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Aplikace matematiky, Vol. 22 (1977), No. 1, 1–13

Persistent URL: `http://dml.cz/dmlcz/103674`

RELAXATION LENGTHS AND NON-NEGATIVE SOLUTIONS IN NEUTRON TRANSPORT

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(Received August 5, 1975)

INTRODUCTION

Most of the problems connected with a stationary system of neutrons in a homogeneous medium without external sources leads to the study of the linearized Boltzmann equation of the form

$$(1)\qquad [\mathbf{v}\nabla + v\,\Sigma(\mathbf{v})]\,N(r,\mathbf{v}) = \int_\omega \mathrm{d}^3\mathbf{v}'v'\,S(\mathbf{v}'\to\mathbf{v})\,N(r,\mathbf{v}') + \int_\omega \mathrm{d}^3\mathbf{v}'v'\,\chi(\mathbf{v}'\to\mathbf{v})\,\nu(\mathbf{v}')\,\Sigma_f(\mathbf{v}')\,N(\mathbf{r},\mathbf{v}')\,.$$

The following standard notation is used:

$\mathbf{r}$, $\mathbf{v}$ – space radius and velocity vectors;
v – modulus of $\mathbf{v}$;
ω – three dimensional velocity space;
$N(\mathbf{r},\mathbf{v})$ – neutron density;
$S(\mathbf{v}'\to\mathbf{v})$ – macroscopic differential scattering cross-section;
$\Sigma(\mathbf{v})$ – macroscopic total cross-section;
$\chi(\mathbf{v}'\to\mathbf{v})$ – fission spectrum;
$\nu(\mathbf{v})$ – number of fission neutrons originated by a neutron with velocity $\mathbf{v}$;
$\Sigma_f(\mathbf{v})$ – macroscopic fission cross-section.

Since the functions S, χ, Σ do not depend on space coordinates, one can expect that some of the particular solutions of equation (1) will be of the form

$$(2)\qquad N(\mathbf{r},\mathbf{v};\varkappa) = e^{-\varkappa x}\,\varphi(\mathbf{v};\varkappa)\,,$$

where x is the first component of the vector $\mathbf{r} = (x, y, z)$ and $\varkappa$ is any complex parameter.

The knowledge of the behavior of particular solutions (2) gives worthwhile information for solving various stationary neutron transport problems. Actually, one can see that for some important cases the collection of all of the solutions (2) contains a system which is complete in a certain sense. Since the construction of the solutions (2) is a relatively easy matter, one can apply the complete system for solving transport problems by the method of superposition (see, e.g., Case [2], Zelazny [27], Mika [19], Cercignani [3]). In connection with the method mentioned, the parameter has a quite natural physical interpretation. Let us consider for example, a point source of neutrons in a slab geometry located at the origin, and let us investigate the stationary neutron distribution propagating from the source in an unbounded homogeneous medium. By expressing the solution of this problem as a superposition of (2) with various $\varkappa$'s, we can observe that those terms would dominate at long distances from the origin for which the real part of $\varkappa$ is minimal. In non-multiplying media one can expect, moreover, that the solution tends to zero as the distance tends to infinity; in multiplying media the solution will assume a non-zero value at infinity. Thus, in agreement with the diffusion approximation, the reciprocal to $\varkappa$ is the diffusion length.

The example shown above leads to the following idea: To classify the media according to the properties of the set of $\varkappa$'s occuring in (2). As usual the value $\varkappa^{-1}$ is called the relaxation length. If there is a purely imaginary $\varkappa$ such that the corresponding expression (2) is a solution of (1), then this medium is called multiplying; in the opposite case absorbing (see [4]). The relation between the properties of the medium considered and the relaxation lengths is not too satisfactory; some natural claims have more in common with assumptions than with assertions. One usually admits that the relaxation lengths in non-multiplying media are real. The proof, however, has been given for some more or less very special cases ([11], [15]). The situation is even less clear in the case of multiplying media [4]. In this paper we consider a rather general case; we emphasize the multiplying case for which all of the results are new, and we also re-prove and complete the theory of relaxation lengths in non-multiplying media. In particular we give an affirmative answer to a conjecture formulated in [15].

FORMULATION OF THE PROBLEM AND BASIC ASSUMPTIONS

The macroscopic quantities $\Sigma(\boldsymbol{v})$, $S(\boldsymbol{v}' \to \boldsymbol{v})$, etc. occuring in equation (1) characterize the following processes: elastic scattering, absorption of neutrons and creation of neutrons by fission of nuclei. How precisely one has to consider these particular processes depends essentially on the material and on the geometrical properties of the medium under consideration.

A general theory of the potential elastic scattering was built up by van Hove [24] on the basis of a time-space correlation function. For the free gas model, i.e., for the model in which, with the exception of the spin, the inner structure of the scattering particles and also the forces between these particles are neglected, this theory gives the following expression for the kernel S:

$$S(\mathbf{v}' \to \mathbf{v}) = \frac{C}{v'}\left(\frac{A}{2\pi T \mathbf{K}^2}\right)^{1/2} \exp\left\{-\frac{A}{2T\mathbf{K}^2}\left(\varepsilon - \frac{\mathbf{K}^2}{2A}\right)^2\right\} \tag{3}$$

where $\mathbf{K} = \mathbf{k} - \mathbf{k}'$, $\varepsilon = E - E'$ denote the difference of impulse and energy of a neutron respectively before and after the collision, A, T, C being certain constants (for details see [26]). The formula (3) remains valid for liquids and solids as well as for energies exceeding 1 eV, because the effects connected with the chemical structure of the medium will be negligible for the approaching neutrons with these energies, and hence the solids and liquids will behave like a free gas in this case.

