2\int_{t_3}^{t} x(s)\,ds\right] + \\ + \int_{t_3}^{t} K(t, s)\left[(K_1 - K_2U_0)\,x(s) - K_4\,u^2(s)\right] ds\,, \quad t_0 < t < \infty\,,$$

where $K = K(t, s)$ is the Cauchy function of the differential equation $\dot{u} = [K_3 - K_2X_0 - 2K_4U_0 - K_2\,x(t)]\,u$ and, hence, positive.

Owing to the inequality $K_1 - K_2U_0 < 0$ as well as to the fact that the zeros of x, u, c are isolated, we conclude from (27)–(29):

2. *If $u(t) \geqq 0$ ($u(t) \leqq 0$) in $[t_1, t_2]$, then either c does not posses any zero in $[t_1, t_2]$ or there is a point $t_3 \in [t_1, t_2]$ such that $c(t_3) = 0$ and, if $t_3 < t_2$, then $c(t) > 0$ ($c(t) < 0$) in $(t_3, t_2]$ and, if $t_1 < t_3$, then $c(t) < 0$ ($c(t) > 0$) in $[t_1, t_3)$.*

3. *If $c(t) \geqq 0$ ($c(t) \leqq 0$) in $[t_1, t_2]$, then either x does not possess any zero in $[t_1, t_2]$, or there is a point $t_3 \in [t_1, t_2]$ such that $x(t_3) = 0$ and, if $t_3 < t_2$, then $x(t) > 0$ ($x(t) < 0$) in $(t_3, t_2]$ and, if $t_1 < t_3$, then $x(t) < 0$ ($x(t) > 0$) in $[t_1, t_3)$.*

4. *If $x(t) \geqq 0$ in $[t_1, t_2]$, then either u does not have any zero in $[t_1, t_2]$, or there is a point $t_3 \in [t_1, t_2]$ such that $u(t_3) = 0$ and, if $t_3 < t_2$, then $u(t) < 0$ in $(t_3, t_2]$ and, if $t_1 < t_3$, then $u(t) > 0$ in $[t_1, t_3)$.*

The next statement is based on the properties of the vector field (6).

5. *If $x(t) \leqq 0$ in $[t_1, t_2]$, then either u does not possess any zero in $[t_1, t_2]$ or there is a point $t_3 \in [t_1, t_2]$ such that $u(t_3) = 0$ and, if $t_3 < t_2$, then $u(t) < 0$ in $(t_3, t_2]$ and, if $t_1 < t_3$, then $u(t) > 0$ in $[t_1, t_3)$.*

Proof. Consider the curve (x, u) in the phase plane x, u. If u_p has the same meaning as in the proof of Lemma 5, then by the statement c) preceding Lemma 3, the function u is decreasing (increasing) in the intervals for which the points (x, u) lie above (below) the graph of u_p.

Let t_3 be the first zero of u in $[t_1, t_2]$. If $x(t_3) = 0$ then by (6), $\dot{u}(t_3) = 0$. As t_3 cannot be a double zero of both functions x, u, either $\dot{x}(t_3) > 0$ and then $t_3 = t_2$ or $\dot{x}(t_3) < 0$ and then $t_3 = t_1$, with respect to $x(t) \leqq 0$ in $[t_1, t_2]$. In the first case ($\dot{x}(t_3) > 0$, $x(t_3) = u(t_3) = \dot{u}(t_3) = 0$), the properties of the vector field imply that $\ddot{u}(t_3) > 0$ cannot happen. Hence $\ddot{u}(t_3) < 0$ and it follows again by these properties that $u(t) < 0$ must hold in $[t_1, t_3)$. In the second case $\ddot{u}(t_3) < 0$ cannot happen, therefore $\ddot{u}(t_3) > 0$ and hence $u(t) > 0$, first in a right neighbourhood of t_3 and then, by using the properties of the vector field (6) as well as the result from the first case, we come to the conclusion that $u(t) > 0$ in $(t_3, t_2]$.

If $x(t_3) < 0$, then we conclude again by the properties of the vector field (6) that $u(t) < 0$ in $[t_1, t_3)$ and by the above result from the first case, $u(t) > 0$ in $(t_3, t_2]$. This completes the proof of the statement 5.

