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THE TAIL σ-FIELDS OF RECURRENT MARKOV PROCESSES

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1. INTRODUCTION

The purpose of this article is the proof of a general representation theorem for the tail σ-field of a discrete parameter Markov process on general state space (Theorem 1) and then the development of the structure of such σ-fields for the most general recurrent processes. A self-contained treatment appears here, and even where there is some overlap with other authors (part of Theorem 2 can be obtained by combining results in [14] and [15], and we have recently discovered the ancillary Theorem 4 in [14]) our proofs are different and our techniques and point of view remain probabilistic throughout.

Theorem 5 presents another proof of the Jamison-Orey [11] generalization to Harris processes of the Blackwell-Freedman [1] description of the tail σ-field of persistent Markov chains; the new proof is again based on the representation Theorem 1.

Let $\{X_n, -\infty < n < \infty\}$ be a Markov process with stationary transition probabilities $P^n(x, E)$ having σ-finite stationary measure π satisfying:

$$(1.1) \qquad \pi(E) > 0 \quad \text{implies} \quad P\{X_n \in E \text{ i.o.} \mid X_0 = x\} = 1 \text{ a.e. } (\pi)$$

on state space Ω.

(1.1) is weaker than the condition of Harris; it is equivalent to conservativity and ergodicity of the process. The bilateral representation provides a one-one shift $T: X_n(T\omega) = X_{n+1}(\omega)$, and a handy symmetry. The helpfulness of this symmetry was observed in [6] where it was seen that the structure of the "infinite past", $\mathcal{T}_{-\infty}$, (defined below) determines the limiting behavior ("the infinite future") of the functions $P^n(x, E)$.

The process $\{X_n, -\infty < n < +\infty\}$ is called the *forward* process. The *backward* process is the process $\{Y_n, -\infty < n < +\infty\}$ where $Y_n = X_{-n}$ for each n. The transition functions for the backward process will be denoted by $Q^n(x, E)$. If π is stationary for the $\{X_n\}$ process, π is stationary for the $\{Y_n\}$ process; a similar symmetry is easy to prove for (1.1). Later (Theorem 4) it will be noted that the condition of Harris is also true for both processes if it is true for one of them.

There are two tail σ-fields of interest:

$$\mathscr{T}_{+\infty} = \bigcap_{n=0}^{\infty} \mathscr{B}(X_n, X_{n+1}, \ldots)$$

$$\mathscr{T}_{-\infty} = \bigcap_{n=0}^{-\infty} \mathscr{B}(\ldots X_{n-1}, X_n) = \bigcap_{n=0}^{\infty} \mathscr{B}(Y_n, Y_{n+1}, \ldots)$$

where $\mathscr{B}(\ldots)$ is the σ-field generated by the random variables in parenthesis. Results will be derived for $\mathscr{T}_{+\infty}$ and may then be applied to $\mathscr{T}_{-\infty}$ by considering the backward process $\{Y_n\}$. $\mathscr{T}_{+\infty}$ and $\mathscr{T}_{-\infty}$ may be quite different even if (1.1) holds, as example 2 in Section 3 shows; but if the condition of Harris holds, $\mathscr{T}_{-\infty} = \mathscr{T}_{+\infty}$ (Theorem 5).

Since π may be infinite, it will be convenient to consider a probability α_0 equivalent to π, and use it to induce a measure on coordinate space for $n \geqq 0$:

$$\alpha^*(U) = \int P(U \mid X_0 = x)\, \alpha_0(dx) \tag{1.2}$$

where $P(U \mid X_0)$ is conditional probability measure determined by the transition functions $P^n(x, E)$. Let α_n be the projection of α^* on x_n-space, i.e., $\alpha_n(S) = \alpha^*(X_n \in S)$. If we substitute π for α_0 in (1.2), α^* becomes π^*, a σ-finite measure invariant under the shift T (see [5], [13], for example).

If π is a probability, we take $\alpha_0 = \pi = \alpha_n$ and $\alpha^* = \pi^*$.

Let $C \in \mathscr{T}_{+\infty}$. Since C may be considered measurable on the sample space of $\{X_k, k \geqq n\}$ for arbitrarily large n (at least, up to equivalence with respect to the underlying measure on bilateral space), the following representation holds for all n:

$$\alpha^*(C) = \int P(C \mid X_n = x)\, \alpha_n(dx)\,. \tag{1.3}$$

Throughout this paper we frequently write "$A = B$" for two sets when these sets may differ by a null set.