At lower energies some quantum effects connected with the chemical binding begin to manifest themselves more strongly. For solids one can see that the macroscopic differential scattering cross-section may be expanded into a sum of elastic and inelastic contributions:

$$S(\mathbf{v}' \to \mathbf{v}) = S_e(\mathbf{v}' \to \mathbf{v}) + S_{in}(\mathbf{v}' \to \mathbf{v}), \tag{4}$$

where

$$S_e(\mathbf{v}' \to \mathbf{v}) = \Sigma_{\text{inc}} v^{-2} e^{-2W_d} \delta(v - v') + \Sigma_{\text{coh}} v^{-2} e \sum_{\tau} \delta(\mathbf{K} - \tau)\, \delta(v - v')\, e^{-2W_d}, \tag{5}$$

where τ is the vector of the reciprocal mesh, W_d – the so-called Debye-Waller factor and Σ_{inc} and Σ_{coh} are certain constants. In a similar manner the inelastic contribution S_{in} can be expanded. In most cases the coherent inelastic contribution is negligible ([23]).

The theory of liquids is not elaborated satisfactorily. Thus, there is no adequate description of the neutron scattering for liquid media. Some kind of description can be obtained using van Hove's theory for small $|\varepsilon|$ and $|\mathbf{K}|$:

$$v'\, S(\mathbf{v}' \to \mathbf{v}) = C_1 \frac{\mathbf{K}^2}{\pi(C_2 \mathbf{K}^4 + \varepsilon^2)} + C_3 \mathbf{K}^2, \tag{6}$$

where C_1, C_2, C_3 are constants [14].

The resonance scattering is connected with the inner structure of the nucleus. In the resonance domain the scattering is comparable with the potential scattering, and outside the resonance domain the scattering is negligibly small. If the resonance is sufficiently narrow, then the resonance scattering can be neglected.

The inelastic scattering is observable only if the energy of the approaching neutron is higher than 10 keV. The macroscopic scattering cross-section either by inelastic or by resonance scattering is mostly independent of the angle [4].

The absorption of neutrons is an isotropic process consisting of the radiation capture and the capture of neutrons joint with the radiation of charged particles. The macroscopic absorption cross-section $\Sigma_a(v)$ is proportional to $1/v$ for low energy of the approaching neutron, and it is bounded for high energy.

The fission process is very closely connected with the inner structure of the nuclei; it has not yet been adequately described. The measurements of the fission spectrum $\chi(\mathbf{v}' \to \mathbf{v})$, i.e., of the probability whether a neutron with velocity $\mathbf{v}'$ creates fission and the creation of a neutron with velocity $\mathbf{v}$, are still rather incomplete. It appears, however, that it is quite acceptable to assume that this process can be considered as isotropic [26]. Assuming that the fission is created by slow neutrons, only the following empirical formula has been derived for the ${}_{92}U^{235}$ nuclei:

$$(7) \qquad \chi(v) = k_0 e^{-v^2/1.93} \sinh\left(\sqrt{(1{\cdot}15)}\, v\right).$$

This formula holds for the velocities corresponding to energies lower than 10 MeV and, up to certain unimportant exceptions, its form is applicable to other fission materials. The behavior of the fission spectrum for higher energies is still not known.

Concerning the fission cross-section $\Sigma_f(v)$ one can say that its behavior is like that of Σ_a, and this is sufficient for our purposes. The function $\nu = \nu(v)$ essentially depends on the atomic weight of the fission material. It remains positive and bounded from above (as usual, its upper bound equals 4).

The total cross-section $\Sigma(\mathbf{v})$ can be expressed as a sum

$$(8) \qquad \Sigma(\mathbf{v}) = \Sigma_s(\mathbf{v}) + \Sigma_a(v) + \Sigma_f(v),$$

where

$$\Sigma_s(\mathbf{v}) = \int_\omega \mathrm{d}^3\mathbf{v}'\, S(\mathbf{v} \to \mathbf{v}').$$

In the following we restrict ourselves to the case of an isotropic medium and do not consider the resonance and inelastic scattering. The differential cross-section will then depend on the angle between $\mathbf{v}$ and $\mathbf{v}'$ by means of the inner product $\mathbf{v} \cdot \mathbf{v}'$ and the following relation, the so-called detailed balance relation [26], will hold in the form

$$(9) \qquad v\, M(v)\, S(\mathbf{v} \to \mathbf{v}') = v'\, M(v')\, S(\mathbf{v}' \to \mathbf{v}),$$

in which

$$M(v) = \left(\frac{m}{2kT}\right)^{3/2} \exp\left\{-\frac{mv^2}{2kT}\right\}$$

with m, k, T being certain constants. This obviously implies for the total cross-section that

$$\Sigma(\mathbf{v}) = \Sigma(v).$$

It is quite natural to assume that

$$(10) \qquad \varkappa^* = \inf\{\Sigma(v) : v \in [0, +\infty)\} > 0;$$

we also require the kernel S to satisfy the following: There is an $\varepsilon > 0$ such that for every $\mathbf{v}$ one has that

$$S(\mathbf{v} \to \mathbf{v}') > 0 \tag{11}$$

for almost all $\mathbf{v}'$ for which $|\mathbf{v} - \mathbf{v}'| < \varepsilon$.

