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ON TWO-DIMENSIONAL AND THREE DIMENSIONAL AXIALLY-SYMMETRIC ROTATIONAL FLOWS OF AN IDEAL INCOMPRESSIBLE FLUID

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In the investigation of flows of an ideal (i.e. non-viscous) fluid the assumption that the flow is irrotational plays an important role because it simplifies considerably the fundamental system of equations. A large series of papers has been devoted to the solution of irrotational stream fields but the study of rotational flows of an ideal fluid has been neglected.

As the demands of praxis have become more and more complex, the necessity to consider more complicated models, among others the models which take into account the vorticity of stream fields, has gradually proved obvious.

In this paper, we shall deal with the solvability of equations describing *stationary two-dimensional and three-dimensional axially-symmetric rotational flows of an ideal incompressible fluid* with respect to problems of inner hydrodynamics. Partially we follow the works [2, 3], but here we study a more general situation.

1. FUNDAMENTAL CONCEPTS

Let E_n denote the Euclidean n-dimensional space. If $M \subset E_n$, then by $\overline{M}$ we denote the closure of the set M in E_n. The Cartesian coordinates in the space E_2 will be denoted by x, y.

Let $\Omega \subset E_2$ be $r + 1$-multiply connected bounded domain $(r \geqq 0)$, whose boundary $\partial\Omega$ has $r + 1$ components $C_0, \ldots, C_r$. Let us suppose that $C_i \subset \operatorname{Int} C_0$ $(i = 1, \ldots, r)$, C_0 is a positively oriented Jordan curve, C_i $(i = 1, \ldots, r)$ are negatively oriented Jordan curves, $C_0, \ldots, C_r$ have finite lengths.

In the case of a two-dimensional flow the set $\Omega^2 = \Omega$ represents the region filled by the fluid. As an example we can consider the flow round profiles $\operatorname{Int} C_i$ $(i = 1, \ldots, r)$, situated in the space $\operatorname{Int} C_0$.

If we want to consider three-dimensional axially-symmetric flows, then let x, y, φ be the cylindrical coordinates, and let x be the axis of symmetry of a stream field. The region in E_3 filled by the fluid is in cylindrical coordinates given as

$$\Omega^3 = \{(x, y, \varphi); (x, y) \in \Omega, \varphi \in < 0, 2\pi)\} .$$

We get this set by rotating the domain Ω round the axis of symmetry x. In this paper, we shall assume that the axis x and the domain Ω^3 are disjoint, and thus

$$(1.1) \qquad y > 0 \quad \text{for all points} \quad (x, y) \in \bar{\Omega} .$$

If we consider a two-dimensional or a three-dimensional flow, we put $k = 2$ or $k = 3$ respectively.

Let the set $\delta\Omega$ be devided into three parts $\delta\Omega_n$, $\delta\Omega_t$ and K:

$$(1.2) \qquad \partial\Omega = \partial\Omega_n \cup \partial\Omega_t \cup K ,$$

where the union at the right hand side is disjoint, the set K is finite and the components of the sets $\partial\Omega_n \cap C_i$ and $\partial\Omega_t \cap C_i$, which we shall denote by N_i^j $(j = 1, \ldots, n_i)$ and T_i^j $(j = 1, \ldots, t_i)$ respectively, are open arcs in C_i with the same orientation as C_i $(i = 0, \ldots, r)$. Let $C_i \cap \partial\Omega_n \neq \emptyset$ for $i = 0, \ldots, r$.

The initial point and the terminal point of an arbitrary curve Γ will be denoted by i.p. Γ and t.p. Γ respectively.

Let i.p. $C_i \in K$ $(i = 0, \ldots, r)$.

The stationary flow of an ideal incompressible fluid is described by the following system of equations:

$$(1.3) \qquad \operatorname{div} \mathbf{V} = 0 ,$$

$$(1.4) \qquad \mathbf{V} \times \operatorname{rot} \mathbf{V} = \operatorname{grad} H ,$$

where $\mathbf{V}$ is the velocity vector of the fluid (for $k = 2$, $\mathbf{V} = (v_x, v_y)$ and for $k = 3$ we have in the cylindrical coordinates $\mathbf{V} = (v_x, v_y, v_\varphi))$, H is the so-called generalized enthalpy,

$$(1.5) \qquad H = \frac{p}{\varrho} + \tfrac{1}{2}\mathrm{V}^2 + U ,$$

p denotes pressure, ϱ density, V the absolute value of velocity and U the potential of the exterior volume force. Let us remark that (1.3) is the continuity equation and (1.4) represents the Euler equations of motion written in the so-called energetic form.

In the case $k = 3$ it follows from the assumption of axial symmetry that all quantities characterizing the stream field depend only on the coordinates x, y, and thus $\partial/\partial\varphi \equiv 0$. This means that the solution of the three-dimensional axially-symmetric flow can be reduced to a two dimensional problem on the domain Ω.

Remark 1. So far we have not intentionally specified the smoothness of the functions $\boldsymbol{V}$, H, the boundary $\partial\Omega$ and the curves or surfaces we shall speak about. If nothing else is explicitly stated, we shall assume tacitly that the functions, curves and surfaces mentioned are "smooth enough" and that the Green theorem holds (see e.g. [5, 6]):

$$\int_\sigma \operatorname{div} \boldsymbol{V} \, d\sigma = \int_{\partial\sigma} \boldsymbol{V} . \boldsymbol{N} \, dS \tag{1.6}$$

for an arbitrary domain $\sigma \subset \Omega^k$, with a "sufficiently smooth" boundary $\partial\sigma$. $\boldsymbol{N}$ is the unit vector of the outer normal to $\partial\sigma$. The integrals can be taken as Lebesgue integrals.

