Vladimír Janovský; Petr Procházka

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CONVERGENCE ANALYSIS OF A NONCONFORMING FINITE ELEMENT METHOD SOLVING A PLATE WITH RIBS

Vladimír Janovský, Petr Procházka

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1. INTRODUCTION

The present paper may be considered a continuation of our work [1] where we proposed a finite element method solving a problem of a clamped plate with ribs which are stiff against bending and torsion. We derived error estimates and convergence assertions depending on the regularity of the solution. It has remained to prove the convergence of our method to a weak solution (without the assumption of its regularity) in the case of intersecting ribs.

The main difficulty consisted in proving the assertion that each weak solution of our problem can be approximated by a "smooth enough" function. In [1] we succeeded in solving this problem for not interesecting ribs. The goal of this paper is an extension of this result to the case of intersecting ribs. As a consequence we obtain a general convergence theorem.

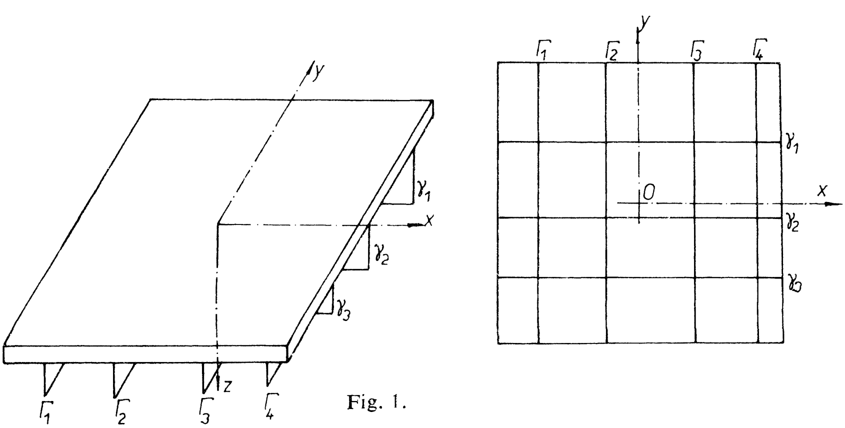

Fig. 1.

We recall the mathematical formulation of the model considered: Let G be a rectangle in the plane; we introduce a Cartesian coordinate system whose axes are perpendicular to the sides of G. Let I and J be the sets of ribs parallel to y and x-axis, respectively – see Fig. 1. We define the space V of admissible "shift" functions as follows:*)

$$V = \{w;\ w \in W_0^{2,2}(G),\ w \in W_0^{2,2}(\Gamma),\ w \in W_0^{2,2}(\gamma),\ \partial w/\partial x \in W_0^{1,2}(\Gamma),$$
$$\partial w/\partial y \in W_0^{2,2}(\gamma) \quad \text{for each} \quad \Gamma \in I,\ \gamma \in J\}$$

and equip it with the norm $|||.|||$:

$$|||w||| = |w|_{2,G} + \sum_{\Gamma \in I} \left(|w|_{2,\Gamma} + \left|\frac{\partial w}{\partial x}\right|_{1,\Gamma} \right) + \sum_{\gamma \in J} \left(|w|_{2,\gamma} + \left|\frac{\partial w}{\partial y}\right|_{1,\gamma} \right).$$

Problem: *Given $f \in L_2(G)$, find $u \in V$ such that*

$$a(u, v) + \sum_{\Gamma \in I} \int_\Gamma \left\{ \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial x\, \partial y} \frac{\partial^2 v}{\partial x\, \partial y} \right\} \mathrm{d}y + \tag{1.1}$$
$$+ \sum_{\gamma \in J} \int_\gamma \left\{ \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x\, \partial y} \frac{\partial^2 v}{\partial x\, \partial y} \right\} \mathrm{d}x = 2 \int_G f v \, \mathrm{d}x\, \mathrm{d}y$$

for all $v \in V$, *where*

$$a(u, v) = \int_G \left(\Delta u\, \Delta v + \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x\, \partial y} \frac{\partial^2 v}{\partial x\, \partial y} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\delta y^2} \right) \mathrm{d}x\, \mathrm{d}y\,.$$

The solution u of the problem (1.1) exists and is unique.

2. NUMERICAL METHOD AND ERROR ESTIMATIONS (SUMMARY)

For each $h \in (0, 1)$ we define a division $G_h \equiv \{G_{ih}\}_{i=1}^{k(h)}$ of the rectangle G into rectangular elements G_{ih}. We assume that the system of divisions G_h is regular. This means:

a) $\bar{G} = \bigcup_{i=1}^{k(h)} \bar{G}_{ih}$,

b) $G_{ih} \cap G_{jh} = \emptyset$; $i \neq j$; $i, j = 1, \ldots, k(h)$,

c) $G_{ih} \cap \Gamma = \emptyset, G_{ih} \cap \gamma = \emptyset$; $\Gamma \in I$, $\gamma \in J$, $i = 1, \ldots, k(h)$,

*) For the detailed notation of all functional spaces and their norms see [1].

d) *if* $h(G_{ih}) = \operatorname{diam} G_{ih}$ *and* $\varrho(G_{ih}) =$ *supremum of diameters of circles inscribed into* G_{ih} *then there exists a constant* C *such that*

$$\min_{i=1,\dots,k(h)} \frac{\varrho(G_{ih})}{h(G_{ih})} \geqq C > 0\,, \quad h \in (0.\ 1)\,,$$

e) $h = \max\limits_{i=1,\dots,k(h)} h(G_{ih})$.

If A is a vertex of $G_{ih} \in G_h$ then we say that A is a nodal point of the division G_h.

Let R be a nondegenerate fixed rectangle and let

$$A(R) = \{\varphi;\ \varphi = \sum_{0 \leqq i+j \leqq 3} a_{ij}x^i y^j + a_{31}x^3y + a_{13}xy^3\}$$

be the set of the so called Ari-Adini's polynomials. For each $h \in (0, 1)$ and $G_{ih} \in G_h$ there exists a regular affine mapping $F_{ih} : G_{ih} \to R$.

For $h \in (0, 1)$ we introduce a space V_h which approximates the space V:

$V_h = \{\varphi;\ \varphi_0 F_{ih}^{-1} \in A(R)$ for $i = 1, \dots, k(h)$; *if* A *is a nodal point of division* G_h *then*

a) φ, $\partial\varphi/\partial x$, $\partial\varphi/\partial y$ *is continuous at the point* A,

b) $\varphi(A) = \partial\varphi/\partial x(A) = \partial\varphi/\partial y(A) = 0$ *for* $A \in \partial G\}$.

Remark. $V_h \not\subset V$.