We emphasize the fact that our main goal is to consider multiplying media; thus, we assume that

$$\Sigma_a(v) + \Sigma_f(v) \not\equiv 0 . \tag{12}$$

The fission spectrum is considered to be independent of the angles; thus

$$\chi(\mathbf{v} \to \mathbf{v}') = \chi(v, v') . \tag{13}$$

Let us set $\varphi(\mathbf{v}; \varkappa) = M(v)\,\psi(\mathbf{v})$ in (2) and let us substitute this expression into (1). Using (9) we then obtain

$$(\Sigma(v) - \varkappa\mu)\,\psi(\mathbf{v}) = \int_\omega d^3\mathbf{v}'\, S(\mathbf{v} \to \mathbf{v}')\,\psi(\mathbf{v}') + \tag{14}$$

$$+ \int_\omega d^3\mathbf{v}'\, \frac{v'\chi(v', v)}{v\, M(v)}\, \Sigma_f(v')\, \nu(v') M(v')\, \psi(\mathbf{v}') ,$$

where $\mu = v_x/v$.

Our problem is to consider the equation (14). For this purpose we introduce the complex Hilbert space $\mathscr{H}$ consisting of measurable functions ψ such that

$$\int_\omega d^3\mathbf{v} v\, M(v)\, |\psi(\mathbf{v})|^2 < +\infty .$$

Obviously $\mathscr{H}$ is a Hilbert space with the inner product

$$(\varphi, \psi) = \int_\omega d^3\mathbf{v} v\, M(v)\, \varphi(\mathbf{v})\, \overline{\psi(\mathbf{v})}, \quad \varphi, \psi \in \mathscr{H} . \tag{15}$$

The equation (14) can be written symbolically as

$$[\Sigma - \varkappa\mu]\,\psi = (S + F)\,\psi , \tag{16}$$

where the meaning of the operators Σ, S, F is obvious according to (14). We summarize some of the properties of these operators.

The operators Σ and μ are obviously self-adjoint and μ and Σ^{-1} are bounded. The operator S is bounded and symmetric. Moreover, S is compact for the free gas model, and so is $K \equiv \Sigma^{-1/2} S\, \Sigma^{-1/2}$ for liquids.

For the case of solids we write $S = S_e + S_{in}$ (according to (4)). It is known that S_{in} is compact and S_e is bounded (see, e.g., [10], [9], [20], [1]).

Concerning the operator F, the situation remains unclear because there is no adequate description of the fission spectrum (7) for high energies (>15 MeV). Fortunately, the experimental data show with good exactness that one can consider only neutrons with energies up to 10 MeV [25] for all of the reactor calculations. This is our case, too, because we already have restricted ourselves to such energies by neglecting the inelastic scattering. As a consequence of this assumption, we can let e.g. $\chi \equiv 0$ for $E \geqq 20$ MeV. In this case the operator F is obviously bounded, and since it has one-dimensional range, it is compact.

The previous arguments also justify our assumption that F is compact in the case of more general fission spectra (13).

SPECTRAL PROPERTIES

We say that a complex number $\varkappa_0$ is an element of the continuous (residual, point) spectrum of the operator pencil $A - \varkappa B$, if $\lambda = 0$ is an element of the continuous (residual, point) spectrum of the operator $A - \varkappa_0 B$, where A and B are linear operators mapping a Banach space $\mathscr{X}$ into itself. The spectrum of A is denoted by $\sigma(A)$.

Before we formulate our result concerning the spectral properties of (14) we shall state some auxiliary assertions needed in our proofs. We refer the reader to the paper [8] where some more general assertions can be found. For the reader's convenience we formulate some results of Gokhberg and Krein and of others in the form we shall use.

Let A be a densely defined linear operator mapping its domain $\mathscr{D}(A) \subset \mathscr{X}$ into $\mathscr{Y}$, where $\mathscr{X}$ and $\mathscr{Y}$ are Banach spaces. The operator A is called normally solvable if the equation $Ax = y$ has a solution if and only if $y'(y) = 0$ for all $y' \in \mathfrak{N}^+(A)$ where $\mathfrak{N}^+(A)$ is the defect space of A. Let us set $\beta(A) = \dim \mathfrak{N}^+(A)$ and $\alpha(A) = \dim \mathfrak{N}(A)$ where $\mathfrak{N}(A)$ is the nullspace of A. A closed operator A is called an **F**-operator if it is normally solvable and its characteristic $d = (\alpha(A), \beta(A))$ contains finite integers. We denote by ind $(A) = \beta(A) - \alpha(A)$ the index of A. A complex number λ is called an **F**-point of A if $A - \lambda I$ is an **F**-operator. The set of all **F**-points of A is called the **F**-set of A and is denoted by $\mathbf{F}(A)$.