2. RESULTS

The first result is a useful representation theorem for $\mathscr{T}_{+\infty}$-sets, and does not require (1.1).

Theorem 1. *Let* $C \in \mathscr{T}_{+\infty}$, $\alpha^*(C) > 0$.

(a) *There is a sequence of sets* $\{U_n, n \geqq 0\}$ *in state space* Ω *given by* (2.2) *below such that C has the representation (up to equivalence):*

$$(2.1) \qquad C = \{X_n \in U_n \ \textit{for all but a finite number of} \ n \geqq 0.\}$$

$$(b) \qquad \lim_{n\to\infty} \alpha_n(U_n) = \alpha^*(C).$$

$$(c) \qquad \lim_{n\to\infty} \int_{U_n} P(C \mid X_n = x)\, \alpha_n(\mathrm{d}x) = \alpha^*(C).$$

Proof. By the Lévy 0–1 theorem,

$$P(C \mid X_0, X_1, \ldots, X_n) \to 1_C \quad \text{a.e.} \quad (\alpha^*) \quad \text{as} \quad n \to \infty,$$

where 1_C is the indicator of C. By the Markov property and the fact that $C \in \mathscr{T}_{+\infty}$, it follows that $P(C \mid X_n) \to 1_C$ a.e. For each $n \geqq 0$, define

$$(2.2) \qquad U_n = \{x : P(C \mid X_n = x) > \tfrac{1}{2}\}.$$

Almost all points ω in C satisfy $\omega \in \liminf_{n\to\infty} \{X_n \in U_n\}$, and conversely, by the above, proving (a). Now let

$$V_n = \{X_k \in U_k \quad \text{for all} \quad k \geqq n\},$$

so that $\{V_n\}$ is increasing with limit C. Thus, N can be chosen so large that

$$\alpha^*(C) \leqq \alpha^*(V_n) + \varepsilon \leqq \alpha^*(X_n \in U_n) + \varepsilon = \alpha_n(U_n) + \varepsilon, \quad n \geqq N,$$

and so for any subsequence m

$$(2.3) \qquad \liminf_{m\to\infty} \alpha_m(U_m) \geqq \alpha^*(C).$$

If C' is the complement of C, we may suppose $\alpha^*(C') > 0$; otherwise (b) is immediate. Let W_n be the sets obtained in (2.2) for C' in place of C. The sets W_n and U_n are disjoint for each n and the analog of (2.3) holds. Then

$$1 \geqq \liminf \alpha_m(U_m \cup W_m) \geqq \liminf \alpha_m(U_m) + \liminf \alpha_m(W_m) \geqq$$
$$\geqq \alpha^*(C) + \alpha^*(C') = 1,$$

proving

$$\liminf_{m\to\infty} \alpha_m(U_m) = \alpha^*(C).$$

Since the subsequence $\{m\}$ is arbitrary, (b) follows. Finally, letting $\{V_n\}$ be the sequence of sets defined above, we obtain

$$\liminf \int_{U_n} P(C \mid X_n = x)\, \alpha_n(\mathrm{d}x) \geqq \liminf \int_{V_n} P(C \mid X_n(\omega))\, \alpha^*(\mathrm{d}\omega) \geqq$$
$$\geqq \liminf \left\{ \int_C P(C \mid X_n(\omega))\, \alpha^*(\mathrm{d}\omega) - \alpha^*(C - V_n) \right\} \geqq$$
$$\geqq \int_C \liminf P(C \mid X_n)\, \alpha^*(\mathrm{d}\omega) = \alpha^*(C).$$

On the other hand,

$$\limsup \int_{U_n} P(C \mid X_n = x)\, \alpha_n(dx) \leqq \limsup \alpha_n(U_n) = \alpha^*(C)$$

by (b), and so (c) follows.

Corollary 1. *The sets $\{U_n\}$ of (2.1) are unique. If C_1 and C_2 are disjoint $\mathscr{T}_{+\infty}$ sets, and if $U_n^{(i)}$, $i = 1, 2$, are the corresponding sets obtained in (2.1), then for each n, $U_n^{(1)} \cap U_n^{(2)} = \emptyset$.*

Proof: Immediate from (2.2).

Definition. The relation (2.1) is called a *representation* of $C \in \mathscr{T}_{+\infty}$, and the sets $\{U_n\}$ are called *representation sets for C*.

From now on (1.1) will be assumed. The next theorem shows how Theorem 1 is strengthened when $\mathscr{T}_{+\infty}$ is known to be atomic.