Approximate problem. *Given* $h \in (0, 1)$, *find* $u_h \in V_h$ *such that*

$$(1.2) \qquad a_h(u_h, v) + \sum_{\Gamma \in I} \left\{ \int_\Gamma \frac{\partial^2 u_h}{\partial y^2} \frac{\partial^2 v}{\partial y^2}\, \mathrm{d}y + \right.$$

$$\left. + \tfrac{1}{2} \sum_{i=1}^{k(h)} \int_{\Gamma \cap \partial G_{ih}} \left\{ \frac{\partial}{\partial y} \left(\mathscr{L}_{ih} \frac{\partial u_h}{\partial x} \right) \frac{\partial}{\partial y} \left(\mathscr{L}_{ih} \frac{\partial v}{\partial x} \right) \right\} \mathrm{d}y \right\} +$$

$$+ \sum_{\gamma \in J} \left\{ \int_\gamma \frac{\partial^2 u_h}{\partial x^2} \frac{\partial^2 v}{\partial x^2}\, \mathrm{d}x + \tfrac{1}{2} \sum_{i=1}^{k(h)} \int_{\gamma \cap \partial G_{ih}} \left\{ \frac{\partial}{\partial x} \left(\mathscr{L}_{ih} \frac{\partial u_h}{\partial y} \right) \frac{\partial}{\partial x} \left(\mathscr{L}_{ih} \frac{\partial v}{\partial y} \right) \right\} \mathrm{d}x \right\} =$$

$$= 2 \int_G f v \,\mathrm{d}x\,\mathrm{d}y$$

for all $v \in V_h$, where

$$a_h(u_h, v) = \sum_{i=1}^{k(h)} \int_{G_{ih}} \left(\Delta u_h\, \Delta v + \frac{\partial^2 u_h}{\partial x^2} \frac{\partial^2 v}{\partial x^2} + 2 \frac{\partial^2 u_h}{\partial x\, \partial y} \frac{\partial^2 v}{\partial x\, \partial y} + \right.$$

$$\left. + \frac{\partial^2 u_h}{\partial y^2} \frac{\partial^2 v}{\partial y^2} \right) \mathrm{d}x\,\mathrm{d}y$$

and $\mathscr{L}_{ih}$ is a linear interpolation operator defined along the edges of G_{ih} in the following way: Let a function ψ be defined along the boundary of the rectangle G_{ih}.

Let A and B be nodal points which are connected by one of the four edges of the rectangle G_{ih}. Then the value of $\mathscr{L}_{ih}\psi$ at an arbitrary point $X = At + (1 - t)\,B$, $t \in (0, 1)$ of the side AB is defined as follows:

$$\mathscr{L}_{ih}(\psi)\,X = t \lim_{\tau \to 1-} \psi(\tau A + (1 - \tau)\,B) + (1 - t) \lim_{\tau \to 0+} \psi(\tau A + (1 - \tau)\,B)$$

provided the limits on the right hand side exist.

We introduce the norm $|||\cdot|||_h$ on the space V_h in the natural way:

$$|||u|||_h = \left(\sum_{i=1}^{k(h)} |v|^2_{2,G_{ih}} + \sum_{\Gamma\in I}\left(|v|^2_{2,\Gamma} + \tfrac{1}{2}\sum_{i=1}^{k(h)} \left\| \frac{\partial}{\partial y}\left(\mathscr{L}_{ih}\frac{\partial v}{\partial x}\right)\right\|^2_{L_2(\Gamma\cap\partial G_{ih})}\right) + \right.$$

$$\left. + \sum_{\gamma\in J}\left(|v|^2_{2,\gamma} + \tfrac{1}{2}\sum_{i=1}^{k(h)} \left\| \frac{\partial}{\partial x}\left(\mathscr{L}_{ih}\frac{\partial v}{\partial y}\right)\right\|^2_{L_2(\gamma\cap\partial G_{ih})}\right)\right)^{1/2}.$$

There exists a unique solution u_h of the problem (1.2) for each $h \in (0, 1)$.

Theorem: 1.2. *Let u and u_h be the solutions of the problem (1.1) and (1.2), respectively. Let*

$\mathfrak{M}_h = \{w;\ w \in L_2(G),\ w$ is polynomial of the second degree on each $G_{ih},\ i = 1, \ldots$
$\ldots, k(h)\}$.

Then the following estimate holds):*

$$|||u - u_h|||_h \leqq C\{\inf_{\varphi\in V_h} |||u - \varphi|||_h + \inf_{\tilde{u}_h\in\mathfrak{M}_h} |u - \tilde{u}_h|_{2,h} + h^2\} \tag{2.2}$$

where

$$|\cdot|_{2,h} = (\sum_{i=1}^{k(h)} (|\cdot|_{2,G_{ih}})^2)^{1/2}.$$

Proof. See [1], Remark 2 and (3.16).

Now we show that if the solution u is "smooth enough" then the right hand side of (2.2) converges towards zero. What is the meaning of the word "enough"?

Definition. *Let G_1 be a regular division of G; $G_1 = \{G_{i1}\}_{i=1}^{k(1)}$. Let us suppose that each side of an arbitrary G_{i1} coincides either with a rib or with the boundary of G – see Fig. 2.*

Then we define

$$W = \{w;\ w \in V \bigcap_{i=1}^{k(1)} W^{3,2}(G_{i1}),\ w \in W^{3,2}(\Gamma) \text{ for each } \Gamma \in I,$$
$$w \in W^{3,2}(\gamma) \text{ for each } \gamma \in J\}.$$

*) Here as well as in the following C denotes a genetic constant independent of h.

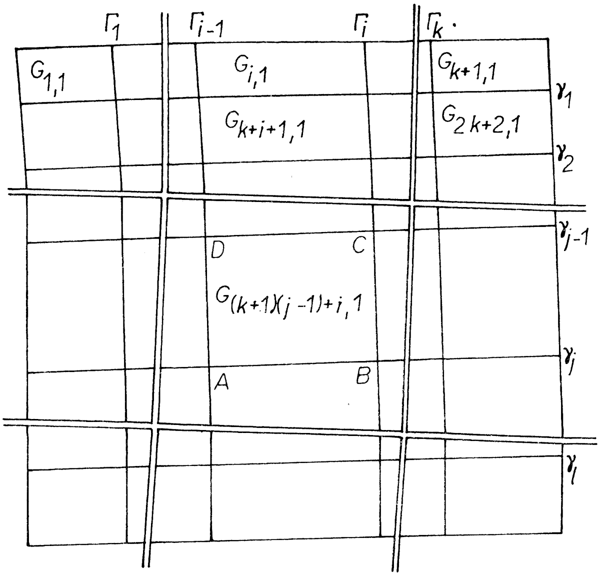

Fig. 2.

Lemma 1.2. *Let $w \in W$. Then the following estimates hold:*

$$\inf_{\varphi \in V_h} |||w - \varphi|||_h \leqq Ch\left(\sum_{i=1}^{k(1)} |w|^2_{3,G_{i1}} + \sum_{\Gamma \in I} |w|^2_{3,\Gamma} + \sum_{\gamma \in J} |w|^2_{3,\gamma}\right)^{1/2}, \tag{3.2}$$

$$\inf_{\varphi \in \mathfrak{M}_h} |w - \varphi|_{2,h} \leqq Ch\left(\sum_{i=1}^{k(1)} |w|^2_{3,G_{i1}}\right)^{1/2}. \tag{4.2}$$

Proof. This lemma can be proved in the same way as Lemma 4.1 in [1].

Remark: If the solution u belongs to the space W then

$$|||u - u_h|||_h \leqq Ch\,.$$

Theorem 2.2. *Let u and u_h be the solutions of the problem (1.1) and (1.2), respectively. Let us denote the closure of the space W with respect to the norm $|||\cdot|||$ by $\overline{W}$. If $u \in \overline{W}$ then*

$$\lim_{h \to 0} |||u - u_h|||_h = 0\,. \tag{5.2}$$

First we recall a very important auxiliary assertion:

Lemma 2.2. *The norm $|||\cdot|||_h$ can be extended onto the space V for each $h \in (0, 1)$. Moreover, the estimate*

$$|||w|||_h \leqq C|||w||| \tag{6.2}$$

holds for each $w \in V$ and $h \in (0, 1)$; the constant C is independent of w.