The symbol $\int_{\mathscr{S}} f \, dS$ will denote the curvilinear or surface integral, if $\mathscr{S}$ is a curve or surface, respectively. In all cases it will be evident which integral is meant so that no confusion may arise.

Now, let us rewrite the system of equations (1.3), (1.4) in the coordinates x, y or x, y, φ and for $k = 3$ let us take into account the assumption of axial symmetry:

$$\frac{\partial}{\partial x}(y^{k-2} v_x) + \frac{\partial}{\partial y}(y^{k-2} v_y) = 0 . \tag{1.7}$$

Let us put

$$\omega = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} . \tag{1.8}$$

Then, for $k = 2$, we have

$$\omega v_y = \frac{\partial H}{\partial x} , \qquad -\omega v_x = \frac{\partial H}{\partial y} , \tag{1.9$_2$}$$

and for $k = 3$

$$\omega v_y = \frac{\partial H}{\partial x} - \frac{1}{2y^2} \frac{\partial (y v_\varphi)^2}{\partial x} , \qquad -\omega v_x = \frac{\partial H}{\partial y} - \frac{1}{2y^2} \frac{\partial (y v_\varphi)^2}{\partial y} , \qquad 0 = v_x \frac{\partial (y v_\varphi)}{\partial x} + v_y \frac{\partial (y v_\varphi)}{\partial y} . \tag{1.9$_3$}$$

To the system (1.7), (1.8) and (1.9) we shall add boundary value conditions, which are divided into three groups.

1) Let $\boldsymbol{n}$ denote the unit vector of the outer normal to $\partial\Omega$, let $\mathbf{t}$ be the tangential vector to $\partial\Omega$ oriented in the same sense as the respective curve C_i. The vector $\boldsymbol{n} = (n_x, n_y)$ exists at a given point of the set $\partial\Omega$ if and only if the vector $\mathbf{t}$ exists at this point and $\mathbf{t} = (-n_y, n_x)$. We shall denote

$$V_n = (v_x, v_y) \, . \, \boldsymbol{n} \, ,$$
$$V_t = (v_x, v_y) \, . \, \mathbf{t} \, .$$

Let two functions $\varphi_n : \partial\Omega_n \to E_1$ and $\varphi_t : \partial\Omega_t \to E_1$ be given. The term *fundamental boundary value conditions* will be used for the conditions

$$V_n \mid \partial\Omega_n = \varphi_n \, , \quad V_t \mid \partial\Omega_t = \varphi_t \, . \tag{1.10}$$

2) Further, let a set $\mathscr{H} \subset \partial\Omega$ (it may be $\mathscr{H} = \emptyset$) and a function $h : \mathscr{H} \to E_1$ (for $k = 3$, moreover, $v : \mathscr{H} \to E_1$) be given. The conditions

$$\begin{aligned} &H \mid \mathscr{H} = h && \text{for} \quad k = 2 \, , \\ &H \mid \mathscr{H} = h \, , \quad v_\varphi \mid \mathscr{H} = v && \text{for} \quad k = 3 \end{aligned} \tag{1.11}$$

will be called *auxiliary boundary value conditions.*

The conditions (1.11) determine the vorticity of the stream field. Under some assumptions it is possible to prove that it is equivalent to prescribe either $H \mid \mathscr{H}$ or $\omega \mid \mathscr{H}$.

3) Here the question arises whether the conditions introduced above are, at least in some cases (e.g. for irrorational flows), complete in the sense of the unique determination of the solution of the system (1.7)–(1.9). It appears that it is not so and it is necessary to add some further conditions.

It follows from the equations (1.3) and (1.6) that

$$\int_{\partial\Omega^k} \mathbf{V} \, . \, \mathbf{N} \, \mathrm{d}S = 0 \, ,$$

which is thus a necessary condition for solvability of the system (1.3), (1.4). In this paper we shall consider a stronger condition

$$\int_{\mathscr{S}} \mathbf{V} \, . \, \mathbf{N} \, \mathrm{d}S = 0$$

fulfilled for every component $\mathscr{S}$ of the boundary $\partial\Omega^k$. This condition, which is satisfied e.g. if the boundary $\partial\Omega^k$ consists of fixed, impermeable walls, can be in our case written in the form

$$\int_{C_i} y^{k-2} V_n \, \mathrm{d}S = 0 \, , \quad i = 0, \ldots, r \, . \tag{1.12}$$

However, if $\partial\Omega_t \neq \emptyset$, it is impossible to verify this condition in advance.

To every arc T_i^j, let us assign a real number Q_i^j and let us consider conditions

$$(1.13) \qquad \int_{T_i^j} y^{k-2} V_n \,\mathrm{d}S = Q_i^j\,, \quad j = 1, \ldots, t_i\,, \quad i = 0, \ldots, r\,.$$

It is evident that the quantity $(2\pi)^{k-2}\, Q_i^j$ represents the flow through the part of the boundary $\partial\Omega^k$ given by the arc T_i^j. With respect to (1.13), the conditions (1.12) are equivalent to the relations

$$(1.14) \qquad \sum_{j=1}^{n_i} \int_{N_i^j} y^{k-2} V_n \,\mathrm{d}S + \sum_{j=1}^{t_i} Q_i^j = 0\,, \quad i = 0, \ldots, r\,.$$

Further, we shall have to give the data which are equivalent to the determination of the "flow between separate components of the set $\partial\Omega^k$". Let q_i $(i = 1, \ldots, r)$ be real numbers, Γ_i $(i = 1, \ldots, r)$ arcs with i.p. $\Gamma_i =$ i.p. C_0 and t.p. $\Gamma_i =$ i.p. C_i. Let the interior points of the arc Γ_i lie in the domain Ω.