Without assuming that t_1 is a zero of one of the functions x, u, c we shall prove the statement

6. *If either x or u or c, respectively, has a double zero at t_2, then the functions x, u, c do not vanish at any $t < t_2$ and $x(t)\,u(t) > 0$, $x(t)\,c(t) < 0$ for all t such that $t_0 < t < t_2$.*

Proof. Let $u(t_2) = \dot{u}(t_2) = 0$. Then, by the statement 1, $x(t_2) = 0$, $c(t_2)\,\dot{x}(t_2)\,\ddot{u}(t_2) \neq 0$. Statements 4 and 5 imply

$$\dot{x}(t_2)\,\ddot{u}(t_2) < 0 \tag{30}$$

(here the behaviour of the functions x and u to the right from t_2 is considered). The statement 3 implies

$$\dot{x}(t_2)\,c(t_2) > 0\,. \tag{31}$$

Suppose that u vanishes to the left from t_2, hence there exists a $t_1 < t_2$ such that $u(t) \neq 0$ in (t_1, t_2) and $u(t_1) = 0$. As $x(t_2) = u(t_2) = 0$, by the statements 4 and 5 we have that there is a t_3 such that $t_1 < t_3 < t_2$ with $x(t) \neq 0$ in (t_3, t_2) and $x(t_3) = 0$. Because $x(t_2) = 0$ and $c(t_2) \neq 0$, by the statement 3 there is a t_4 with $t_3 < t_4 < t_2$ such that $c(t) \neq 0$ in (t_4, t_2) and $c(t_4) = 0$. This contradicts the statement 2. Similarly we come to a contradiction when we suppose the existence of a point t_3 or t_4, respectively, with the above properties (i.e. $x(t_3) = 0$, $x(t) \neq 0$ in (t_3, t_2), $c(t_4) = 0$ and $c(t) \neq 0$ in (t_4, t_2)). Hence $x(t)\,u(t)\,c(t) \neq 0$ for all t, $t_0 < t < t_2$ and on the basis of (30), (31) we can state more precisely that $x(t)\,u(t) > 0$, $x(t)\,c(t) < 0$ for $t_0 < t < t_2$.

If $x(t_2) = \dot{x}(t_2) = 0$, then $c(t_2) = 0$ by the statement 1 and using statements 3 and 2 we get $\dot{c}(t_2)\,\ddot{x}(t_2) > 0$ and $\dot{c}(t_2)\,u(t_2) > 0$.

Let there exist a point $t_1 < t_2$ such that $x(t_1) = 0$, $x(t) \neq 0$ in (t_1, t_2). As $x(t_2) = c(t_2) = 0$, by 3 there is a t_3, $t_1 < t_3 < t_2$ such that $c(t_3) = 0$, $c(t) \neq 0$ in (t_3, t_2). Because $c(t_2) = 0$ and $u(t_2) \neq 0$, by the statement 2 there is a t_4, $t_3 < t_4 < t_2$ such that $u(t_4) = 0$, $u(t) \neq 0$ in (t_4, t_2). This contradicts the statement 4. Hence the statement 6 is true in this case.

When $c(t_2) = \dot{c}(t_2) = 0$, we have $\ddot{c}(t_2)\,\dot{u}(t_2) > 0$, $x(t_2).\ \dot{u}(t_2) < 0$ in virtue of the statements 2, 4 and 5. If there is a $t_1 < t_2$ such that $c(t_1) = 0$, $c(t) \neq 0$ in (t_1, t_2), then there are $t_1 < t_3 < t_4 < t_2$ with $u(t_3) = 0$, $x(t_4) = 0$, $u(t) \neq 0$ in (t_3, t_2), $x(t) \neq 0$ in (t_4, t_2). This leads to contradiction with the statement 3. Again 6 is true.

7. ***Between** two successive zeros of x and u and c, there exists in (t_1, ∞) exactly one zero of u and c and x, respectively, and all zeros of the functions x, u, c are simple.*

Only the first assertion will be proved. The other two can be proved in a similar way. Since t_1 is a zero of one of the functions x, u, c, the statement 6 implies that all zeros of x, u, c in (t_1, ∞) are simple and hence (6) implies that none two of the functions x, u, c have a common zero-point.

Let $t_1 < s_1 < s_2$, $x(s_1) = x(s_2) = 0$ and $x(t) \neq 0$ in (s_1, s_2). By the statements 4 and 5, u can have at most one zero in (s_1, s_2). Suppose it has none. Then the same is true for $[s_1, s_2]$ and two cases will be distinguished:

$$\text{i)}\quad \operatorname{sgn} x(t) = \operatorname{sgn} u(t) \quad (t \in (s_1, s_2))\,.$$

Let $s_3 > s_2$ be the next zero of x, i.e. $x(s_3) = 0$, $x(t) \neq 0$ in (s_2, s_3). By the statements 4 and 5, $u(t) \neq 0$ in $[s_1, s_3]$. Then 2 gives that c possesses at most one zero in $[s_1, s_3]$, either in (s_1, s_2) or in (s_2, s_3). In all three cases (the first case occurs when c does not vanish in $[s_1, s_3]$) we come with regard to the statement 3 to contradiction.

$$\text{ii)}\quad \operatorname{sgn} x(t) \neq \operatorname{sgn} u(t) \quad (t \in (s_1, s_2))\,.$$

Here two subcases may occur. If x has no zero smaller than s_1 and greater or equal to t_1, then by statements 4 and 5, neither u has. By 2, c possesses at most one zero in $[t_1, s_2]$ and, by the meaning of t_1, t_1 is the unique zero of c in $[t_1, s_2]$. But $x(s_1) = x(s_2) = 0$ leads to a contradiction with the statement 3.