Proof. See [1], Lemma 4.3.

Proof of Theorem 2.2: Using the error estimate (2.2) we can state

$$\tag{7.2} |||u - u_h|||_h \leqq C\{ \inf_{\varphi \in V_h} |||w - \varphi|||_h + |||u - w|||_h + |u - w|_{2,h} + + \inf_{\psi \in \mathfrak{M}_h} |w - \psi|_{2,h} + h^2\}$$

for any $w \in W$; the constant C is independent of w. Let us notice that $|u - w|_{2,h} \leqq \leqq |||u - w|||_h$. Hence, in virtue of (6.2) and the fact that $u \in \overline{W}$ we can state: For an arbitrary $\varepsilon > 0$ there exists $w \in W$ such that

$$C\{|||u - u_h|||_h + |u - w|_{2,h}\} < \varepsilon/2$$

for each $h \in (0, 1)$. According to (3.2) and (4.2), we can choose $h_0 \in (0, 1)$ such that

$$C\{ \inf_{\varphi \in V_h} |||w - \varphi|||_h + \inf_{\psi \in \mathfrak{M}_h} |w - \psi|_{2,h} + h^2\} < \varepsilon/2$$

for each $h \in (0, h_0)$. This means that the right hand side of (7.2) can be estimated by an arbitrary $\varepsilon > 0$ for sufficiently small h.

Remark. Now we are to verify the assumption $u \in \overline{W}$ of Theorem 2.2. In [1] we succeeded in the case $J = \emptyset$ (or $I = \emptyset$, respectively) only.

3. REGULARITY. CONVERGENCE

First we introduce the most important assertion of this chapter concerning an "apriori" information about regularity of a (weak) solution u of our problem.

Theorem 1.3. *If a function u belongs to V then its traces u and $\partial u/\partial x$ and $\partial u/\partial y$ are continuous with respect to $\partial G \bigcup_{\Gamma \in I} \Gamma \bigcup_{\gamma \in J} \gamma$.*

The proof of the theorem will be based on some auxiliary lemmas.

Lemma 1.2. *Let $\Omega = \{(x, y) \in R_2, x \in (0, a_1), \; y \in (0, a_2)\}$, $a_1 \neq 0, a_2 \neq 0$ be a rectangle and $A = [0, 0]$, $B = [a_1, 0]$, $C = [a_1, a_2]$, $D = [0, a_2]$ its vertices. Let us suppose that*

a) $w \in W^{1,2}(\Omega)$,

b) $w \equiv 0$ on $\overline{AB} \cup \overline{BC} \cup \overline{CD} \setminus \{A\}$,

c) $w(A) = \varepsilon$,

d) *w is a linear function along $\overline{AD}$.*

Then $\varepsilon = 0$, i.e. $w \equiv 0$ on $\partial\Omega$.

Proof. We extend the function w onto the set $\mathbb{R}_2^+ = \{(x, y) \in \mathbb{R}_2, x \geqq 0\}$. For $x > a_1$ or $y > a_2$ we set $w(x, y) = 0$ and for $y < 0$ we set $w(x, y) = w(x, -y)$. Using the fact that $w(x, 0) \equiv 0$ for $x < 0$, we can easily verify that $w \in W^{1,2}(\mathbb{R}_2^+)$. It is well known (see e.g. [2]) that $w(0, .) \in W^{1/2,2}(\mathbb{R}_1)$. This is equivalent to the condition $T < +\infty$, where

$$T = \int_{-\infty}^{\infty} (1 + |\xi|^2)^{1/2} |\hat{w}(\xi)|^2 \, d\xi$$

and

$$\hat{w}(\xi) = \int_{-\infty}^{\infty} w(0, y) \, e^{-iy\xi} \, dy$$

(see [2], Theorem 1.2). By the direct calculation we obtain

$$\hat{w}(\xi) = \frac{2i\varepsilon}{\xi}\left(1 - \frac{\sin a_2\xi}{a_2\xi}\right).$$

Substituting it into the formula for T, we get $T < +\infty$ iff $\varepsilon = 0$.

Lemma 2.3. *Let Ω be the rectangle described in Lemma 1.3. If $g \in W^{1,2}(\partial\Omega)$ then there exists $p \in W^{1,2}(\Omega)$ such that $p = g$ on $\partial\Omega$ in the sense of traces.*

Proof. We could quote the work [5] but the assumption $g \in W^{1,2}(\partial\Omega)$ is strong enough to allow for a simple proof: Let $\chi(x, y) = \alpha x + \beta y + \gamma xy + \delta$ be the polynomial of the first degree with the coefficients $\alpha, \beta, \gamma, \delta$ uniquely determined by the following conditions:

$\chi = g$ at the points A, B, C, D (vertices of Ω).

We define the function $\psi = g - \chi$ on $\partial\Omega$. It is obvious that $\psi \in W^{1,2}(\partial\Omega)$ and $\psi = 0$ at A, B, C, D. Let ω_{AB} be an infinitely differentiable function with a compact support in $\mathbb{R}_2$, $\omega_{AB} \equiv 1$ on $\overline{AB}$, $\omega_{AB} \equiv 0$ on $\overline{CD}$. Hence, the function $\psi_{AB}(x, y) = \psi(0, x) . \omega_{AB}(x, y)$ has the following properties: $\psi_{AB} \in W^{1,2}(\Omega)$, $\psi_{AB} \equiv 0$ on $\partial\Omega \setminus \overline{AB}$, $\psi_{AB} = \psi$ along the side $\overline{AB}$. In the same way we can define functions ψ_{BC}, ψ_{CD}, ψ_{AD} belonging to $W^{1,2}(\Omega)$ which have the following properties:

$$\begin{array}{llll} \Delta_{BC} \equiv 0 & \text{on} \quad \partial\Omega \setminus \overline{BC}\,, & \psi_{BC} = \psi & \text{on} \quad \overline{BC} \\ \psi_{CD} \equiv 0 & \text{on} \quad \partial\Omega \setminus \overline{CD}\,, & \psi_{CD} = \psi & \text{on} \quad \overline{CD} \\ \psi_{AD} \equiv 0 & \text{on} \quad \partial\Omega \setminus \overline{AD}\,, & \psi_{AD} = \psi & \text{on} \quad \overline{AD}\,. \end{array}$$

Then it is sufficient to set $p = \psi_{AB} + \psi_{BC} + \psi_{CD} + \psi_{AD}$.

Lemma 3.3. *Let Ω be the rectangle described in Lemma 1.3. Let us suppose that $v \in W^{1,2}(\Omega) \cap W^{1,2}(AB) \cap W^{1,2}(BC) \cap W^{1,2}(CD) \cap W^{1,2}(AD)$ and v is continuous on $\partial\Omega \setminus \{A\}$. Then v is continuous on the whole boundary $\partial\Omega$.*

Proof. Let us define:

$$\varepsilon = \lim_{t \to 0_+} v(tB + (1 - t) A) - \lim_{t \to 0_+} v(tD + (1 - t) A) .$$

The existence of both the limits is guaranteed by the fact that $v \in W^{1,2}(AB)$, $v \in W^{1,2}(AD)$. Let us suppose that the assertion of the lemma does not hold, i.e. $\varepsilon \neq 0$.