Let us consider the conditions

$$(1.15) \qquad \int_{\Gamma_i} y^{k-2}(v_x, v_y) \cdot (t_y, -t_x) \,\mathrm{d}S = q_i\,, \quad i = 1, \ldots, r\,,$$

where $\mathbf{t} = (t_x, t_y)$ is the unit tangential vector to the arc Γ_i oriented in the same sense as Γ_i. The quantity $(2\pi)^{k-2}\, q_i$ is the flow through the arc Γ_i $(k = 2)$ or the flow through the surface obtained by the rotation of Γ_i round the axis x $(k = 3)$. From the relation (1.6) and the equation (1.3) it follows that q_i does not depend on the arc Γ_i, but only on i.p. Γ_i and t.p. Γ_i.

The conditions (1.13) and (1.15) will be called the *complementary conditions*. As an example let us consider an irrotational stream field in an annulus given by a potential vortex. This field will be determined uniquely, if we give the intensity of this vortex, which is equivalent to the determination of the flow between the circles that form the boundary of the region filled by the fluid.

On the basis of the preceding considerations let us formulate the problem.

Problem (A). *Let functions* $\varphi_n : \partial\Omega_n \to E_1$, $\varphi_t : \partial\Omega_t \to E_1$ *and real numbers* Q_i^j $(j = 1, \ldots, t_i, i = 0, \ldots, r)$, q_i $(i = 1, \ldots, r)$ *be given and let the equalities* (1.14) *with* $V_n = \varphi_n$ *be satisfied. For* $k = 2$ *or* $k = 3$, *functions* v_x, v_y, H *or* v_x, v_y, v_φ, H *respectively are called classical solution of Problem* (A) *if they are "sufficiently smooth" in the set* $\bar{\Omega}$ *and satisfy the system of equations* (1.7), (1.8) *and* (1.9) *in* Ω *and the conditions* (1.10), (1.11), (1.13) *and* (1.15).

Remark 2. The concept "sufficiently smooth" is not specified, in accordance with Remark 1. It is possible e.g. to take the functions v_x, v_y, v_φ, H as elements of the space $C(\bar{\Omega}) \cap C^1(\Omega)$. However, this assumption is too strong in most cases.

In weaker formulations, it is reasonable to assume from the physical point of view that the total kinetic energy of the fluid and the potential energy of the pressure and the volume force are finite and thus

$$(1.16, a) \qquad \frac{1}{2}\int_\Omega y^{k-2}(v_x^2 + v_y^2 + (k-2)\, v_\varphi^2)\, d\Omega < +\infty ,$$

$$(1.16, b) \qquad -\infty < \int_\Omega y^{k-2} H \, d\Omega < +\infty .$$

In virtue of the boundedness of the domain Ω and the assumption (1.1), the inequality (1.16, a) is fulfilled if and only if

$$(1.17, a) \qquad \int_\Omega (v_x^2 + v_y^2)\, d\Omega < +\infty ,$$

$$(1.17, b) \qquad \int_\Omega v_\varphi^2 \, d\Omega < +\infty \quad (\text{only for } k = 3) .$$

2. STREAM FUNCTION

If we confine ourselves to the case of irrotational flows, then $\omega = 0$, $H = \text{const.}$ and for $k = 3$ also $yv_\varphi = \text{const.}$ in Ω. The system of equations (1.7) and $\omega = 0$ can be solved among others with help of the theory of analytic or generalized analytic functions (see e.g. [8]). For the investigation of rotational stream fields it is convenient to introduce the so called stream function ψ, which satisfies the relations

$$(2.1) \qquad \frac{\partial \psi}{\partial y} = y^{k-2}\, v_x , \quad \frac{\partial \psi}{\partial x} = -y^{k-2} v_y .$$

If the functions $v_x, v_y \in C(\bar{\Omega}) \cap C^1(\Omega)$ satisfy the relations (2.1), then they obviously fulfil the equations (1.7) and (1.12). Conversely, the equation (1.7) and the assumptions (1.12) imply that to the functions $v_x, v_y \in C(\bar{\Omega}) \cap C^1(\Omega)$ there exists a stream function which is unique up to an additive constant.

Under the assumption that there exist functions $A, B \in C^1(E_1)$ such that

$$(2.2) \qquad \begin{aligned} &H = A \circ \psi && \text{for} \quad k = 2 , \\ &H = A \circ \psi , \quad yv_\varphi = B \circ \psi && \text{for} \quad k = 3 , \end{aligned}$$

it is possible to transform our problem to the equation

$$(2.3) \qquad \frac{\partial}{\partial x}\left(a(x, y)\, \frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(a(x, y)\, \frac{\partial \psi}{\partial y}\right) = f(y, \psi)$$

where

$$a = y^{2-k} \tag{2.4}$$

and

$$f(y, \psi) = A'(\psi) \quad \text{for} \quad k = 2\,, \tag{2.5}$$

$$f(y, \psi) = yA'(\psi) - \frac{1}{2y} C'(\psi) \quad \text{with} \quad C = B^2 \quad \text{for} \quad k = 3\,.$$

The equation (2.3) was in the case $k = 3$ derived in [2], for $k = 2$ we can proceed quite analogously.