Lemma 1. *Let $A = A(\lambda)$ be an analytic operator-valued function in a domain G of the complex plane and let each of the operators $A(\lambda)$, $\lambda \in G$, be a closed and normally solvable operator mapping its domain $\mathscr{D}(A) \subset \mathscr{X}$ into $\mathscr{X}$, where $\mathscr{X}$ is a Banach space, such that* 0 *is an* **F**-*point of $A(\lambda)$ for all $\lambda \in G$.*

(a) *Then the quantities $\beta(A(\lambda)) = \dim \mathfrak{N}^+(A(\lambda))$ and $\alpha(A(\lambda)) = \dim \mathfrak{N}(A(\lambda))$ are finite and the function* ind $(A(\lambda)) = \beta(A(\lambda)) - \alpha(A(\lambda))$ *is constant in G.*

(b) *If, moreover, $A(\lambda)$ is compact for $\lambda \in G$, then the operator $I - A(\lambda)$ is either not boundedly invertible at any point of G, or the bounded inverse exists and $(I - A(\lambda))^{-1}$ is analytic in G with possibly an exceptional set σ which is at most countable. Moreover, $\alpha(I - A(\lambda)) = \alpha$ for $\lambda \in G \setminus \sigma$, where α is a constant, and $\alpha(I - A(\lambda_j)) > \alpha$ for $\lambda_j \in \sigma$.*

Lemma 2. *Let A be a closed operator mapping its domain $\mathscr{D}(A) \subset \mathscr{X}$ into $\mathscr{X}$, let B be a compact map of $\mathscr{X}$ into $\mathscr{X}$. Then the* **F**-*sets of A and of $A + B$ coincide*: $\mathbf{F}(A + B) = \mathbf{F}(A)$.

Let us note that Lemma 1 is a consequence of Theorems 3.5 and 3.7 in [8]. Lemma 2 is actually Theorem 3.4 in [8].

As a consequence of Lemma 2 we obtain:

Proposition 1. *The point $\lambda = 0$ is an* **F**-*point of $C(\varkappa) = \Sigma - S - F - \varkappa\mu$ if and only if it is an* **F**-*point of $B(\varkappa) = \Sigma - S - \varkappa\mu$.*

The main result of this section is the following

Theorem 1. (a) *The spectrum of the pencil $C(\varkappa) = \Sigma - S - F - \varkappa\mu$ coincides with the set*

$$M_1 = (-\infty, -\varkappa^*] \cup [\varkappa^*, +\infty) \cup N_1 \tag{17}$$

for the case of gas and liquid media, and is a part of the set

$$M_2 = (-\infty, -\gamma\varkappa^*] \cup [\gamma\varkappa^*, +\infty) \cup N_2 \tag{18}$$

for the case of solid materials. Here in (17, 18) *the sets N_1 and N_2 are certain sets of isolated points whose limits belong to* $(-\infty, -\varkappa^*] \cup [\varkappa^*, +\infty)$ *and* $(-\infty, -\gamma\varkappa^*] \cup [\gamma\varkappa^*, +\infty)$ *respectively, and*

$$\gamma = 1 - \sup \left\{ \frac{1}{\Sigma(v)} \int_\omega \mathrm{d}^3\mathbf{v}'\ S_e(\mathbf{v} \to \mathbf{v}') : v \in [0, +\infty) \right\} > 0\,. \tag{19}$$

(b) *The operator- valued function $(\Sigma - S - F - \varkappa\mu)^{-1}$ is analytic with respect to $\varkappa$ outside the sets M_1 and M_2 respectively; the singularities of $(\Sigma - S - F - \varkappa\mu)$ belonging to N_1 and N_2 are poles.*

(c) *The dimensions of the null-spaces of $C(\varkappa)$ and of its adjoint coincide.*

Proof. Let $\varkappa = \mathrm{i}\nu$ be purely imaginary. Then we have for $\varphi \in \mathscr{H}$ that

$$\|(I - \varkappa\mu\Sigma^{-1})^{-1}\,\Sigma^{-1/2}\,F\Sigma^{-1/2}\varphi\|^2 = \tag{20}$$
$$= \frac{4\pi}{|\nu|}\int_0^\infty \mathrm{d}v \arctan \frac{|\nu|}{\Sigma(v)}\, v^3\, M(v)\, |F\Sigma^{-1/2}\,\varphi(v)|^2 \leqq 1|\nu|^{-1}\,\tau\|F\Sigma^{-1/2}\varphi\|^2\,,$$

where τ is a positive constant, $\tau < +\infty$. Furthermore, for the same $\varkappa$'s

$$|\varphi - (I - \varkappa\mu\Sigma^{-1})^{-1}\,\Sigma^{-1/2}S\Sigma^{-1/2}\varphi| \geqq$$
$$\geqq |\varphi| - \frac{1}{|1 - \varkappa\mu\Sigma^{-1}|}\,|\Sigma^{-1/2}S\Sigma^{-1/2}\varphi| \geqq$$
$$\geqq |\varphi| - \Sigma^{-1/2}S\Sigma^{-1/2}|\varphi| = (I - \Sigma^{-1/2}S\Sigma^{-1/2})\,|\varphi|\,.$$