Theorem 2. *Let $\mathscr{T}_{+\infty}$ be atomic. Then*
(2.4) *$\mathscr{T}_{+\infty}$ is X_n measurable for any fixed integer n.*
(2.5) *There is a partition of Ω into cyclically moving classes: there is an integer $r \geqq 1$ and disjoint sets $C_0, C_1, \ldots, C_{r-1}$ with $\bigcup_{i=0}^{r-1} C_i$ and Ω differing by a π-null set F. Each atom of $\mathscr{T}_{+\infty}$ is equivalent to one of the sets $[X_0 \in C_i]$, $0 \leqq i \leqq r-1$. Defining C_n for an arbitrary integer n as the unique set C_i, $0 \leqq i \leqq r-1$ with $n = i \pmod r$, we have*

$$P(x, C_{n+1}) = \begin{cases} 1, & x \in C_n \\ 0, & x \notin C_n . \end{cases}$$

The decomposition into sets $\{C_i\}$ is unique in the sense that any other such decomposition consists of sets equivalent to the sets $\{C_i\}$.
(2.6) *$\alpha_n(C_n)$ is constant for each $n \geqq 0$, where C_n is defined as above.*

Proof. Let C be an atom of $\mathscr{T}_{+\infty}$, $\alpha^*(C) > 0$. For any integer n, T^nC is an atom of $\mathscr{T}_{+\infty}$, so by (1.1) there is a smallest positive integer r with $\alpha^*(T^rC \cap C) > 0$, and then $T^rC \cap C = C = T^rC$. Stationarity of transition probabilities yields for any integer s,

$$(2.7) \qquad U_s = \{x\colon P(C \mid X_s = x) > \tfrac{1}{2}\} = \{x\colon P(T^rC \mid X_s = x) > \tfrac{1}{2}\} = $$
$$= \{x\colon P(C \mid X_{s+r} = x) > \tfrac{1}{2}\} = U_{s+r},$$

and so the representation sets are periodic: $U_m = U_n$ if $m = n \pmod r$. It follows that C is measurable with respect to the random variables $\{X_k, k \leqq n\}$ for any fixed integer n, since knowledge of $P(C \mid X_k(\omega))$ for $k \leqq n$ is sufficient, by (2.7) and (2.1), to determine whether $\omega \in C$. Thus C is $\mathscr{T}_{-\infty}$ measurable. Then the Markov property shows

$$1_C = E(1_C \mid \dots X_{n-1}, X_n) = E(1_C \mid X_n)$$

and (2.4) has been proved. Now we resort to an idea used in [6] and [7]. Let $\Delta_n = \mathscr{B}(\dots X_{-n-1}, X_{-n})$ for $n \geqq 0$, $f = 1_{[X_0 \in A]}$, the indicator of $X_0 \in A$, any set of finite measure, and put $E(f \mid \Delta_n) = f_n$. The invariance of T and the Markov property easily show (see [6] and [7])

$$(2.8) \qquad T^n f_n = T^n E(f \mid \Delta_n) = E(1_{[X_n \in A]} \mid \dots X_{-1}, X_0) = E(T^n f \mid X_0) = \\ = P^n(X_0, A).$$

Since C is X_0 measurable, there is a set $A \subset \Omega$ with $f = 1_{[X_0 \in A]} = 1_C$, and (2.8) and the $\mathscr{T}_{-\infty}$ measurability of C imply by (2.8)

$$(2.9) \qquad 1_{[X_n \in A]} = T^n f = T^n E(f \mid \Delta_n) = T^n f_n = P^n(X_0, A).$$

(2.9) immediately proves the basic fact

(2.10) $\qquad P^n(X_0, A)$ assumes only the values 0 or 1 a.e. (α^*).

Now let $C_0 = A$ and define C_{-n} by

$$C_{-n} = [x: P^n(x, C_0) = 1], \quad n \geqq 1.$$

From (2.10) it readily follows that

$$C_{-n} = [x: P(x, C_{-n+1}) = 1], \quad n \geqq 1$$

and that

$$(2.11) \qquad C = [X_0 \in C_0] = [X_{-n} \in C_{-n}].$$

Since $C = T^{-r}C = [X_{-n+r} \in C_{-n}]$, it is clear that the sets $\{C_{-n}, n \geqq 0\}$ are periodic with period r and are precisely the r distinct representation sets. By (1.1), iterates of C generate all the atoms of $\mathscr{T}_{+\infty}$; thus each atom can be expressed in terms of a set C_{-n}, $n \geqq 0$:

$$T^{-n}C = [X_n \in C_0] = [X_0 \in C_{-n}].$$

Relabel the sets so that the r distinct sets are indexed by the integers modulo r with $P(x, C_{n+1}) = 1$ or 0 depending upon whether x is or is not an element of C_n The uniqueness assertion is clear, and (2.5) has been shown. (2.6) is immediate from the relation

$$\alpha_n(C_n) = \int_{C_{n-1}} P(x, C_n)\, \alpha_{n-1}(\mathrm{d}x) = \alpha_{n-1}(C_{n-1}).$$

This completes the proof of Theorem 2.