The boundary value conditions (1.10) imposed on the components of the velocity v_x, v_y can be transformed into the conditions for the stream function:

$$\frac{\partial \psi}{\partial n} = -y^{k-2} V_t\,, \tag{2.6, a}$$

$$\frac{\partial \psi}{\partial t} = y^{k-2} V_n\,. \tag{2.6, b}$$

$\partial/\partial n$ and $\partial/\partial t$ denote here the derivatives in the direction $\boldsymbol{n}$ and $\mathbf{t}$ respectively. Hence, the function ψ satisfies the Neumann boundary value condition on the set $\partial\Omega_t$:

$$a \frac{\partial \psi}{\partial n} \bigg| \partial\Omega_t = y^{2-k} \frac{\partial \psi}{\partial n} \bigg| \partial\Omega_t = -\varphi_t\,. \tag{2.7}$$

On the part $\partial\Omega_n$ of the boundary the values of ψ can be determined by integration. It is evident that the function ψ is given by (2.6, b) on every component N_i^j of the set $\partial\Omega_n$ up to an additive constant. Nevertheless, with respect to the complementary conditions, the function ψ can be determined uniquely on the set $\partial\Omega_n$. Let us put $q_0 = 0$ and consider condition

$$\psi \mid \partial\Omega_n = \tilde{\psi}\,, \tag{2.8}$$

where

$$\tilde{\psi} \mid \partial\Omega_n \cap C_i = \psi_i(\tau) = q_i + \int_{C_i(\tau)} y^{k-2} V_n \, \mathrm{d}S\,, \quad i = 0, \ldots, r\,. \tag{2.9}$$

Here $C_i(\tau)$ is a curve which is part of C_i, oriented in the same sense as C_i, with i.p. $C_i(\tau) =$ i.p. C_i and t.p. $C_i(\tau) = \tau \in \partial\Omega_n \cap C_i$. Further, $V_n \mid \partial\Omega_n \cap C_i = \varphi_n \mid \partial\Omega_n \cap$ $\cap\, C_i$. Moreover, we substitute the values Q_i^j into (2.9) instead of the integrals over the components T_i^j of the set $\partial\Omega_t \cap C_i$ where φ_n is not given.

If the functions A and (for $k = 3$ also) B are known, then we obtain the problem (2.3), (2.7), (2.8) which we shall call the problem for the stream function and denote it as

Problem (B). *Let functions $\varphi_n : \partial\Omega_n \to E_1$, $\varphi_t : \partial\Omega_t \to E_1$ and $A, B \in C^1(E_1)$ be given. We shall call a function ψ, the classical solution of Problem (B), if it is "smooth enough" in $\bar{\Omega}$ and satisfies the equation (2.3) in Ω and the conditions (2.7), (2.8) on the boundary $\partial\Omega$.*

Remark 3. The solution ψ can be considered "smooth enough" if e.g. $\psi \in C^1(\bar{\Omega}) \cap C^2(\Omega)$. In weaker formulations let us assume (see (1.17, a)) that

$$\int_\Omega (\nabla\psi)^2 \, d\Omega < +\infty \tag{2.10}$$

(we use the notation $\nabla\psi = \operatorname{grad} \psi = (\partial\psi/\partial x, \partial\psi/\partial y)$).

Now, let us consider the connection between Problems (A) and (B).

Theorem 1. *Let ψ be a classical solution of the problem for the stream function. Then the functions*

$$\begin{aligned} H &= A \circ \psi\,, \\ v_x &= y^{2-k} \frac{\partial\psi}{\partial y}\,, \\ v_y &= -y^{2-k} \frac{\partial\psi}{\partial x}\,, \\ v_\varphi &= \frac{1}{y}(B \circ \psi) \quad (\textit{only if } k = 3) \end{aligned} \tag{2.11}$$

form a classical solution of Problem (A) if and only if

$$\begin{aligned} (A \circ \psi) \,|\, \mathscr{H} &= h\,, \\ \frac{1}{y}(B \circ \psi) \,|\, \mathscr{H} &= v \quad (\textit{only if } k = 3)\,. \end{aligned} \tag{2.12}$$

Remark 4. If $\mathscr{H} \subset \partial\Omega_n$, then with respect to (2.8) and (2.9) the conditions (2.12) can be verified immediately, without determining the solution of Problem (B). The conditions (2.12) are evidently equivalent to the relations

$$\begin{aligned} (A \circ \psi_i) \,|\, \mathscr{H} \cap C_i &= h \,|\, \mathscr{H} \cap C_i\,, \quad i = 0, \ldots, r \\ \frac{1}{y}(B \circ \psi_i) \,|\, \mathscr{H} \cap C_i &= v \,|\, \mathscr{H} \cap C_i\,, \quad i = 0, \ldots, r \quad (\textit{only if } k = 3)\,. \end{aligned} \tag{2.13}$$

Proof of Theorem 1. First, it is evident that the vector (v_x, v_y) satisfies the boundary value conditions (1.10) – see the relations (2.6). Further, since $\partial^2\psi/\partial x\, \partial y = \partial^2\psi/\partial y\, \partial x$, the equation (1.7) is satisfied.