Therefore,

$$\text{(21)} \qquad \left\| I - (I - \varkappa\mu\Sigma^{-1})^{-1} \Sigma^{-1/2} S \Sigma^{-1/2} \right\| \geqq$$

$$\geqq \sup_{\|\varphi\|=1} \left((I - \Sigma^{-1/2} S \Sigma^{-1/2}) |\varphi|, |\varphi| \right) \geqq \alpha^2 > 0 \,, \quad \text{where} \quad \alpha^2 = \inf_{\substack{\varphi \in \mathscr{D}(\Sigma) \\ \|\varphi\|=1}} \left((\Sigma - S)\, \varphi, \varphi \right).$$

Let us consider first the case of the liquid and gas materials. In this case the operator $\Sigma^{-1/2}(S + F)\,\Sigma^{-1/2} = K + L$ is compact and we have that

$$\Sigma - S - F - \varkappa\mu = \Sigma^{1/2}[I - (K + L) - \varkappa\mu\Sigma^{-1}]\,\Sigma^{1/2}\,.$$

The number $\lambda = 0$ obviously belongs to the spectrum of $I - \varkappa\mu\Sigma^{-1}$ for all $\varkappa \in (-\infty, -\varkappa^*] \cup [\varkappa^*, +\infty)$; the inverse to $I - \varkappa\mu\Sigma^{-1}$ exists and is bounded for $\varkappa$ outside this set. Proposition 1 then implies that $\lambda = 0$ belongs to the spectrum of $\Sigma - S - F - \varkappa\mu$ for the same $\varkappa$'s as it does for $I - K - L - \varkappa\mu\Sigma^{-1}$. The inverse $[I - (I - \varkappa\mu\Sigma^{-1})^{-1}(K + L)]^{-1}$ is bounded for all purely imaginary $\varkappa$'s with sufficiently large absolute values $|\varkappa|$ as can be observed from (20). The assertions (a) and (c) then follow according to Lemma 1 and Lemma 2. Assertion (b) is a consequence of (b) in Lemma 1 and the relationship

$$\text{(22)} \qquad \Sigma - S - F - \varkappa\mu =$$

$$= \Sigma^{1/2}(I - \varkappa\mu\Sigma^{-1})\,[I - (I - \varkappa\mu\Sigma^{-1})^{-1}(K + L)]\,\Sigma^{1/2}\,.$$

For the case of solid materials the operator $\Sigma^{-1/2}(S_i + F)\,\Sigma^{-1/2}$ is compact and $\Sigma^{-1/2}S_e\Sigma^{-1/2}$ is bounded, and

$$\text{(23)} \qquad \Sigma - S - F - \varkappa\mu = \Sigma^{1/2}[I - \Sigma^{-1/2}S_e\Sigma^{-1/2} - \varkappa\mu\Sigma^{-1}]\,.$$

$$.\,\{I - [I - \Sigma^{-1/2}S_e\Sigma^{-1/2} - \varkappa\mu\Sigma^{-1}]^{-1}\,\Sigma^{-1/2}(S_i + F)\,\Sigma^{-1/2}\}\,\Sigma^{1/2}\,.$$

For real $\varkappa$ we have (see (10)) that $\left\|(I - \Sigma^{-1/2}S_e\Sigma^{-1/2} - \varkappa\mu\Sigma^{-1})\,\varphi\right\| \geqq \geqq (\gamma - \varkappa/\varkappa^*)\,\|\varphi\|$. Hence, $I - \Sigma^{-1/2}S_e\Sigma^{-1/2} - \varkappa\mu\Sigma^{-1}$ has a bounded inverse for all $\varkappa \in (-\gamma\varkappa^*, \gamma\varkappa^*)$ and, since it is selfadjoint, the inverse exists and is bounded for all $\varkappa$ with $\operatorname{Im}\varkappa \neq 0$. Furthermore, the inverse $(\Sigma - S - F - \varkappa\mu)^{-1}$ exists and is bounded for purely imaginary $\varkappa$'s with sufficiently large absolute values as can be observed from (20) and the relationship

$$\text{(24)} \qquad \Sigma - S - F - \varkappa\mu =$$

$$= \Sigma^{1/2}(I - \varkappa\mu\Sigma^{-1})\,[I - (I - \varkappa\mu\Sigma^{-1})^{-1}\,\Sigma^{-1/2}(S + F)\,\Sigma^{-1/2}]\,\Sigma^{1/2}\,.$$

The validity of all of the assertions (a)–(c) then follows similarly as in the previous case according to Lemmas 1 and 2. Theorem 1 is thus completely proved.

We already know that the set $(-\infty, -\varkappa^*) \cup (\varkappa^*, +\infty)$ is a part of the continuous spectrum for liquid and gas media. For solid materials the question concerning the qualitative structure of this set remains open. For practical purposes the following theorem may be useful.