It $C \in \mathscr{T}_{+\infty}$ is $\{X_n, n \leqq N\}$ measurable for some fixed N, it turns out that C has a pleasant representation even in the case where $\mathscr{T}_{+\infty}$ is not atomic. The following theorem describes this situation in part (a), and more generally gives a rather complete description of $\mathscr{T}_{+\infty}$.

Theorem 3. *Either $\mathscr{T}_{+\infty}$ is atomic or $\mathscr{T}_{+\infty}$ is non-atomic. In the atomic case Theorem 2 gives a complete description of the structure of $\mathscr{T}_{+\infty}$. If $\mathscr{T}_{+\infty}$ is non-atomic, the sets in $\mathscr{T}_{+\infty}$ can be of two types:* (a) *sets C which are X_n measurable for any fixed integer n. If $\{U_n\}$ are the representation sets for C, the following statements are satisfied for all $n \geqq 0$:*

$$P(x, U_{n+1}) = \begin{cases} 1\,, & x \in U_n\,. \\ 0\,, & x \notin U_n \end{cases}$$

$$C = [X_n \in U_n]\,.$$

$\alpha_n(U_n)$ is constant and has common value $\alpha^*(C)$.
(b) *sets C which are not measurable on the sample space of $\{X_n, n \leqq N\}$ for any finite integer N.*

If at least one of $\mathscr{T}_{-\infty}$ or $\mathscr{T}_{+\infty}$ is non-atomic, then the forward and backward processes are singular in the sense that for almost all (π) x there exist π-null sets $N_x^{(n)}$, $M_x^{(n)}$ for each integer $n \geqq 1$, with

$$P^n(x, N_x^{(n)}) = 1$$

$$Q^n(x, M_x^{(n)}) = 1$$

for the forward and backward n-step transition functions.

If $\mathscr{T}_{-\infty}$ and $\mathscr{T}_{+\infty}$ are both atomic, then $\mathscr{T}_{-\infty} = \mathscr{T}_{+\infty} = \mathscr{T}$.

Proof. If there is one atom C, $\alpha^*(C) > 0$, (1.1) assures the existence of an integer r such that $\bigcup_{n=0}^{r-1} T^{-n}C$ is equivalent to the whole space, where each set $T^{-n}C$ is clearly seen to be an atom. In this case, then, there is no non-atomic part, so that $\mathscr{T}_{+\infty}$ is atomic. Otherwise, there is no atom, and $\mathscr{T}_{+\infty}$ is non-atomic. In the non-atomic case suppose $C \in \mathscr{T}_{+\infty}$ is measurable with respect to $\{X_n, n \leqq N\}$ for some fixed integer N. The Markov property used in the first part of the proof of Theorem 2 may be applied here to prove C X_n-measurable for $n \geqq N$. Moreover, it involves no loss of generality to suppose $N = 0$. If (2.8) is considered for the backward process (by substituting $-n$ for n and putting $\Delta_{-n} = \mathscr{B}(X_n, X_{n+1}, \ldots)$, $n \geqq 0$), the analog of (2.9) implies the analog of (2.10), namely, that $Q^n(X_0, A) = 1$ or 0 a.e. for all $n \geqq 1$, where $1_{X_0 \in A} = 1_C$. This follows from the X_n-measurability of C for $n \geqq 0$. The sets U_n, analogs of the sets C_{-n}, can be defined, and then it is not hard to obtain all the statements of part (a), using the relation $\alpha_n(U_n) = \alpha_{n+1}(U_{n+1}) =$

$= \int_{U_{n+1}} Q(x, U_n)\, \alpha_{n+1}(\mathrm{d}x) = \int_{U_n} P(x, U_{n+1})\, \alpha_n(\mathrm{d}x)$. We note that the periodicity of the representation sets now fails in general because the atomic property no longer holds. The only alternative to (a) in the non-atomic case is given by (b).