Let w be the trace of the function defined in b), c), d) of Lemma 1.3, where the constant ε is given by the above equation ($\varepsilon \neq 0$). Then the function $v + w$ belongs evidently to $W^{1,2}(\partial\Omega)$*).

According to Lemma 2.3 there exists a function $U \in W^{1,2}(\Omega)$, $U = v + w$ on $\partial\Omega$. It means that the function $z = U - v$ belongs to $W^{1,2}(\Omega)$ and $z = w$ on $\partial\Omega$ in the sense of traces.

Using Lemma 1.3, we obtain a contradiction: $\varepsilon = 0$.

Proof of Theorem 1.3.

a) According to the well known embedding theorem (see [4]), the function u is continuous even on $\overline{\Omega}$.

b) Let v be equal to $\partial u/\partial x$ or $\partial u/\partial y$ on $\partial G \bigcup_{\Gamma \in I} \Gamma \bigcup_{\gamma \in J} \gamma$ in the sense of traces. Let us construct a basic division $G_1 \equiv \{G_{i1}\}_{i=1}^{k(1)}$ in the natural way – see Fig. 2. If $\Omega = G_{(k+1)(j-1)+i,1}$ is an arbitrary rectangle of the division G_1 then we denote its vertices by A, B, C and D – see Fig. 2. According to the definition of the space V, the function v belongs to $W^{1,2}(\Omega) \cap W^{1,2}(AB) \cap W^{1,2}(BC) \cap W^{1,2}(CD) \cap W^{1,2}(AD)$. It means that v is continuous on each side of the rectangle Ω. Hence it remains to show that v is continuous at the vertices A, B. C, D with respect to $\partial\Omega$. (Evidently this assertion would complete the proof of Theorem 1.3).

We make the following assumption:

(A) v is continuous at the vertices A, C, D with respect to $\partial\Omega$. Using Lemma 3.3, we can state that v is continuous at B (with respect to $\partial\Omega$).

If A, C and D coincides with ∂G then v is continuous at A, C and D, respectively. This fact follows from the definition of the space V immediately. Hence, the assumption (A) is fulfilled in the case $\Omega = G_{1,1}$. It starts the induction of (A).

Theorem 2.3. *Let $u \in V$ be the solution of the problem* (1.1). *Then $u \in \overline{W}$ (see Theorem* 2.2).

To prove the above assertion we need some auxiliary lemmas.

*) $v + w$ is continuous along $\partial\Omega$ and belongs to $W^{1,2}$ along each side of $\partial\Omega$.

Lemma 4.3. *Let Ω be a nondegenerate rectangle; let $F_1 \in L_2(\Omega)$ be given. Then there exists a (unique) solution $u_1 \in W^{3,2}(\Omega) \cap W_0^{2,2}(\Omega)$ of the problem*

$$a(u_1, \varphi) = 2 \int_\Omega F_1 \varphi \, \mathrm{d}x \, \mathrm{d}y^{*)}$$

for each $\varphi \in W_0^{2,2}(\Omega)$.

Proof. See [3].

Lemma 5.3. *Let Ω be a nondegenerate rectangle (the same as in Lemma 1.3). Let the functions φ_0, φ_1 be given on $\partial\Omega$ so that*

a) *φ_0, φ_1 are infinitely differentiable along each edge of $\partial\Omega$,*

b) *the supports of φ_0 and φ_1 do not contain vertices A, B, C, D (i.e. $\varphi_0, \varphi_1 \in D(AB) \cap D(BC) \cap D(DC) \cap D(AD)$).*

Then there exists an infinitely differentiable function Ψ on Ω such that $\Psi = \varphi_0$, $\partial\Psi/\partial\nu = \varphi_1$ on $\partial\Omega$ in the sense of traces (ν is the outward normal vector).

Proof. Let $\omega_{AB} = \omega_{AB}(x, y)$ be an infinitely diffetentiable function with a compact support in $\mathbb{R}_2$; $\omega_{AB} \equiv 1$ in a neighbourhood of the $\overline{AB}$, $\omega_{AB} \equiv 0$ on $\overline{CD}$. The values of functions φ_0 and φ_1 on $\overline{AB}$ are denoted by $\varphi_0(x, 0)$ and $\varphi_1(x, 0)$, where $x \in \langle 0, a_1 \rangle$ (see Lemma 1.3); we notice that $\varphi_0(., 0)$ and $\varphi_1(., 0)$ belong to $D((0, a_1))$ according to the assumption b).

It can be easily verified that the function

$$\Psi_{AB}(x, y) = (\varphi_0(x, 0) + y\, \varphi_1(x, 0))\, \omega_{AB}(x, y)$$

is infinitely differentiable on Ω, $\Psi_{AB} \equiv \partial\Psi_{AB}/\partial\nu \equiv 0$ on $\partial\Omega \dot{-} \overline{AB}$, $\Psi_{AB} = \varphi_0$ and $\partial\Psi_{AB}/\partial\nu = \varphi_1$ on $\overline{AB}$. In the same way it is possible to define functions Ψ_{BC}, Ψ_{CD}, Ψ_{AD} having the same property as described above, when replacing AB by BC, CD and AD, respectively.

Now it is sufficient to set

$$\Psi = \Psi_{AB} + \Psi_{BC} + \Psi_{CD} + \Psi_{AD}\,.$$

This completes the proof.

Lemma 6.3. *Let Ω be a nondegenerate rectangle with vertices A, B, C, D. Let $F \in L_2(\Omega)$ and $\varphi_0, \varphi_1 \in D(AB) \cap D(BC) \cap D(CD) \cap D(AD)$ be given. Then there exists a unique solution $w \in W^{3,2}(\Omega)$ of the problem: Find $w \in W^{2,2}(\Omega)$ such that $w = \varphi_0$, $\partial w/\partial\nu = \varphi_1$ on $\partial\Omega$ in the sense of traces and*

$$a(w, \varphi) = 2 \int_\Omega F\varphi \, \mathrm{d}x \, \mathrm{d}y$$

for each $\varphi \in W_0^{2,2}(\Omega)$.

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*) The bilinear form $a(., .)$ is defined over Ω; see (1.1) with G replaced by Ω.

Proof. Let F_1 be equal to $F - \Delta^2\Psi$; Ψ is an infinitely differentiable function on Ω, $\Psi = \varphi_0$, $\partial\Psi/\partial\nu = \varphi_1$ on $\partial\Omega$ (see Lemma 5.3). According to Lemma 4.3, there exists $u_1 \in W^{3,2}(\Omega) \cap \overset{\circ}{W}{}^{2,2}_0(\Omega)$ such that

$$a(u_1, \varphi) = 2\int_\Omega F_1\varphi \, dx \, dy$$

for each $\varphi \in W_0^{2,2}(\Omega)$. Integrating by parts, we obtain the equality

$$2\int_\Omega F_1\varphi \, dx \, dy = 2\int_\Omega F\varphi \, dx \, dy - a(\Psi, \varphi); \; \varphi \in W_0^{2,2}(\Omega) .$$

Hence, we get finally:

$$a(u_1 + \Psi, \varphi) = 2\int F\varphi \, dx \, dy$$

for each $\varphi \in W_0^{2,2}(\Omega)$. Evidently, we can set $w = u_1 + \Psi$.