Theorem 2. *Let the kernel* $S(\mathbf{v} \to \mathbf{v}')$ *in* (1) *have the following form*:

$$S(\mathbf{v} \to \mathbf{v}') = \frac{1}{4\pi} \sum_{j=1}^{M} (2j + 1)\, S_j(v, v')\, P_j(\tilde{\mu})\, v'^2 ,$$

where $\tilde{\mu}$ *is the cosine of the angle between the vectors* $\mathbf{v}$ *and* $\mathbf{v}'$, P_j *denotes the Legendre polynomial and* M *is a positive integer. Then the set* $(-\infty, -\varkappa^*) \cup \cup (\varkappa^*, +\infty)$ *contains no points of either the residual or the point spectrum of the pencil* $\Sigma - S - F - \varkappa\mu$.

The proof of this theorem does not differ from that given in [15] for non-multiplying media and hence it is omitted.

NON-NEGATIVE EIGENFUNCTIONS

Non-negative solutions of the equation (14) are of great importance in physical applications. In this section we show that certain existence assertions concerning non-negative eigenfunctions of (14) are implied by the fact that the operators $K = \Sigma^{-1/2} S \Sigma^{-1/2}$ and $L = \Sigma^{-1/2} F \Sigma^{-1/2}$ leave invariant the cone of function-classes in $\mathscr{H}$ having non-negative representatives; the uniqueness assertions are then consequences of the indecomposability of $K + L$.

We call a bounded linear operator T in a Banach space $\mathscr{X}$ positive if it leaves invariant a normal reproducing cone $\mathscr{C} \subset \mathscr{X}$ [13]. A positive operator T is called indecomposable [22] if, for every couple $x \in \mathscr{C}$, $x \neq 0$, $x' \in \mathscr{C}'$, $x' \neq 0$, where $\mathscr{C}'$ is the dual cone [13], there is a positive integer $\mathrm{p} = \mathrm{p}(x, x')$ such that $x'(T^{\mathrm{p}} x) > 0$.

Proposition 2. *Let* T *be an operator of Radon-Nikolskii type, i.e. let* $T = U + V$, *where* U *is compact and* V *bounded, and their spectral radii satisfy the following relation*: $r(T) > r(V)$. *Furthermore, let both the operators* U *and* V *be positive and* $U + V$ *indecomposable with respect to a generating and normal cone* $\mathscr{C}$. *Then there is an eigenvector* x_0 *of* T *corresponding to* $r(T)$ *such that* $x_0 \in \mathscr{C}$ *and, up to scalar multiples of* x_0, *there are no other eigenvectors of* T *in* $\mathscr{C}$. *Moreover,* $x'(x_0) > 0$ *for every* $x' \in \mathscr{C}'$, $x' \neq 0$.

This proposition is proved in [16] (see also [18]); we state it here because it is the main tool for avoiding certain difficulties connected with the non-compactness of S.

Before stating our main result, we introduce some notation.

Let $\varphi \in \mathscr{H}$, $\varphi \geqq 0$ a.e., $\varphi \neq 0$. We set [16] $r_\varphi(K + L) = \sup \{ v \in (-\infty, +\infty) : ((K + L)\, \varphi, \varphi') \geqq v(\varphi, \varphi'),\ \forall \varphi' \in \mathscr{H},\ \varphi' \geqq 0 \text{ a.e.},\ \varphi' \neq 0 \}$, $r^\varphi(K + L) = \inf \{ \tau \in (-\infty, +\infty) : ((K + L)\, \varphi, \varphi') \leqq \tau(\varphi, \varphi'),\ \varphi' \in \mathscr{H},\ \varphi' \geqq 0 \text{ a.e. } \varphi' \neq 0 \}$.

Theorem 3. (a) *If* $\sup_{\varphi>0} r^{\varphi}(K + L) \leqq 1$, *then there exists a value* $\varkappa_0 \in [0, \varkappa^*)$ *and a function* $\varphi_0 > 0$ *a.e. such that* $(\Sigma - S - F - \varkappa_0\mu)\,\varphi_0 = 0$. *Moreover, the functions* $\varphi^{\pm}(\mathbf{v}) = \varphi_0(\pm\mathbf{v})$ *are the only non-negative solutions to* (14) *for* $\varkappa$ *in the complex plane.*

(b) *If*

$$\inf\{r_{\varphi}(K + L) : \varphi \in \mathscr{H},\ \varphi \geqq 0 \text{ a.e.},\ \varphi \neq 0\} \geqq 1\,,$$

then there are no non-negative solutions to (14) *for any* $\varkappa$ *in the complex plane.*

Proof. Let us consider the operator

$$T(\varkappa) = I + \varkappa\mu\Sigma^{-1} + K\,,\quad \varkappa \in [0, +\infty)\,.$$

This operator is bounded and symmetric, and for $\varkappa \in [0, \varkappa^*]$ leaves invariant the cone $\mathscr{C} = \{\varphi \in \mathscr{H};\ \varphi \geqq 0 \text{ a.e.}\}$. This cone is obviously reproducing and normal. According to our assumption (11), the operator K is indecomposable and so is $T(\varkappa)$. By setting $U = K$ for liquids and gases, and $U = K + \Sigma^{-1/2}S_{in}\Sigma^{-1/2}$ for solids, and

$$V(\varkappa) = \begin{cases} I + \varkappa\mu\Sigma^{-1} & \text{for liquid and gas media} \\ I + \varkappa\mu\Sigma^{-1} + \Sigma^{-1/2}S_e\Sigma^{-1/2} & \text{for solid media}\,, \end{cases}$$

we see that $T(\varkappa) = U + V(\varkappa)$, and we can show that $T(\varkappa)$ is a Radon-Nikolskii operator for every $\varkappa \in [0, \varkappa^*]$. To prove this, some simple criteria derived in [17] can be used, e.g. Theorem 2 and Theorem 5, in particular. It follows that $T(\varkappa) + L$ is an indecomposable Radon-Nikolskii operator.