If at least one of $\mathscr{T}_{-\infty}$ or $\mathscr{T}_{+\infty}$ is non-atomic, it will follow from Theorems 4 and 5 that (1.1′), the condition of Harris, fails both for the forward and for the backward process. The singularity of the processes is then obtained from [10].

If $\mathscr{T}_{+\infty}$, say, is atomic, (2.4) shows $\mathscr{T}_{+\infty}$ measurable on $\{X_n, n \leqq N\}$ for all fixed $N \leqq 0$, so that $\mathscr{T}_{+\infty}$ is $\mathscr{T}_{-\infty}$ measurable: $\mathscr{T}_{+\infty} \subset \mathscr{T}_{-\infty}$. If $\mathscr{T}_{-\infty}$ is also atomic, then $\mathscr{T}_{-\infty} \subset \mathscr{T}_{+\infty}$, and so $\mathscr{T} = \mathscr{T}_{-\infty} = \mathscr{T}_{+\infty}$.

This completes the proof of all the assertions of Theorem 3.

Now we introduce the condition of T. E. Harris, a strengthening of (1.1):

$$(1.1') \qquad \pi(E) > 0 \quad \text{implies} \quad P\{X_n \in E \text{ i.o.} \mid X_0 = x\} = 1 \text{ for all } x \in \Omega .$$

We shall prove that (1.1′) implies the atomicity of $\mathscr{T}_{-\infty}$ and $\mathscr{T}_{+\infty}$ so that one can speak of a single tail σ-field $\mathscr{T}$ by Theorem 3, and $\mathscr{T}$ satisfies the assertions of Theorem 2. Thus we will show the preceding theorems contain the results of [1] and [11].

It is known that under (1.1′) there is an integer $r \geqq 1$, a $\delta > 0$, and a set $V \subset \Omega$, $0 < \pi(V) < \infty$, such that the density $p^r(x, y)$ of $P^r(x, E)$ with respect to π satisfies

$$(2.12) \qquad \inf_{(x,y) \in V \times V} p^r(x, y) \geqq \delta$$

(see [13], p. 7).

We want to see that the symmetry of the forward and backward processes persists even under (1.1′).

Theorem 4. *If the forward process satisfies* (1.1′), *so does the backward process, except perhaps for a fixed null set of x.*

Proof. The backward process satisfies (1.1); if it does not satisfy (1.1′) a.e., then for almost all $(\pi)\, x$, the n-step backward transition function has its support on a π-null set, for each n (see [10]). Therefore, to show (1.1′), it will be sufficient to prove the existence of a positive integer r satisfying

$$(2.13) \qquad \frac{\mathrm{d}(Q^r(x, \cdot))}{\mathrm{d}\pi} > 0$$

on an x-set of positive (π) measure, where (2.13) is the Radon-Nikodym derivative of $Q^r(x, \cdot)$ with respect to π. Let r be the positive integer and V the set of (2.12) for the forward process transition function. Let $A \subset V$, $B \subset V$. Then

$$P(X_{-r} \in A, X_0 \in B) = \int_B Q^r(x, A)\, \pi(\mathrm{d}x) + \int_B \mu_x(M_x \cap A)\, \pi(\mathrm{d}x) =$$

$$= \int_A P^r(x, B)\, \pi(\mathrm{d}x) + \int_A \lambda_x(N_x \cap B)\, \pi(\mathrm{d}x) \geqq \delta\, \pi(B)\, \pi(A) .$$

Here $\mu_x(\cdot)$, $\lambda_x(\cdot)$ are measures singular with respect to π and having supports on π-null sets M_x and N_x respectively. From the above, it follows that on $V \times V$

$$\frac{\mathrm{d}P(X_{-r} \in \cdot, X_0 \in \cdot)}{\mathrm{d}\pi \times \mathrm{d}\pi} = \frac{\mathrm{d}Q^r(x, \cdot)}{\mathrm{d}\pi} \geqq \delta .$$

This proves (2.13) and completes the proof of Theorem 4.