Now, let us put the functions (2.11) into the equations (1.9). Let us consider the case $k = 3$, for $k = 2$ the situation is still simpler. We obtain

$$-\frac{\partial\psi}{\partial x}\,\omega = y(A' \circ \psi)\frac{\partial\psi}{\partial x} - \frac{1}{2y}(C' \circ \psi)\frac{\partial\psi}{\partial x},$$

$$-\frac{\partial\psi}{\partial y}\,\omega = y(A' \circ \psi)\frac{\partial\psi}{\partial y} - \frac{1}{2y}(C' \circ \psi)\frac{\partial\psi}{\partial y},$$

$$0 = \frac{\partial\psi}{\partial y}(B' \circ \psi)\frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial x}(B' \circ \psi)\frac{\partial\psi}{\partial y}.$$

The validity of the last equation is evident immediately, the first two equations are satisfied in view of (1.8) and (2.3)–(2.5):

$$-\omega = \frac{\partial}{\partial x}\left(\frac{1}{y}\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial\psi}{\partial y}\right) = y(A' \circ \psi) - \frac{1}{2y}(C' \circ \psi).$$

Finally, it remains to verify that the conditions (1.13), (1.15) and (1.11) are fulfilled. It follows from the relation (2.9) that $\psi(\text{i.p. } C_i) = q_i$ $(i = 0, \ldots, r)$, where of course $q_0 = 0$. If Γ_i is an arc whose inner points lie in the domain Ω, i.p. $\Gamma_i =$ $=$ i.p. C_0, t.p. $\Gamma_i =$ i.p. C_i, then by (2.1) we have

$$\int_{\Gamma_i} y(v_x, v_y) \,.\, (t_y, -t_x)\,\mathrm{d}S = \int_{\Gamma_i}\left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right).\,(t_y, -t_x)\,\mathrm{d}S =$$

$$= \int_{\Gamma_i}\frac{\partial\psi}{\partial t}\,\mathrm{d}S = \psi(\text{t.p. } \Gamma_i) - \psi(\text{i.p. } \Gamma_i) = q_i - q_0 = q_i\,.$$

Similarly, if we use the formula (2.6, b), we conclude

$$\int_{T_i^j} yV_n\,\mathrm{d}S = \int_{T_i^j}\frac{\partial\psi}{\partial t}\,\mathrm{d}S = \psi(\text{t.p. } T_i^j) - \psi(\text{i.p. } T_i^j)\,.$$

From the comments to the relation (2.9) it follows that $\psi(\text{t.p. } T_i^j) - \psi(\text{i.p. } T_i^j) = Q_i^j$. Now, it is evident that the conditions (1.11) will be valid if and only if the conditions (2.12) are valid.

Remark 5. In paper [2] the following assertion was proved: Let v_x, v_y, v_φ, $H \in C^1(\Omega)$ and a function ψ satisfy the relations (2.1) in Ω. Further, let $A, B \in C^1(E_1)$ and $A \circ \psi = H$, $B \circ \psi = yv_\varphi$. If $(v_x, v_y) \neq 0$ in Ω and v_x, v_y, v_φ, H is a solution of the system (1.7), (1.8) and (1.9$_3$), then ψ is a solution of the equation (2.3). The assumption $(v_x, v_y) \neq 0$ can be weakened, if we admit that the set of elements of Ω where $(v_x, v_y) =$ $= 0$ is isolated.

It follows from the preceding considerations that the functions A and B must be necessarily known if we want to solve Problem (A) with help of the stream function. It is evident that these functions depend on the auxiliary boundary value conditions. If $\mathscr{H} = \emptyset$ (the auxiliary conditions vanish), then it is natural to choose the functions A, B as convenient and simple as possible – e.g. $A = \text{const.}$, $B = \text{const.}$, which corresponds to an irrotational flow (rot $\mathbf{V} = 0$).

Quite a different situation arises if some auxiliary conditions are given. Then the complete problem must be formulated as follows:

Problem (C). *We look for functions* $A, B : E_1 \to E_1$ *and* $\psi : \bar{\Omega} \to E_1$ *such that* (2.3), (2.7), (2.8) *and* (2.12) *hold.*

Under the assumption that Problem (B) has solutions for all functions A, B from a wide enough class, the solution of Problem (C) (and thus also of Problem (A)) is reduced to the investigation of the possibility to determine these functions. In this paper we shall deal with the solution of the above mentioned problem in the case

$$\mathscr{H} \subset \partial\Omega_n . \tag{2.14}$$

Let us determine $\psi \mid \partial\Omega_n$ by (2.9) and denote

$$M = \psi(\mathscr{H}) . \tag{2.15}$$

The necessary and sufficient condition for the existence of functions A and B with properties (2.12) is the implication

$$z_1, z_2 \in \mathscr{H}\,, \quad \psi(z_1) = \psi(z_2) \Rightarrow h(z_1) = h(z_2)\,, \quad (yv)(z_1) = (yv)(z_2)\,.$$

In other words, this condition is valid if $h(\psi_{-1}(t))$ and $(yv)(\psi_{-1}(t))$ are one-point sets for every $t \in M$. Then we can put $A(t) = h(\psi_{-1}(t))$, $B(t) = (yv)(\psi_{-1}(t))$ for every $t \in M$. For a wide class of practical cases it is possible to extend these functions to the set E_1 so that $A, B \in C^1(E_1)$ and moreover, to satisfy some further demands concerning the behaviour of these functions which follow from physical considerations (see e.g. (1.16)). We cannot go into details. The situation is quite clear, if $\mathscr{H}$ is an arc, $V_n \mid \mathscr{H}$ is a continuous bounded function whose sign does not change, and the functions h and v have a continuous derivative along the arc $\mathscr{H}$. The case when the condition (2.14) is not valid, remains open.