Lemma 7.3. *Let Ω be a nondegenerate rectangle with vertices A, B, C, D. There exists $w \in W^{2,2}(\Omega)$ satisfy the following conditions:*

a) $a(w, \varphi) = 0$ *for each* $\varphi \in W_0^{2,2}(\Omega)$,

b) $w = \varphi_0$, $\partial w/\partial \nu = \varphi_1$ *on* $\partial\Omega$ *in the sense of traces where* $\varphi_0 \in W_0^{2,2}(AB) \cap \cap W_0^{2,2}(BC) \cap W_0^{2,2}(CD) \cap W_0^{2,2}(AD)$ *and* $\varphi_1 \in W_0^{1,2}(AB) \cap W_0^{1,2}(BC) \cap W_0^{1,2}(CD) \cap \cap W_0^{1,2}(AD)$.

Furthermore, there exists a constant C independent of φ_0, φ_1 such that

$$\|w\|_{2,\Omega} \leqq C\{|\varphi_0|_{2,AB} + |\varphi_0|_{2,BC} + |\varphi_0|_{2,CD} + |\varphi_0|_{2,AD} + $$
$$+ |\varphi_1|_{1,AB} + |\varphi_1|_{1,BC} + |\varphi_1|_{1,CD} + |\varphi_1|_{1,AD}\} = \alpha C$$

Proof. According to [5]*), we can find a function $\Psi \in W^{2,2}(\Omega)$, $\Psi = \varphi_0$ and $\partial\Psi/\partial\nu = \varphi_1$ ob $\partial\Omega$ in the sense of traces, $\|\Psi\|_{2,\Omega} \leqq C\alpha$, where the constant C does not depend on φ_0, φ_1. Let $w_1 \in W_0^{2,2}(\Omega)$ satisfy the equation

$$a(w_1, \varphi) = -a(\Psi, \varphi)$$

for each $\varphi \in W_0^{2,2}(\Omega)$. It is well known that w_1 exists and

$$\|w_1\|_{2,\Omega} \leqq C\|\Psi\|_{2,\Omega}$$

where C is independent of Ψ (i.e. of φ_0, φ_1)**). Setting $w = w_1 + \Psi$, we obtain a function with the properties a), b). The estimate $\|w\|_{2,\Omega} \leqq C \,.\, \alpha$ is evident.

---

*) See Lemma 3.1 in [1], where we have quoted Jakovlev's theorem.

**) With regard to [4], it is evident that the bilinear form $a(.,.)$ is $W_0^{2,2}(\Omega)$ — eliptic and $a(\Psi, .)$ is a linear continuous functional over $W_0^{2,2}(\Omega)$. The existence of w_1 and the corresponding estimate is a consequence of the Lax-Milgram theorem.

Proof of Theorem 2.3. Let G_1 be the "basic" division of G; let Ω be an arbitrary rectangle of the division G_1 with vertices A, B, C, D – see Fig. 2. According to Theorem 1.3, the functions u and $\partial u/\partial x$ and $\partial u/\partial y$ are continuous at the points A, B, C, D with respect to $\partial G \bigcup_{\Gamma \in I} \Gamma \bigcup_{\gamma \in J} \gamma$. We define on each Ω a polynomial

$$\omega = \omega(x, y) = \sum_{i,j=0}^{5} a_{ij} x^i y^j$$

satisfying the following conditions:

$$\omega(X) = u(X) \qquad \frac{\partial^2 \omega}{\partial x^2}(X) = \frac{\partial^2 \omega}{\partial y^2}(X) = 0$$

$$\frac{\partial \omega}{\partial x}(X) = \frac{\partial u}{\partial x}(X)$$

$$\frac{\partial \omega}{\partial y}(X) = \frac{\partial u}{\partial y}(X) \qquad \frac{\partial^3 \omega}{\partial x^3\, \partial y}(X) = \frac{\partial^3 \omega}{\partial x\, \partial y^2}(X) = 0$$

$$\frac{\partial^2 \omega}{\partial x\, \partial y}(X) = 0 \qquad \frac{\partial^4 \omega}{\partial x^2\, \partial y^2}(X) = 0$$

for $X = A, B, C, D$, respectively (36 conditions).

The above conditions determine the polynomial ω on Ω uniquely. The function ω is twice continuously differentiable over G and vanishes together with its first-order derivatives over ∂G. It means that $\omega \in W_0^{2,2}(G) \cap W^{3,2}(G)$. The traces ω, $\partial\omega/\partial x$ on $\Gamma \in I$ and ω, $\partial\omega/\partial y$ on $\gamma \in J$ are piecewise polynomials of the 5-th degree which are continuous with its second-order derivatives over Γ and γ, respectively (see [6]). We can easily verify that $\omega \in V \cap W$.

Using (1.1), we get the following condition:

$$a(u, \varphi) = 2 \int_\Omega f\varphi \,\mathrm{d}x\,\mathrm{d}y$$

for each $\varphi \in W_0^{2,2}(\Omega)$, where $a(.,.)$ is restricted onto Ω*).

We set

$$U = u - \omega$$

(on the whole G). It is evident that the function U satisfies the following conditions on Ω:

a) $a(U, \varphi) = 2 \int_\Omega (f - \Delta^2 \omega)\, \varphi \,\mathrm{d}x\,\mathrm{d}y$

for each $\varphi \in W_0^{2,2}(\Omega)$,

---

*) If $\varphi \in D(\Omega)$ then evidently $\varphi \in V$. Hence the conditions hold for each $\varphi \in D(\Omega)$. Using the density of $D(\Omega)$ in $W_0^{2,2}(\Omega)$ and the continuity of $a(u,.)$, we obtain the present condition.

b) $U = \varphi_0$ and $\partial U/\partial \nu = \varphi_1$, where

$$\varphi_0 \in W_0^{2,2}(AB) \cap W_0^{2,2}(BC) \cap W_0^{2,2}(CD) \cap W_0^{2,2}(AD)$$

and

$$\varphi_1 \in W_0^{1,2}(AB) \cap W_0^{1,2}(BC) \cap W_0^{1,2}(CD) \cap W_0^{1,2}(AD).$$

In virtue of the density of $D(\cdot)$ in $W_0^{2,2}(\cdot)$ and $W_0^{1,2}(\cdot)$, there exist sequences $\{\varphi_{0n}\}_{n=1}^{\infty}$, $\{\varphi_{1n}\}_{n=1}^{\infty}$ such that

$$\varphi_{0n} \in D(AB) \cap D(BC) \cap D(CD) \cap D(AD),$$

$$\varphi_{1n} \in D(AB) \cap D(BC) \cap D(CD) \cap D(AD)^{*})$$

Using (1.1), we get the following condition:

$$a(u, \varphi) = 2 \int_{\Omega} f\varphi \, dx \, dy$$

for each $\varphi \in W_0^{2,2}(\Omega)$, where $a(.,.)$ is restricted onto Ω**).

