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Aplikace matematiky, Vol. 21 (1976), No. 4, 290–295

Persistent URL: `http://dml.cz/dmlcz/103648`

ON KLOTZ'S RESULT ON THE ASYMPTOTIC EFFICIENCY FOR THE SIGNED RANK TESTS

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(Received January 27, 1976)

1. INTRODUCTION

The most important problem in investigating the asymptotic efficiency in the Bahadur sense of a sequence of statistics $\{S_n\}$ is to find the exponential rate of convergence to zero of the probabilities of large deviations under the hypothesis $\mathscr{H}$, namely to compute

$$(1.1)\qquad \lim_{n\to\infty}\left[-\frac{1}{n}\log\sup\{Pr(S_n > nr_n \mid f) : f\in\mathscr{H}\}\right] = K(r)\,,\quad \text{say}\,,$$

where $\{r_n\}$ is a sequence of constants tending to some $r > 0$, (see, e.g., [1], [3]). If $\mathscr{H}$ and S_n are nonparametric, the limit gets much simpler, and it is

$$(1.2)\qquad \lim_{n\to\infty}\left[-\frac{1}{n}\log P(S_n > nr_n)\right] = K(r)\,,$$

where P indicates the probability measure under $\mathscr{H}$, and this notation will be kept throughout the paper.

Let $X_1, \ldots, X_n$ be a sequence of independent random variables having the same continuous cdf $F(x)$. Let $X_{(1)}, \ldots, X_{(n)}$ be the rearrangement of $X_1, \ldots, X_n$ ordered by magnitude of their absolute values, i.e. $|X_{(1)}| \leqq \ldots \leqq |X_{(n)}|$, and let $R_1^+, \ldots, R_n^+$ be the corresponding ranks of $|X_1|, \ldots, |X_n|$. Let $U_1, \ldots, U_n$ denote the signs of $X_{(1)}, \ldots, X_{(n)}$. The nonparametric hypothesis $\mathscr{H}$ in the paper consists of all $F(x)$ which are symmetric about 0 but otherwise arbitrary.

Klotz [4] has investigated the limit (1.2) for

$$(1.3)\qquad S_n = \sum_{i=1}^{n} E_{ni} U_i\,,$$

where E_{ni}, $1 \leqq i \leqq n$, are the expected values of the i-th smallest order statistics from a sample with cdf $G(x)$ on $(0; \infty)$ satisfying

$$\int_0^\infty x^3 \, dG(x) < \infty . \tag{1.4}$$

Note that, under $\mathscr{H}$, the variables $U_1, \ldots, U_n$ are independent and

$$P(U_i = 1) = P(U_i = -1) = 1/2 , \quad 1 \leqq i \leqq n . \tag{1.5}$$

The same problem has been explored in [3] for the most general case of the linear signed rank tests

$$S_n = \sum_{i=1}^n \alpha_n(i/(n+1) , \quad R_i^+/(n+1) , \ W_i) ,$$

from which Klotz's result follows.

In this paper the author will give a different direct proof of one result in [3] which generalized Klotz's one.

2. RESULTS

Theorem. *Let*

$$S_n = \sum_{i=1}^n a_{ni} U_i , \tag{2.1}$$

with $a_{ni} = a_n(i)$ *satisfying*

$$\int_0^1 |a_n(1 + [nu]) - \varphi(u)| \, du \to 0 \quad \text{as} \quad n \to \infty , \tag{2.2}$$

for some $\varphi(u) \in L_1(0; 1)$, *where* $[\cdot]$ *indicates the integer function. Let*

$$r_n \to r , \quad 0 < r < \int_0^1 |\varphi(u)| \, du = M , \quad \text{say} . \tag{2.3}$$

Then

$$\lim_{n\to\infty} \left[-\frac{1}{n} \log P(S_n > nr_n) \right] = K(r) , \tag{2.4}$$

where $K(r)$ *is evaluated from*

$$K(r) = br - \int_0^1 \log \cosh (b \, \varphi(u)) \, du \tag{2.5}$$

with $b > 0$ *being a unique solution of*

$$\int_0^1 \varphi(u) \tanh (b \, \varphi(u)) \, du = r . \tag{2.6}$$

Proof. First note that

$$h(b) = \int_0^1 \varphi(u) \tanh(b\,\varphi(u))\, du = \int_0^1 |\varphi(u)| \tanh(b|\varphi(u)|)\, du$$

is an increasing function of b, and takes value 0 at $b = 0$ and M at $b = \infty$. Thus for r satisfying (2.3) there is always a unique solution $b > 0$ of (2.6).