Let us add that if $C_i \cap \partial\Omega_n = \emptyset$ for some $i \in \{1, \ldots, r\}$, then we do not prescribe q_i, ψ fulfills (2.7) on C_i and all our results remain valid.

3. SOLUTION OF THE PROBLEM FOR THE STREAM FUNCTION

In this section we shall study the solvability of Problem (B) under the assumption that the functions A and B are known. We emphasize that we shall confine ourselves to the investigation of the so-called weak solutions.

We shall start with the recapitulation of some concepts already mentioned which can be considered here in a weaker sense, and introduce some new essential concepts and notation.

Let the integral and measure considered be the Lebesgue integral and measure. Let $\Omega \subset E_2$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. The set $\partial\Omega$ is a disjoint union

$$\partial\Omega = \partial\Omega_n \cup \partial\Omega_t \cup K \tag{3.1}$$

of sets measurable with respect to one-dimensional measure μ_1 defined on $\partial\Omega$, where

$$\mu_1(K) = 0\,, \quad \mu_1(\partial\Omega_n) > 0\,. \tag{3.2}$$

Let us denote

$$R_1 = \inf\{y; \exists x(x, y) \in \bar{\Omega}\}\,, \quad R_2 = \sup\{y; \exists x(x, y) \in \bar{\Omega}\}\,.$$

We suppose that $R_1 > 0$ for $k = 3$ (see (1.1)).

Let $W_2^1(\Omega)$ be the well-known Sobolev space of all (equivalent classes of) functions $u \in L_2(\Omega)$ whose first generalized derivatives $\partial u/\partial x$, $\partial u/\partial y \in L_2(\Omega)$. The space $W_2^1(\Omega)$ is equipped with the norm

$$\|u\| = \left(\int_\Omega \left(u^2 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right) \mathrm{d}\Omega\right)^{1/2} = \left(\int_\Omega (u^2 + (\nabla u)^2)\, \mathrm{d}\Omega\right)^{1/2}. \tag{3.3}$$

We define the subspace of $W_2^1(\Omega)$

$$V = \{u \in W_2^1(\Omega)\,; \;\; u \mid \partial\Omega_n = 0\} \tag{3.4}$$

(the relation $u \mid \partial\Omega_n = 0$ is considered in the sense of traces – see [5, 6]). $W_2^1(\Omega)$ and V are Hilbert spaces.

Let $\psi_0 \in W_2^1(\Omega)$ denote a function such that

$$\psi_0 \mid \partial\Omega_n = \tilde{\psi}\,, \tag{3.5}$$

where $\tilde{\psi}$ is the function from the boundary value condition (2.8). Further, let the function φ_t from the condition (2.7) be an element of the space $L_2(\partial\Omega_t)$. If we put $g \mid \partial\Omega_t = -\varphi_t$ and $g \mid \partial\Omega - \partial\Omega_t = 0$, then $g \in L_2(\partial\Omega)$.

We shall solve our problem in the space $W_2^1(\Omega)$. Then the condition (2.10) will be fulfilled. Let us suppose that the functions A and B have continuous and bounded derivatives in E_1. From here it follows that the function f is continuous in the set $\langle R_1, R_2\rangle \times E_1$ and

$$|f| \leqq c_1 \quad \text{in} \quad \langle R_1, R_2\rangle \times E_1\,. \tag{3.6}$$

(Symbols $c_1, c_2, \ldots$ denote constants.) Let us remark that the function a defined in (2.4) satisfies the inequalities

$$0 < c_2 \leqq a \leqq c_3 \quad \text{in} \quad \bar{\Omega}\,. \tag{3.7}$$

Now, let us introduce the following definition: *We shall say that* $\psi \in W_2^1(\Omega)$ *is a weak solution of Problem* (B), *if it holds*:

$$\text{(3.8, a)} \qquad \psi - \psi_0 \in V,$$

$$\text{(3.8, b)} \qquad \int_\Omega (a\nabla\psi \,.\, \nabla v + f(., \psi)\, v)\, \mathrm{d}\Omega = \int_{\partial\Omega} gv\, \mathrm{d}S \quad \forall v \in V.$$

It is evident that both integrals in (3.8, b) are convergent. By using the Green theorem we can easily find out that a classical solution of Problem (B) satisfies (3.8) and conversely, a sufficiently smooth weak solution is a classical solution.

The problem (3.8) is obviously equivalent to the equation

$$\text{(3.9)} \qquad \int_\Omega (a\nabla(\psi_0 + u) \,.\, \nabla v + f(., \psi_0 + u)\, v)\, \mathrm{d}\Omega = \int_{\partial\Omega} gv\, \mathrm{d}S \quad \forall v \in V$$

for the unknown $u \in V$ (see (3.5)). In the following paragraphs we shall deal with the solvability of the problem (3.9). For this purpose let us introduce several concepts.

If V is a real Banach space, then V^* will denote its dual, i.e. the Banach space of all real continuous linear functionals defined on the space V, with the norm

$$\|f\| = \sup_{\substack{v \in V \\ \|v\| = 1}} |\langle f, v\rangle|$$

for every $f \in V^*$. The symbol $\langle f, v\rangle$ denotes the value of a functional f at a point $v \in V$.