We set

$$U = u - \omega$$

(on the whole G). It is evident that the function U satisfies the following conditions on Ω:

a) $a(U, \varphi) = 2 \int_{\Omega} (f - \Delta^2 \omega) \varphi \, dx \, dy$ for each $\varphi \in W_0^{2,2}(\Omega)$,

b) $U = \varphi_0$ and $\partial U/\partial \nu = \varphi_1$, where

$$\varphi_0 \in W_0^{2,2}(AB) \cap W_0^{2,2}(BC) \cap W_0^{2,2}(CD) \cap W_0^{2,2}(AD) \text{ and}$$

$$\varphi_1 \in W_0^{1,2}(AB) \cap W_0^{1,2}(BC) \cap W_0^{1,2}(CD) \cap W_0^{1,2}(AD).$$

In virtue of the density of $D(\cdot)$ in $W_0^{2,2}(\cdot)$ and $W_0^{1,2}(\cdot)$, there exist sequences $\{\varphi_{0n}\}_{n=1}^{\infty}$, $\{\varphi_{1n}\}_{n=1}^{\infty}$ such that

$$\varphi_{0n} \in D(AB) \cap D(BC) \cap D(CD) \cap D(AD),$$

$$\varphi_{1n} \in D(AB) \cap D(BC) \cap D(CD) \cap D(AD)^{***})$$

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*) If φ_0 and φ_1 are identically zero over a side then we set φ_{0n} and φ_{1n} equal to zero along that side. This is the case of the sides coinciding with the boundary $\partial\Omega$.

**) If $\varphi \in D(\Omega)$ then evidently $\varphi \in V$. Hence the conditions hold for each $\varphi \in D(\Omega)$. Using the density of $D(\Omega)$ in $W_0^{2,2}(\Omega)$ and the continuity of $a(u, .)$, we obtain the present condition.

***) If φ_0 and φ_1 are identicaly zero over a side then we set φ_{0n} and φ_{1n} equal to zero along that side. This is the case of the sides coinciding with the boundary $\partial\Omega$.

$(n = 1, 2, \ldots)$ and defining

$$\alpha_n = |\varphi_0 - \varphi_{0n}|_{2,AB} + |\varphi_0 - \varphi_{0n}|_{2,BC} + |\varphi_0 - \varphi_{0n}|_{2,CD} + |\varphi, - \varphi_{0n}|_{2,AD},$$

$$\beta_n = |\varphi_1 - \varphi_{1n}|_{1,AD} + |\varphi_1 - \varphi_{1n}|_{1,BC} + |\varphi_1 - \varphi_{1n}|_{1,CD} +$$

$$+ |\varphi_1 - \varphi_{1n}|_{1,AD}$$

we have

$$\alpha_n \to 0\,, \quad \beta_n \to 0 \quad \text{for} \quad n \to \infty\,.$$

For each n we solve an auxiliary problem: Find $U_n \in W_0^{2,2}(\Omega)$ such that

$$U_n = \varphi_{0n}\,, \quad \partial U_n/\partial \nu = \varphi_{1n} \quad \text{on} \quad \partial\Omega \quad \text{in the sense of traces}\,,$$

$$a(U_n, \varphi) = 2\int_\Omega (f - \Delta^2 \omega)\, \varphi \,\mathrm{d}x\,\mathrm{d}y$$

for each $\varphi \in W_0^{2,2}(\Omega)$.

According to Lemma 6.3, the solution U_n exists and

$$U_n \in W^{3,2}(\Omega)\,.$$

We define $u_n = U_n + \omega$. The function $u - u_n = U - U_n$ satisfies the condition

$$a(u - u_n, \varphi) = 0$$

for each $\varphi \in W^{2,2}(\Omega)$ and $u - u_n = \varphi_0 - \varphi_{0n}$, $(\partial/\partial\nu)(u - u_n) = \varphi_1 - \varphi_{1n}$ on $\partial\Omega$ in the sense of traces. Using Lemma 7.3, we conclude

$$\|u - u_n\|_{2,\Omega} \leqq C(\alpha_n + \beta_n) \to 0$$

for $n \to \infty$.

If we realize that U_n belongs to V then evidently $u_n \in V$, too. Moreover, u_n belongs to W. It remains to show that $u_n \to u$ with respect to the norm $|||\cdot|||$.

According to the definition of $|||\cdot|||$, we have the inequality

$$|||u - u_n||| \leqq C\Big[\sum_{i=1}^{k(1)} \big(|u - u_n|^2_{2,G_{i1}} + \alpha_n^2 + \beta_n^2\big)\Big]^{1/2}\,.$$

It was shown that the right hand side converges towards zero for $n \to \infty$.

Theorem 3.3. *Let u and u_h be the solutions of the problem* (1.1) *and* (1.2), *respectively. Then our method converges in the following sense*:

$$\lim_{h \to 0} |||u - u_h|||_h = 0\,.$$

Proof. See Theorem 2.3, (5.2) (Theorem 2.2) and (6.2).

4. NUMERICAL EXAMPLE

(One case of a more general boundary condition.)

So far we have studied the clamped plate with ribs – see the previous chapters and our paper [1]. The present nonconforming method can be applied to more general cases of boundary conditions. The following example was tested practically and the results are given in this chapter.

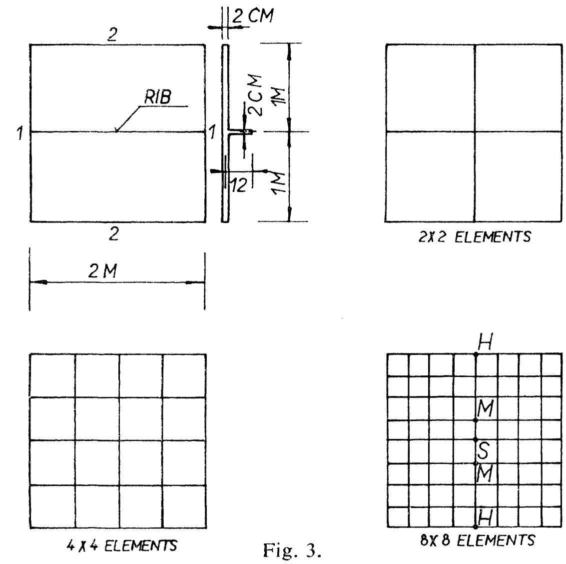

Fig. 3.

Let a plate be clamped along the edges AD, BC and simply supported along the edges AB, CD – see Fig. 3. We shall consider two perpendicular systems of ribs, as usual:

$$I = \{\Gamma_i\}_{i=1}^{k}$$

– see Fig. 2

$$J = \{\gamma_j\}_{j=1}^{l}$$

It is easy to derive the following variational formulation of the problem considered: To find $u \in V$ such that the equation (1.1) holds for each $v \in V$, where

$$V = \{w;\ w \in W^{2,2}(G),\ D_\alpha w = 0 \quad \text{a.e.} \quad \text{on} \quad \overline{AD} \cup \overline{BC} \quad \text{for} \quad |\alpha| \leqq 1\,,$$

$$w = 0 \quad \text{a.e.} \quad \text{on} \quad \overline{AB} \cup \overline{CD},\ w \in W^{2,2}(\Gamma) \cap W_0^{1,2}(\Gamma),\ \partial w/\partial x \in W_0^{1,2}(\Gamma)$$

$$\text{for each} \quad \Gamma \in I,\ w \in W_0^{2,2}(\gamma),\ \partial w/\partial y \in W_0^{1,2}(\gamma) \quad \text{for each} \quad \gamma \in J\}\,.$$

The space V is equipped with the norm $|||\cdot|||$ defined in Chapter 1.