Before proceeding to Theorem 5, we make an observation: let $C \in \mathscr{T}_{+\infty}$ have $\alpha^*(C) > 0$. Theorem 1 gave a representation of C by (2.1) where the sets U_n are defined in (2.2). The number $\frac{1}{2}$ there could have been replaced with any fixed number $a \geqq \frac{1}{2}$, and all the results of Theorem 1 and its corollary would follow. (If $0 < a < \frac{1}{2}$ is chosen, the representation so obtained is valid, but representation sets for disjoint tail sets are not necessarily disjoint for each n.) Thus we define for each $\varepsilon \leqq \frac{1}{2}$,

$$U_n^\varepsilon = \{x : P(C \mid X_n = x) > 1 - \varepsilon\} . \tag{2.14}$$

Theorem 5. (1.1′) *implies* $\mathscr{T}_{+\infty}$ *and* $\mathscr{T}_{-\infty}$ *are atomic. Therefore,* $\mathscr{T}_{+\infty} = \mathscr{T}_{-\infty} = \mathscr{T}$ *by Theorem* 3, *and the results of Theorem 2 hold. In particular there is a decomposition of* $\mathscr{T}$ *into cyclically moving classes.*

Proof. We shall show $\mathscr{T}_{+\infty}$ atomic; then by considering the backward process and Theorem 4, it will follow that $\mathscr{T}_{-\infty}$ is also atomic. First the case of finite π will be considered; in fact, suppose $\pi(\Omega) = 1$. Let r, δ and V be any items specified in (2.12). $\mathscr{T}_{+\infty}$ will be shown atomic by proving that there do not exist more than r disjoint $\mathscr{T}_{+\infty}$ sets. To this end, suppose $\{C_i, 1 \leqq i \leqq r + 1\}$ is a partition of the entire space into disjoint $\mathscr{T}_{+\infty}$ sets, each with positive α^* measure. Choose ε subject to the following restrictions:

$$\varepsilon \leqq \frac{1}{16} \tag{2.15}$$

$$\frac{r\varepsilon^{1/4}}{(1 - \varepsilon^{1/4})\,\delta} \leqq \frac{\pi(V)}{20} \tag{2.16}$$

$$\frac{\varepsilon^{1/4}}{\delta} \leqq \frac{\pi(V)}{20} . \tag{2.17}$$

For each set C_i, let $\{{}_iU_n^\varepsilon\}$ be the representation sets as described in (2.14). Put

$$F_n^\varepsilon = \Omega - \bigcup_{i=1}^{r+1} {}_iU_n^\varepsilon .$$

$\sum_{i=1}^{r+1} \pi({}_iU_n^\varepsilon) \to 1$ as $n \to \infty$ since $\pi(\Omega) = 1$, by (b) of Theorem 1 (note: $\alpha_n = \pi$, here), hence $\pi(F_n^\varepsilon) \to 0$ as $n \to \infty$. This implies:

(2.18) there is an integer N such that if $n \geqq N$, $V \cap \bigcup_{i=1}^{r+1} {}_iU_n^\varepsilon$ is not empty.

We will now prove that for any set C_i there is an integer $l(i) \geqq N$, with

$$\pi({}_iU^{\beta}_{l(i)+kr} \cap V) \geqq .9\pi(V), \quad \text{where} \quad \beta = \varepsilon^{1/2} \tag{2.19}$$

for all positive integers k. It will be sufficient to do this for C_1. Observe first that if $0 < a < b \leqq \frac{1}{2}$, then for each fixed i and n, ${}_iU^a_n \subset {}_iU^b_n$. Let $l \geqq 0$ be any integer with $x \in {}_1U^{\varepsilon}_l \cap V$; an infinite number of such exist by (1.1). Then $x \in {}_1U^{\beta}_l$, and so $P(C_1 \mid X_l = x) > 1 - \beta$. Fix $i \neq 1$, and put $W = {}_iU^{\beta}_{l+r} \cap V$. Then

$$\beta \geqq P(C_i \mid X_l = x) \geqq \int_W P^r(\mathrm{d}y \mid X_l = x)\, P(C_i \mid X_{l+r} = y) > \delta\, \pi(W)(1 - \beta)$$

by (2.12), so that

$$\pi(W) < \frac{\beta}{(1-\beta)\delta}.$$

Then (2.16) implies

$$\pi\Big(\big(\bigcup_{i \neq 1} {}_iU^{\beta}_{l+r}\big) \cap V\Big) < \frac{r\beta}{(1-\beta)\delta} \leqq \frac{\pi(V)}{20}. \tag{2.20}$$

Let C_1' be the complement of C_1, $x \in {}_1U^{\varepsilon}_l \cap V$ and put $M = F^{\beta}_{l+r} \cap V$. Then we obtain

$$\varepsilon \geqq P(C_1' \mid X_l = x) \geqq \int_M P^r(\mathrm{d}y \mid x_l = x)\, P(C_1' \mid X_{l+r} = y) > \delta\, \pi(M)\, \beta,$$

implying

$$\pi(M) < \frac{\beta}{\delta} \leqq \frac{\pi(V)}{20}, \tag{2.21}$$

by (2.17). (2.20) and (2.21) yield

$$\pi({}_1U^{\beta}_{l+r} \cap V) \geqq \pi(V) - 2\,.\,\frac{\pi(V)}{20} = .9\pi(V). \tag{2.22}$$