If $s_0 < s_1$ is the greatest zero of x smaller than s_1, then by the statements 4 and 5, $u(t) \neq 0$ in $[s_0, s_2]$. This leads to a contradiction similarly as in the case i).

The contradiction shows that between s_1 and s_2 there exists a unique zero of $\boldsymbol{u}$.

3. Finally we shall prove existence of a periodic solution to (1). To that aim we shall apply Hopf's theorem and the method from the paper [4].

Theorem 7. *Let K_1, K_3, K_5 be such that H_1 determined by (11) is positive. Let K_4 be arbitrary and $K_2 = K_2^0$ be such that $H_1 - H_2 = 0$, where H_2 is defined by (12). Then there exists an $\varepsilon_0 > 0$ such that for $K_2 \in (K_2^0 - \varepsilon_0, K_2^0 + \varepsilon_0)$ the system (1) has a periodic solution different from the equilibrium point whose orbit lies in R exactly in one of the three cases: Either only for each $K_2 \in (K_2^0 - \varepsilon_0, K_2^0)$ or only for each $K_2 \in (K_2^0, K_2^0 + \varepsilon_0)$ or only for $K_2 = K_2^0$.*

Proof. Put $K_2 = K_2^0 + \varepsilon$. As the coefficients of (7) determined by (8_1) are continuous in K_2, the eigenvalues of the matrix of the system of the first approximation are continuous in K_2. Hence, by Lemma 1, there is an $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ one eigenvalue $-\bar{a}(\varepsilon)$ is real and negative, while the other two are complex conjugate. Denote them $e(\varepsilon) \pm i(\bar{b}(\varepsilon))^{1/2}$, where $\bar{a}(0) = a(0)$, $\bar{b}(0) = b(0)$ and $a(0)$, $b(0)$ stand for the coefficients a, b in (8_1) for $\varepsilon = 0$.

Differentiating (7) with respect to ε, we get

$$\lambda'(\varepsilon) = [-a'(\varepsilon)\,\lambda^2(\varepsilon) - b'(\varepsilon)\,\lambda(\varepsilon) - d'(\varepsilon)]/[3\lambda^2(\varepsilon) + 2a(\varepsilon)\,\lambda(\varepsilon) + b(\varepsilon)]\,.$$

Since for $\varepsilon = 0$ the roots of (7) are distinct, the Implicit Function Theorem guarantees the existence of $\lambda'(\varepsilon)$ for sufficiently small ε. But $\lambda'(\varepsilon) = e'(\varepsilon) \pm (1/2)\,[\bar{b}(\varepsilon)]^{-1/2}\,\bar{b}'(\varepsilon)\,i$, therefore

$$e'(0) = \operatorname{Re}\{[a'(0)\,b(0) - d'(0) - b'(0)\,.\,(\pm i[b(0)]^{1/2})]/[-2\,b(0) + 2\,a(0)\,(\pm i[b(0)]^{1/2})]\} =$$

$$= (2\, b(0)/[4\, b^2(0) + 4\, a^2(0)\, b(0)])\, [-a'(0)\, b(0) + d'(0) - b'(0)\, a(0)]\,.$$

Here $(2\, b(0)/[4\, b^2(0) + 4\, a^2(0)\, b(0)]) > 0$, $a'(0) = -11K_1K_3^2K_4/(K_2^0K_3 + 10K_1K_4)^2$, $d'(0) = 0$ and $b'(0) = -K_1K_3^2K_4(55K_1 + 44K_5)/(K_2^0K_3 + 10K_1K_4)$ and thus $e'(0) >$ > 0. All hypotheses of Hopf's theorem being satisfied, by this theorem there exist periodic solutions of (1) with orbits in a neighbourhood of the equilibrium point a_1 for $K_2 = K_2^0$ exactly in one of the three cases: Either for $\varepsilon > 0$, or for $\varepsilon = 0$, or for $\varepsilon < 0$.

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Súhrn

O EXISTENCII OSCILATORICKÝCH RIEŠENÍ VO WEISBUCHOVOM-SALOMONOVOM-ATLANOVOM MODELE BELOUSOVEJ-ŽABOTINSKÉHO REAKCIE

VALTER ŠEDA

V práci sa pojednáva o stabilite riešení diferenciálneho systému (1), ktorý predstavuje uvažovaný model Belousovej-Žabotinského reakcie. Dokazuje sa existencia oscilatorických riešení tohto systému a veta o oddeľovaní nulových bodov jednotlivých zložiek takýchto riešení. Záverom je dokázaná aj existencia periodického riešenia.

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