Let $G_h = \{G_{ih}\}_{i=1}^{k(h)}$ be a regular division of the region G, let R be a fixed non-degenerate rectangle and let $A(R)$ be the set of Ari-Adini's polynomials over R. We denote by F_{ih} the regular affine mapping from G_{ih} onto R. Let

$$(2.4) \qquad V_h = \{w;\ w \circ F_{ih}^{-1} \in A(R) \ \textit{for each}\ i = 1, \ldots, k(h),\ \textit{if}\ Q \ \textit{is a nodal point of the division}\ G_h\ \textit{then}\ D^\alpha w\ \textit{is continuous at}\ Q\ \textit{with respect to}\ G\ \textit{for}\ |\alpha| \leqq 1,\ \textit{if}\ Q \in \overline{AD} \cup \overline{BC}\ \textit{then}\ D^\alpha w(Q) = 0\ \textit{for}\ |\alpha| \leqq 1,\ \textit{if}\ Q \in \overline{AB} \cup \overline{CD}\ \textit{then}\ w(Q) = 0\}\,.$$

It is easy to verify that $|||\cdot|||_h$ from Chapter 2 is a norm on V_h when V_h is now defined by (2.4).

We define finite element procedure in the natural way: To find $u_h \in V_h$ such that (1.2) holds for each $v \in V_h$.

If we define the space of regular solutions of (1.1) in the following way

$$(3.4) \qquad W = \{w;\ w \in V,\ w \in W^{3,2}(G_{s1}),\ w \in W^{3,2}(\Gamma_i),\ w \in W^{3,2}(\gamma_j),\ s = 1, \ldots, m,\ i = 1, \ldots, k,\ j = 1, \ldots, 1,\ m = (k+1)(1+1\}$$

where V is now defined by (1.4) and the geometrical notation is clear from Fig. 2, we can follow the same lines as in the proof of convergence of approximations to the regular solution in [1]. When the regularity of the solution of the problem (1.1) is not known apriori at all, then we can prove the convergence of the present nonconforming method in a similar manner as in the case of the clamped plate.

As can be readily seen, the analysis of the proposed procedure can be regarded as a combination of a plate flexure analysis and of a beam flexure and torsion in Saint-Venant sense. The computer programme for the analysis of a plate with ribs was thus written in two stages.

In the first stage, the stiffness matrix of the plate was constructed by the non-conforming method – Adini's rectangle. The explicit form of the stiffness matrix for Adini's rectangle has been published in many papers such as e.g. [9]. In [1] and in the previous chapters of this paper it was a type of problem discussed. It is readily seen that the theorems and lemmas remain valid also for a more general form of problem (i.e., when the real values of material constants are taken into account). In our examples we have considered isotropic plates with Hooke's law in the standard form:

$$\{\sigma\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = [D]\{\varepsilon\}\,; \quad \{\varepsilon\} = \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x\,\partial y} \end{Bmatrix}$$

where M_x, M_y, M_{xy} are the bending and twisting moments, respectively, and

$$[D] = \frac{Et^3}{12(1 - v^2)} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & (1 - v)/2 \end{bmatrix}$$

t being the thickness, E Young's modulus and v Poisson's ratio.

In the second stage the stiffness matrices of beams with respect to the torsion and flexure rigidity was found. For details of the torsional properties of a beam see [8]. According to the numerical method proposed, the stiffness matrix of an element of a beam is as follows:

$$\begin{Bmatrix} Q_i \\ M_i \\ T_i \\ Q_j \\ Mj \\ T_j \end{Bmatrix} = EJ \begin{bmatrix} 12/l^3 & 6/l^2 & 0 & -12/l^3 & 6/l^2 & 0 \\ 6/l^2 & 4/l & 0 & -6/l^2 & 2/l & 0 \\ 0 & 0 & \varepsilon/l & 0 & 0 & -\varepsilon/l \\ -12/l^3 & -6/l^2 & 0 & 12/l^3 & -6/l^2 & 0 \\ 6/l^2 & 2/l & 0 & 6/l^2 & 4/l & 0 \\ 0 & 0 & -\varepsilon/l & 0 & 0 & \varepsilon/l \end{bmatrix} \begin{Bmatrix} w_i \\ \varphi_i \\ \xi_i \\ w_j \\ \varphi_j \\ \xi_j \end{Bmatrix}$$

where Q, M, T are the shear force, the bending and the twisting moment, respectively, w, φ, ξ, are the corresponding generalized displacements, $\varepsilon = J_D/(2(1 + v)\,J)$ with J_D standing for the torsion rigidity (see [8]). 1 is the length of an element with the endpoints i and j, J the momentum of inertia.

The two stages were then assembled to form the programme for the analysis of a complete plate with ribs.

To check the accuracy of the method and in particular to determine the rate of convergence with respect to the number of elements employed, several problems were calculated and results compared with results obtained by the folded plate method – see [7].

Because of an extraordinary precision of results obtained by the folded plate method in the case of boundary conditions used in the following examples, this only case was analysed. Our effort was concentrated to test various cases of geometry and types of loading.

The constructions were analysed under various mesh sizes in otder to test the convergence of the method. The results obtained are given in tables together with the solution by the folded plate method. The percentage error of the approximation, calculated as

$$\frac{\text{approximate value} - \text{folded plate solution}}{\text{folded plate solution}}\, 100\%$$

are also given in the tables. According to the formula for percentage errors, an overestimation will be shown by a positive sign and an underestimation by a negative sign. For the sake of simplicity in each of the three examples and for each mesh size the elements of division are of the same geometry.

The first problem considered was that of an isotropic square plate $2 \times 2\ m^2$ with a thickness of 2 cm, clamped along the opposite edges and simply supported along the rest of the boundary, subjected to a uniformly distributed load of 1 kp/cm^2. The only rib ($2 \times 12\ cm^2$) is situated in the middle of the plate and it connects the simply supported edges – see Fig. 3, where also the various grids used are shown. The torsion rigidity of the rib has no influence in this case. The results are given in Table 1. From this table it is seen that the finite elements results are in reasonably good agreement with the solution obtained by the folded plate method and they do converge towards the exact values (we suppose, of course, that the folded plate method yields reasonably accurate values).

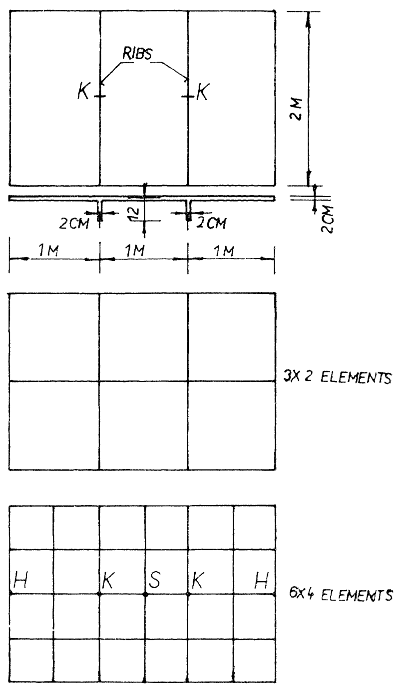

Fig. 4.