The following function ϱ will be examined:

$$\varrho(\varkappa) = \sup\{\lambda \in (-\infty, +\infty) : \lambda \in \sigma(T)\varkappa) + L)\}\,.$$

For every $\varkappa \in [0, \varkappa^*]$, $\varrho(\varkappa)$ is an eigenvalue of $T(\varkappa) + L$ and to this eigenvalue there corresponds, up to a positive factor, a unique non-negative eigenfunction $\varphi_\varkappa$ of $T(\varkappa) + L$ and a non-negative eigenfunction $\varphi_\varkappa^*$ of the adjoint $[T(\varkappa) + L]^*$. Because $\varphi_\varkappa$ is positive a.e., we may assume that $(\varphi_\varkappa, \varphi_\varkappa^*) = 1$. It can be shown easily that the function $\varrho = \varrho(\varkappa)$ is continuous and continuously differentiable, its first derivative being given by

$$\frac{\mathrm{d}}{\mathrm{d}\varkappa}\varrho(\varkappa) = (\mu\Sigma^{-1}\varphi_\varkappa, \varphi_\varkappa^*)\,. \tag{25}$$

Actually, ϱ is analytic as a function of $\varkappa$ in $(0, \varkappa^*)$. Obviously, $\varrho(0) - 1 = r(K + L)$, the spectral radius of $K + L$, and

$$\varrho(\varkappa^*) > 2\,. \tag{26}$$

The last relation is a consequence of the fact that $(-\infty, -\varkappa^*) \cup (\varkappa^*, +\infty)$ always belongs to the spectrum of the pencil $\Sigma - S - \varkappa\mu$. We note that this is proved in [15] for gas and liquid moderators, and in [6] for solid moderators by means of a criterion of Weyl and von Neumann. For general media the required result is obtained by using a monotonicity argument.

First, let us consider the case (a); then $\varrho(0) = r(K + L) + 1 \leqq 2$, and hence, since ϱ is continuous, it assumes the value 2 at a point $\varkappa_0 \in [0, \varkappa^*)$. The uniqueness of $\varkappa_0$ follows from the relation

$$\frac{d}{d\varkappa}\varrho(\varkappa) \geqq 0 \quad \text{for} \quad \varkappa \in [0, \varkappa^*) . \tag{27}$$

To prove the validity of (27), we consider the expression

$$(\mu\Sigma^{-1}\varphi_\varkappa, \varphi_\varkappa^*) = \frac{1}{\varkappa} - \frac{1}{\varkappa}((K + L)\varphi_\varkappa, \varphi_\varkappa^*) \geqq \frac{1}{\varkappa}[1 - r^{\varphi_\varkappa}(K + L)] \geqq 0 .$$

This completes the proof of assertion (a).

In the case (b) we see that

$$\frac{d}{d\varkappa}\varrho(\varkappa) \leqq \frac{1}{\varkappa}[1 - \inf\{r_\varphi(K + L) : \varphi \geqq 0 \text{ a.e. } \varphi \neq 0\}] \leqq 0 \quad \text{for} \quad \varkappa \in [0, \varkappa^*] .$$

Hence $\varrho(\varkappa)$ does not increase for such $\varkappa$'s, and the assertion (b) then follows according to (26). Theorem 3 is thus completely proved.

Remark. The assumptions of Theorem 2 concerning the quantities $r_\varphi(K + L)$ and $r^\varphi(K + L)$ are is entirely theoretical, and the verification of their validity may be a rather difficult matter. On the other hand, continuity arguments show that the condition stated in (a) is always fulfilled if $\|L\|$ is small enough, and the condition in (b) holds whenever the medium under consideration is strongly fissionable.

CONCLUDING REMARKS

Let us note that our results generalize all of the results derived in [15] for non-multiplying media. If, in particular, L is the zero operator, then $\|K\| \leqq 1$ and the conclusion of Theorem 3 completes the uniqueness result for solid moderators conjectured in [15].

As already mentioned above, a relaxation-lengths theory for moderator media has been presented in [15], where the symmetry properties of the operators appearing there have been used in an essential way. However, the proof of the uniqueness result of Theorem III in [15] is based on a lemma which is obviously false. As we show, the conclusions of this theorem remain valid for general media.

We summarize the results.

If the fission in the medium is sufficiently weak, then:

(i) The properties of the relaxation lengths are very similar to those in the corresponding non-multiplying medium characterized by the same cross-sections.

(ii) The possible complex relaxation lengths are located in a strip along the real axis, the thickness of the strip being a multiple of the norm of the fission operator F.

(iii) The diffusion length is (up to symmetry) the only relaxation length to which there corresponds a non-negative eigensolution.