It has therefore been shown that for any $l \geqq 0$ with ${}_1U^{\varepsilon}_l \cap V$ not empty, (2.22) holds. To complete the proof by induction, we show that if $l \geqq N$ (N being the integer described in (2.18)), and if

$$\pi({}_1U^{\beta}_{l+kr} \cap V) \geqq .9\pi(V) \tag{2.23}$$

for some $k \geqq 0$, then ${}_1U^{\varepsilon}_{l+kr} \cap V$ is not empty, so that the argument leading to (2.22) proves (2.23) holds for $k + 1$ substituted for k. To see this, we use (2.18). If ${}_1U^{\varepsilon}_{l+kr} \cap$ $\cap V$ is empty, there must exist $i \neq 1$ with $x \in {}_iU^{\varepsilon}_{l+kr} \cap V$. Then repeat the argument leading to (2.22) on C_i instead of C_1 to show, where we write $\eta = \varepsilon^{1/4} = \beta^{1/2}$

$$\pi({}_iU^{\eta}_{l+(k+1)r} \cap V) \geqq \pi({}_iU^{\beta}_{l+(k+1)r} \cap V) \geqq .9\pi(V). \tag{2.24}$$

On the other hand, the restrictions on ε are such that since there exists $x \in {}_1U^{\beta}_{l+kr} \cap V$ by (2.23), again the argument leading to (2.22) may be repeated with β substituted for ε to show

$$(2.25) \qquad \pi({}_1U^{\eta}_{l+(k+1)r} \cap V) \geqq .9\pi(V) .$$

$\eta \leqq \frac{1}{2}$ by (2.15) so the sets in (2.24) and (2.25) are representation sets and are disjoint for fixed n. Thus (2.24) and (2.25) are contradictory. This proves ${}_1U^{\varepsilon}_{l+kr} \cap V$ non-empty and the assertion (2.19) follows by induction on k, because by (1.1) there is certainly $l \geqq N$ with ${}_1U^{\varepsilon}_l \cap V$ not empty. But (2.19) is incompatible with the existence of more than r values of i because the representation sets in (2.19) are disjoint. This contradicts the supposed partition into $r + 1$ sets and proves $\mathscr{T}_{+\infty}$ atomic when $\pi(\Omega) < \infty$.

To extend to the case of infinite π we make use of the "process on A" approach of Harris (see [4], [13]). Let $A \supset V$ be a set of finite π measure. Let $A^* = [\omega: X_0(\omega) \in A]$. For almost all (π^*) $\omega \in A^*$, the random variable $T_n(\omega)$, the n^{th} value of $X_1(\omega), X_2(\omega), \ldots$ lying in A, is defined, by (1.1). Put $Y_n(\omega) = X_{T_n(\omega)}(\omega)$, $n \geqq 1$; $Y_0(\omega) = X_0(\omega)$, $\{Y_n\}$ is a Markov process with stationary transition probabilities $P^n_A(x, E)$, $E \subset A$, and with stationary probability measure $\pi_A = \pi(\cdot)/\pi(A)$ (see [4]). Let $C \in \mathscr{T}_{+\infty}$; as we have observed at the begining of the proof of Theorem 1, if $\omega \in A^*$

$$P(C \mid Y_n(\omega)) = P(C \mid X_{T_n(\omega)}(\omega)) \to 1_C(\omega) \quad \text{a.e.} \quad (\pi^*) \quad \text{on} \quad A^*$$

by the Lévy $0-1$ theorem and the Markov property, since $T_n(\omega) \to \infty$ a.e. by (1.1). This is equivalent to stating: $P(C \cap A^* \mid Y_n) \to 1_{C \cap A^*}$ a.e. with respect to "process on A" measure induced by π_A and $P^n_A(x, E)$ so that $C \cap A^*$ is measurable with respect to $\mathscr{T}^*_{+\infty}$, the forward tail σ-field for the Y_n process. Since we suppose $A \supset V$ the $\{Y_n\}$ process satisfies (1.1′) and has transition probabilities satisfying [4]