Let V be the Hilbert space defined in (3.4) (V is a reflexive Banach space). Let us define the operator $T : V \to V^*$:

$$\text{(3.10)} \qquad \langle T(u), v\rangle = \int_\Omega (a\nabla(\psi_0 + u) \,.\, \nabla v + f(., \psi_0 + u)\, v)\, \mathrm{d}\Omega \quad (u, v \in V)\,.$$

In virtue of the theorem on traces, the mapping "$v \to \int_{\partial\Omega} gv\, \mathrm{d}S$" is a continuous linear functional defined on V. If we denote it by f, then $\langle f, v\rangle = \int_{\partial\Omega} gv\, \mathrm{d}S$. Hence, the problem (3.9) is equivalent to the solution of the equation

$$\text{(3.11)} \qquad \langle T(u), v\rangle = \langle f, v\rangle \quad \forall v \in V,$$

with respect to $u \in V$, or

$$\text{(3.12)} \qquad T(u) = f\,.$$

We shall use the monotone operators method for the solution of the equation (3.12) (see e.g. [1, 7]). If we define operators $M, N : V \to V^*$ by the relations

$$\text{(3.13)} \qquad \langle M(u), v\rangle = \int_\Omega a\nabla(\psi_0 + u) \,.\, \nabla v\, \mathrm{d}\Omega\,,$$

$$\langle N(u), v\rangle = -\int_\Omega f(., \psi_0 + u)\, v\, \mathrm{d}\Omega \quad (u, v \in V)\,,$$

then $T = M - N$. It follows from general results in the monographs [1, 7] that the operator T is hemicontinuous and the operator N is continuous from the weak topology of V to the strong topology of V^*. Thus, it will suffice to prove that (see e.g. Theorem 6.3 from [1])

$$\langle M(u_1) - M(u_2), u_1 - u_2 \rangle \geqq 0 \quad \forall u_1, u_2 \in V \tag{3.14, a}$$

(M is monotone) and

$$\lim_{\substack{\|u\| \to +\infty \\ u \in V}} \frac{\langle T(u), u \rangle}{\|u\|} = +\infty \tag{3.14, b}$$

(T is coercive).

For the following considerations we shall need the assertion known as the Fridrichs inequality ([6]).

Theorem 2. *Let V be the space defined in* (3.4) *and let $\partial\Omega_n$ satisfy the assumption* (3.2). *Then there exists a constant $c_4 > 0$ such that*

$$\int_\Omega u^2 \, d\Omega \leqq c_4 \int_\Omega (\nabla u)^2 \, d\Omega \quad \forall u \in V. \tag{3.15}$$

From here it follows at once that the norms (3.3) *and*

$$\|u\|_V = \left(\int_\Omega (\nabla u)^2 \, d\Omega \right)^{1/2} \tag{3.16}$$

are equivalent on the space V. It means that there exist constants $c_5, c_6 > 0$ such that

$$c_5 \|u\| \leqq \|u\|_V \leqq c_6 \|u\| \quad \forall u \in V. \tag{3.17}$$

Let us show now that the operators M and T satisfy the conditions (3.14). It holds

$$\langle M(u_1) - M(u_2), u_1 - u_2 \rangle = \int_\Omega a(\nabla(u_1 - u_2))^2 \, d\Omega \geqq$$
$$\geqq c_2 \|u_1 - u_2\|_V^2 \geqq c_2 c_5^2 \|u_1 - u_2\|^2 .$$

Hence, (3.14, a) is valid and the operator M is even strongly monotone. Further,

$$\langle T(u), u \rangle = \int_\Omega [a\nabla(\psi_0 + u) . \nabla u - f(., \psi_0 + u)\, u] \, d\Omega \geqq$$

$$\geqq c_2 \int_\Omega (\nabla u)^2 \, d\Omega - c_3 \left| \int_\Omega \nabla\psi_0 . \nabla u \, d\Omega \right| - c_1 \left| \int_\Omega u \, d\Omega \right| .$$

By using the Cauchy inequality we get

$$\langle T(u), u\rangle \geqq c_2\|u\|_V^2 - c_3\left(\int_\Omega (\nabla\psi_0)^2 \,\mathrm{d}\Omega\right)^{1/2} \|u\|_V - c_1\ \mu(\Omega)\ \|u\|$$

($\mu(\Omega)$ denotes the measure of the set Ω), so that, in view of (3.17),

$$\langle T(u), u\rangle \geqq c_2 c_5^2\|u\|^2 - c_7\|u\|\,,\quad c_7 = c_3 c_6\left(\int_\Omega (\nabla\psi_0)^2 \,\mathrm{d}\Omega\right)^{1/2} + c_1\ \mu(\Omega)\,,$$

which implies immediately the relation (3.14, b). We have proved the following theorem:

Theorem 3. *Let $A, B \in C^1(E_1)$, the functions a and f be given by the formulae* (2.4), (2.5) *and let* (3.2), (3.6) *and* (3.7) *hold. Then the problem* (3.9) *has at least one solution. Hence, the problem for the stream function has at least one weak solution.*

The problem of the uniqueness of the solution remains open. The uniqueness of the solution of Problem (B) is guaranteed e.g. if the function $f(y, t)$ is for every $y \in \langle R_1, R_2\rangle$ a nondecreasing function of the argument t. This assumption implies that the operator T is strictly monotone. However, it is difficult to interpret this assumption from the physical point fo view. E.g., for $k = 2$ it means that the vorticity ω is nonincreasing in dependence on the magnitude of the flow. In the report [4], an example of a "reasonable" stream field was studied, for which this assumption is not satisfied.