In order to test the ability of the method to take more irregular load-functions into consideration, the plate $3 \times 2\ m^2$, with the same thiskness as in the first example, is now subjected to 10 Mp at the points K and two parallel ribs connect the simply supported edges – see Fig. 4. From Table 2 which shows the results obtained we can see again a reasonably good agreement between the results reached by the two methods.

Table 1

Number of elements	Central deflection		Central positive moments				Maximum positive moment		Maximum negative moment	
	w_S		M_{xS}		M_{yS}		M_M		M_H	
	value	% error	value	% error	value	% error	value	% error	value	% error
2 × 2	1·507	18·47	614·03	2·94	1158·13	93·01	—	—	−1807·47	13·09
4 × 4	1·346	5·82	587·79	−1·47	668·23	11·36	—	—	−2002.76	3·70
8 × 8	1·291	1·49	594·03	−0·42	615·27	2·53	789·0	3·89	−2054·93	1·19
folded plate	1·272		596·55		600·09		765·68		−2079·75	

Table 2

Number of elements	Central deflection		Central maximum positive moments				Maximum negative moment	
	w_S		M_{xS}		M_{yS}		M_H	
	value	% error	value	% error	value	% error	value	% error
6 × 4	2·505	3·51	1054·44	−2·66	1062·04	6·55	−2177·86	−5·27
12 × 8	2·455	1·45	1077·36	−0·547	1012·76	1·61	−2261·84	−1·615
folded plate	2·42		1083·29		996·72		−2298·968	

Table 3

Number of elements	Deflection								Maximum negative moments			
	w_A		w_B		w_C		$w_M = w_{max}$		M_D		M_S	
	value	% error	value	% error	value	% error	value	% error	value	% error	value	% error
2 × 2	0·7534	−18·50	—	—	—	—	—	—	−1372·48	−16·22	−435·50	−2·07
4 × 2	0·7475	−17·57	0·7193	−29·65	0·2745	−12·73	—	—	−2062·20	−25·88	−459·31	3·29
2 × 4	0·6764	− 6·39	—	—	—	—	—	—	−1038·22	−36·62	−418·99	−5·78
4 × 4	0·6729	− 5·84	0·5918	− 6·67	0·2541	− 4·35	—	—	−1576·65	− 3·76	−426·28	−4·14
4 × 8	0·6526	− 2·64	0·5687	− 2·51	0·2486	− 2·09	—	—	−1463·33	−10·67	−418·20	−5·96
8 × 4	0·6651	− 4·61	0·5876	− 5·91	0·2511	− 3·12	0·7470	−5·34	−1706·73	− 4·18	−435·69	−2·03
8 × 8	0·6453	− 1·49	0·5643	− 1·71	0·2461	− 1·07	0·7435	−4·85	−1619·25	− 1·16	−444·70	0·00
folded plate	0·6358		0·5548		0·2435		0·7091		−1638·34		−444·70	

Number of elements	Central moments								Max. posit. nom.			
	M_{Ax}		M^-_{Ay}		M^+_{Ay}		M_{Bx}		M_{By}		$M_M = M_{max}$	
	value	% error	value	% error	value	% error	value	% error	value	% error	value	% error
2 × 2	359·40	−19·09	890·50	−208·8	267·59	14·17	—	—	—	—	—	—
4 × 2	367·39	−21·74	234·27	18·74	244·49	21·58	274·79	8·33	830·8	−55·90	—	—
2 × 4	357·43	−18·44	901·40	−212·65	288·50	7·47	—	—	—	—	—	—
4 × 4	319·39	− 5·83	370·61	− 28·55	297·58	4·55	251·14	16·19	692·65	−29·99	—	—
4 × 8	303·76	− 0·65	413·16	− 43·30	290·09	6·96	213·21	28·85	714·96	−34·17	—	—
8 × 4	321·18	− 3·44	297·24	− 3·09	315·10	−1·06	251·26	16·15	575·35	− 7·97	696·82	−4·19
8 × 8	305·94	− 1·37	308·98	− 7·17	306·27	1·77	287·58	4·22	568·46	− 6·68	690·56	−3·25
folded plate	301·79		288·31		311·78		299·66		532·86		668·82	

The third example keeps the geometry of the plate from the first example but now only a half of the plate is subjected to a uniformly distributed load of 1 kp/cm^2 – see Fig. 5. The results obtained are shown in Table 3. It is evident that the finite element solution is quite satisfactory. In order to test the behaviour, a detailed discussion is carried out.

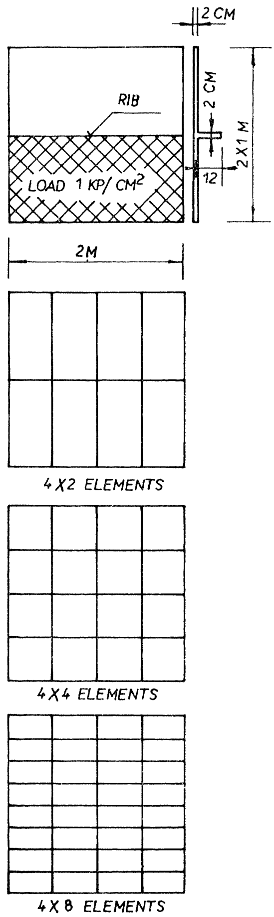

Fig. 5.

Due to the particular deflection function assumed (vertical deflection w), the second derivative values, i.e. $\partial^2 w/\partial x^2$ and $\partial^2 w/\partial y^2$ will be bilinear in x and y over each element of the division and hence the bending moments will vary linearly over the element. In addition, the internal moments calculated from the adjacent elements will be discontinuous at their common node. The discontinuities decrease with the mesh size. In our examples the average values are taken and it is readily seen that the resulting curve is very close to the folded plate method in each of the examples discussed.

CONCLUSION

A finite element solution of a plate with ribs proposed in [1] for a more general geometry of construction (intersecting ribs) has been discussed. Convergence theorems were proved. In order to show the practical use of the method, the analysis of three examples was carried out. The results obtained are shown to be in a good agreement with the folded plate method [7].

From the technical standpoint the finite element solution given can be readily applied to the solution of structures containing openings, elastic supports, supported on pillars etc. On the other hand, the method does not explicitly respect the influence of the excentricity of ribs. Because of the simplicity of the procedure discussed, it can be used to minicomputers.

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ANALÝZA KONVERGENCE NEKONFORMNÍ METODY KONEČNÝCH PRVKŮ PŘI ŘEŠENÍ DESKY SE ŽEBRY

Vladimír Janovský, Petr Procházka

V této práci je studována konvergence jedné nekonformní metody konečných prvků pro problém desky se žebry. Je sledován problém formulovaný v [1] a je přihlédnuto ke konečnému počtu obecně křížících se žeber. Konvergence metody je prokázána teoreticky a na praktických příkladech je prokázána i v aplikacích. Na praktických příkladech jsou zkoumány i vlastnosti metody v závislosti na hustotě sítě. Konvergence numerické metody je dokázána za předpokladu, že není známa „a priorní" regularita řešení výchozího problému.

Authors' addresses: RNDr. *Vladimír Janovský*, CSc., MFF KU, Malostranské náměstí 25, 11 800 Praha 1, Ing. RNDr. *Petr Procházka*, CSc., DP Metroprojekt, 120 00 Praha 2, Na Slovanech.