$$P^r_A(x, E) \geqq P^r(x, E) , \quad E \subset A ,$$

so that on $V \times V$ the respective densities satisfy

$$p^r_A(x, y) \geqq p^r(x, y) \geqq \delta$$

implying that $\mathscr{T}^*_{+\infty}$ has no more than r disjoint atoms by the proof for the finite case. If $\mathscr{T}_{+\infty}$ is non-atomic, there are $r + 1$ disjoint $\mathscr{T}_{+\infty}$ sets $C_1, C_2, \ldots, C_{r+1}$ of positive measure, and then, as noted above, $C_1 \cap A^*, C_2 \cap A^*, \ldots, C_{r+1} \cap A^*$ are $\mathscr{T}^*_{+\infty}$ sets. Since A^* may be considered so large that $\pi^*(C_i \cap A^*) > 0$ for each i, this contradicts $\mathscr{T}^*_{+\infty}$ having at most r disjoint atoms. Therefore $\mathscr{T}_{+\infty}$ is atomic.

Now apply Theorem 2 to obtain a cyclic decomposition of $\mathscr{T}_{+\infty}$. (2.5) and (2.12) make it clear that each atom C_i satisfies: there is an integer $t = t(i)$ with $V \subset {}_iU_n$ for $n \equiv t (\text{mod } r)$, where $\{{}_iU_n\}$ gives a representation of C_i. From this it follows that there are exactly r' disjoint atoms of $\mathscr{T}_{+\infty}$, if r' is the smallest value of r given in (2.12).

Using Theorem 4, $\mathscr{T}_{-\infty}$ may be handled in the same way, and so $\mathscr{T}_{+\infty} = \mathscr{T}_{-\infty} = \mathscr{T}$, by Theorem 3, and the proof of Theorem 5 is complete.

3. EXAMPLES

We conclude by giving two examples for which (1.1) but not (1.1′) hold. These examples are due to Jamison and Orey.

Example 1. State space Ω is the unit circle. To define $P(x, E)$, rotate by an irrational multiple c of the number π. Set $P(x, \{y\}) = 1$ if $y = \exp i(c + \theta)$ where $x = \exp i\theta$, and $P(x, \{y\}) = 0$ otherwise. The orbit of each x is dense in Ω, (1.1) is satisfied for Lebesgue measure which is stationary for the process. (1.1′) is not satisfied, for the orbit of each point x is countable. The process is deterministic: for all sets E, $P(x, E) = 1$ or 0, and $\mathscr{T}_{+\infty}$ is equivalent to the class of all X_0-measurable sets. We are in the situation where all sets are described by Theorem 3(a). In this example $\mathscr{T}_{-\infty} = \mathscr{T}_{+\infty}$.

Example 2. Let Z_n be a sequence of independent, identically distribution random variables with common distribution $P(Z_n = 0) = P(Z_n = 1) = \frac{1}{2}$, and let $-\infty < n < +\infty$. Define the point X_n on $\Omega = [0, 1]$ by the binary expansion

$$X_n = .Z_n Z_{n-1} Z_{n-2} \dots$$

X_n is a Markov process on Ω and the Borel sets, Lebesgue measure is stationary, and (1.1) is satisfied, but not (1.1′). $\mathscr{T}_{-\infty}$ is trivial, for it is measurable with respect to the tail σ-field of the independent $\{Z_n, n \leqq 0\}$. On the other hand, $\mathscr{T}_{+\infty}$ consists of all measurable subsets of bilateral space, since every point in bilateral Z-space may be expressed in terms of the X_n's for $n \geqq N, N$ arbitrarily large and fixed. Hence $\mathscr{T}_{-\infty} \neq \mathscr{T}_{+\infty}$, in fact, they are as "far apart" as possible.

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Souhrn

ZBYTKOVÉ σ-ALGEBRY REKURENTNÍCH MARKOVOVÝCH PROCESŮ

RICHARD ISAAC

Nechť $\{X_n, -\infty < n \infty\}$ je Markovův proces s homogenními pravděpodobnostmi přechodu, mající σ-konečnou stacionární míru a splňující podmínku slabé rekurentnosti. V článku se studuje struktura zbytkových σ-algeber budoucnosti a minulosti $\mathscr{T}_{+\infty}$ a $\mathscr{T}_{-\infty}$ v různých situacích. Hlavním výsledkem je věta o representaci množin v $\mathscr{T}_{+\infty}$; na jejím základě je pak provedeno systematické vyšetřování a odvozeny některé nové i některé známé věty včetně rozkladu na cyklické třídy pro procesy vyhovující Harrisově podmínce. Základní pojetí i metody jsou všude pravděpodobnostní.

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