If the fission in the medium under consideration is sufficiently strong, then:

(iv) There are no non-negative eigensolutions of the relaxation-lengths equation (14). This has the interpretation that the concept of diffusion length loses its meaning.

References

[1] *M. Borysewicz, J. Mika:* Time behavior of thermal neutrons in moderator media. CNEN Documents CEC (68), 1 Bologna (1968).

[2] *K. M. Case:* Elementary solutions of the transport equation and their applications. Ann. Phys. 9 (1960), 1—23.

[3] *C. Cercignani:* Unsteady solutions of the kinetic models with velocity-dependent collision frequency. Ann. Phys. 40 (1966), 454—468.

[4] *B. Davison:* Milne problem in a multiplying medium with a linearly anisotropic acattering. Canadian Report CRT 358 (1946).

[5] *B. Davison:* Neutron Transport Theory. Oxford University Press, London 1957.

[6] *J. J. Duderstadt:* Theory of the propagation of the plane-wave disturbances in a distribution of thermal neutrons. J. Math. Phys. 10 (1969), 266—276.

[7] *N. Dunford, J. T. Schwartz:* Linear operators. Part I. Interscience Publ. New York 1957.

[8] *L. C. Gokhberg, M. G. Krein:* The basic propositions on defect numbers, root numbers and indices of linear operators. Usp. mat. nauk XII (1957), 43—118 (Russian).

[9] *H. Grad:* Asymptotic theory of the Boltzmann equation. I. Proceedings of the Third Intern. Symp. Rarefied Gas Dynamics, Vol. I, Academic Press, New York—London, 1963, pp. 26—59.

[10] *E. Hecke:* Über die Integralgleichung der kinetischen Gastheorie. Math. Zeitschrift 12 (1922), 277—286.

[11] *S. Kaplan:* Properties of the relaxation lengths in P_L, double P_L and angle-space synthetic approximations. Bull. Sci. Eng. 28 (1967), 456.

[12] *T. Kato:* Perturbation Theory for Linear Operators. Springer-Verlag, Berlin—Heidelberg—New York 1966.

[13] *M. G. Krein, M. A. Rutman:* Linear operators leawing invariant a cone in a Banach space. Usp. mat. Nauk III (1948), N 1, 3—95 (Russian), English translation in the Translations of the AMS, Vol. 26 (1952).

[14] *I. Kuščer,* N. *Corngold:* Discrete relaxation times in neutron thermalization. Phys. Rev. 139 (1965), A981—990; 140 (1965), A85.

[15] *I. Kuščer, I. Vidav:* On the spectrum of relaxation lengths of neutron distributions in a moderator. J. Math. Anal. Appl. 25 (1969), 80—92.

[16] *I. Marek:* Frobenius theory of positive operators. Comparison theorems and applications. SIAM J. Appl. Math. 19 (1970), 607—628.

[17] *I. Marek:* On Fredholm points of compactly perturbed bounded linear operators. Acta Univ. Carol. — Mathematica et Physica Vol. 17, (1976) No. 1, 65—72.
[18] *I. Marek:* On some spectral properties of Radon-Nikolskii operators and their generalizations, Comment. Math. Univ. Carol. 3, 1 (1962), 20—30.
[19] *J. Mika:* Neutron transport with anisotropic scattering. Nucl. Sci. Eng. 11 (1961), 415—427.
[20] *B. Montagnini, M. L .Demuru:* Complete continuity of the free gas scattering operator in neutron thermalization theory. J. Math. Anal. Appl. 12 (1965), 49—57.
[21] *W. Rudin:* Functional Analysis, McGraw-Hill Co. New York, 1973.
[22] *I. Sawashima:* On spectral properties of some positive operators. Natur. Sci. Rep. Ochanomizu Univ. 15 (1969), 55—66.
[23] *V. F. Turchin:* Thermal Neutrons. Gosatomizdat, Moscow 1963 (Russian).
[24] *L. Van Hove:* Correlations in space and time and Born approximation scattering in systems of interacting particles. Phys. Rev. 95 (1954), 249.
[25] *A. M. Weinberg, E. P. Wigner:* The Physical Theory of Neutron Chain Reactors. Chicago University Press, Chicago 1958.
[26] *M. M. R. Williams:* The Slowing Down and Thermalization of Neutrons. North-Holland Publ. Company, Amsterdam 1966.
[27] *R. Zelazny:* Exact solution of a critical problem for a slab. J. Math. Phys. 2 (1961), 538 to 542.

Souhrn

RELAXAČNÍ DÉLKY A NEZÁPORNÁ ŘEŠENÍ V TRANSPORTU NEUTRONŮ

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V práci je vybudována teorie tak zvaných relaxačních délek v teorii transportu neutronů. Tato teorie je natolik obecná, že pokrývá případ jak moderátorových tak štěpných prostředí. Ač hlavní výsledky se týkají převážně prostředí štěpných, práce obsahuje nové výsledky i pro prostředí nemnožící. Speciálně, tvrzení o unicitě vyslovené v práci I. Kuščera a I. Vidava „On the spectrum of relaxation lengths of neutron distribution in a moderator" J. Math. Anal. Appl. 25 (1969), 80—92, jakožto hypotéza, je v naší práci důsledkem obecné teorie.

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