The second possibility is that the function f satisfies the condition

$$|f(y, t_1) - f(y, t_2)| \leqq K|t_1 - t_2| \qquad (3.18)$$
$$\forall y \in \langle R_1, R_2\rangle\,,\quad \forall t_1, t_2 \in E_1\,,$$

with a sufficiently small constant K. The detailed calculation shows that it is sufficient if

$$0 \leqq K < c_2 c_5^2\,. \qquad (3.19)$$

This assumption can be interpreted physically in such a way that the vorticity varies only little in dependence on the magnitude of the flow. It does not mean, of course, that the vorticity is small!

Thus, we can formulate a uniqueness theorem of the solution of Problem (B).

Theorem 4. *Let the assumptions of Theorem 3 hold and let either for every $y \in \langle R_1, R_2\rangle$, $f(y, t)$ be a nondecreasing function of t, or* (3.18) and (3.19) *be valid. Then the problem for the stream function has exactly one solution.*

Nevertheless, the problem of the uniqueness of the solution of Problem (A) is not solved since it depends on the extension of the functions A and B from the set M (see (2.15)) to the whole set E_1. When $\psi(\bar{\Omega}) = M$ for every ψ which we get as a solution of Problem (B) with arbitrary extensions of the functions A, B from M to E_1 (satisfying the conditions introduced above), the uniqueness of the solution of Problem (A) in the class of stream fields satisfying the assumption (2.2) is equivalent to the uniqueness of the solution of Problem (B). Here the question arises whether it is possible to accept the hypothesis that every well-founded stream field satisfies the assumption (2.2). It would be useful to carry out a series of experiments and to compare their results with the calculations based on this theory.

For irrotational flows we have $f \equiv 0$. The functions A and B are constant on M and the only possibility is to extend them to E_1 by the same constants. In this case, the solution of Problem (B) is unique. Under some assumptions on regularity of the solutions of Problem (A) which guarantee the existence of the stream function, we get the uniqueness of the solution of Problem (A). This implies that the boundary value conditions considered in the formulation of Problem (A) are in fact complete in the mentioned sense (see Section 1).

In conclusion, let us show in which sense the functions (2.11), given by a weak solution ψ of Problem (B), satisfy the system of equations (1.3) and (1.4). First, let us notice that the functions (2.11) are elements of the space $L_2(\Omega)$. The vorticity ω and the equations (1.7), (1.8) must be considered in the sense of distributions. It is evident that $\partial H/\partial x$, $\partial H/\partial y \in L_2(\Omega)$ and for $k = 3$ also $\partial(yv_\varphi)/\partial x$, $\partial(yv_\varphi)/\partial y \in L_2(\Omega)$. The third equation (1.9_3) is thus fulfilled almost everywhere in Ω, as can be found by substitution. The equation (3.8, b) implies that the distribution ω can be identified with the function $f(y, \psi(x, y))$, which is measurable and bounded in Ω. Thus, the system (1.9) is (after this identification) satisfied almost everywhere in Ω. Let us add that on the basis of the theorem on traces, the boundary value conditions (2.8) can be interpreted as a given flow through the curves $C_i(\tau)$ $(i = 0, \ldots, r)$ defined in the comments to the relation (2.9).

References

[1] *R. W. Carrol:* Abstract methods in partial differential equations, Harper, Row Publishers, New York, 1968.

[2] *M. Feistauer:* Some cases of numerical solution of differential equations describing the vortex-flow through three-dimensional axially-symmetric channels. Apl. mat. 16 (1971), No 4, 265—288.

[3] *M. Feistauer, J. Polášek:* The calculation of axially-symmetric stream fields. Proceedings of the Hydro-Turbo Conference 74, Luhačovice 1974 (in Czech).

[4] *M. Feistauer:* The calculation of some types of stream fields in the model of a unified outlet. Technical research report Tp VZ 11/74, ŠKODA Plzeň, 1974 (in Czech).

[5] *O. John, J. Nečas:* The equations of mathematical physics. SPN Prague, 1972 (in Czech).

[6] *J. Nečas:* Les méthodes directes en théorie des equations elliptiques. Academia, Prague, 1967.

[7] *М. М. Вайнберг:* Вариационный метод и метод монотонных операторов. Наука, Москва, 1972.

[8] *И. Н. Векуа:* Обобщенные аналитические функции, Москва, 1959.

Souhrn

DVOUROZMĚRNÉ A TŘÍROZMĚRNÉ OSOVĚ SYMETRICKÉ ZAVÍŘENÉ PROUDĚNÍ IDEÁLNÍ NESTLAČITELNÉ TEKUTINY

MILOSLAV FEISTAUER

V článku je vyšetřován obecný problém dvourozměrného a třírozměrného osově symetrického stacionárního zavířeného proudění ideální nestlačitelné tekutiny v omezené oblasti. Soustava diferenciálních rovnic sestávající z rovnice kontinuity a pohybových rovnic zapsaných v energetické formě byla doplněna třemi druhy okrajových podmínek, které jsou úplné v tom smyslu, že pro nevířivé proudění zaručují jednoznačnost řešení. Tato úloha pak byla po zavedení proudové funkce přetransformována na okrajovou úlohu pro kvazilineární parciální diferenciální rovnici druhého řádu se smíšenými okrajovými podmínkami, jejíž řešitelnost byla vyšetřena pomocí teorie monotonních a pseudomonotonních operátorů. Nakonec byla diskutována jednoznačnost řešení a otázka, v jakém smyslu splňují zmíněnou soustavu rovnic funkce dané slabým řešením úlohy pro proudovou funkci.

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