Clearly (2.2) implies

(2.7) (a) $a_n(1 + [nu]) \to \varphi(u)$ in the Lebesgue measure $\mathscr{L}$ on $(0, 1)$

$$\text{(b)} \quad \int_0^1 a_n(1 + [nu])\, du = \frac{1}{n} \sum_{i=1}^{n} a_{ni} \to \int_0^1 \varphi(u)\, du\,,$$

$$\text{(c)} \quad \int_0^1 |a_n(1 + [nu])|\, du = \frac{1}{n} \sum_{i=1}^{n} |a_{ni}| \to \int_0^1 |\varphi(u)|\, du = M\,, \quad \text{as} \quad n \to \infty\,.$$

By Lebesgue's theorem (cf. [5], Th. 3, p. 137), for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left|\int_\Delta a_n(1 + [nu])\, du\right| \leqq \int_\Delta |a_n(1 + [nu])|\, du < \varepsilon$$

for all $\Delta \subset (0; 1)$ with $\mathscr{L}(\Delta) < \delta$, and for all $n = 1, 2, \ldots$. Hence

$$\frac{1}{n}|a_{ni}| = \int_{(i-1)/n}^{i/n} |a_n(1 + [nu])|\, du < \varepsilon$$

for $1 \leqq i \leqq n$ if $n > 1/\delta$. Therefore

$$\lim_{n\to\infty} \max_{1 \leqq i \leqq n} \left\{\frac{1}{n} |a_{ni}|\right\} = 0\,. \tag{2.8}$$

Denote

$$V_i = a_{ni} U_i\,, \quad 1 \leqq i \leqq n\,. \tag{2.9}$$

Let $F_i(x)$ be the cdf of V_i under $\mathscr{H}$. For $b > 0$ and $1 \leqq i \leqq n$, put

$$p_i = E(e^{bV_i} \mid \mathscr{H}) = \int_{-\infty}^{\infty} e^{bx}\, dF_i(x) = \cosh(ba_{ni})\,,$$

and

$$H_i(x) = \frac{1}{p_i} \int_{-\infty}^{x} e^{by}\, dF_i(y)\,.$$

Thus H_i may be considered as a "new" cdf of V_i in accordance with a "new" probability measure Q given by

$$(2.10)\qquad Q(V_i = a_{ni}) = e^{ba_{ni}}/2\cosh(ba_{ni}) = \alpha_i\,,\quad \text{say}\,,$$
$$Q(V_i = -a_{ni}) = e^{-ba_{ni}}/2\cosh(ba_{ni}) = 1 - \alpha_i = \beta_i\,,\quad \text{say}\,.$$

Let H be the corresponding "new" cdf of S_n provided $V_1, \ldots, V_n$ are independent under Q. Using Feller's transformation (see (3.9), [2]), one has

$$(2.11)\quad P(S_n \leqq x) = \prod_{i=1}^{n} p_i \int_{-\infty}^{x} e^{-by}\,\mathrm{d}H(y) = \prod_{i=1}^{n}\cosh(ba_{ni})\int_{-\infty}^{x} e^{-by}\,\mathrm{d}H(y)\,.$$

It follows from (2.10) that

$$(2.12)\quad E(S_n \mid H) = \sum_{i=1}^{n} a_{ni}(\alpha_i - \beta_i) = \sum_{i=1}^{n} a_{ni}\tanh(ba_{ni}) = \mu_n(b) = \mu_n\,,\ \text{say,}$$

and

$$(2.13)\quad \operatorname{Var}(S_n \mid H) = 4\sum_{i=1}^{n} a_{ni}^2\alpha_i\beta_i = \sum_{i=1}^{n} a_{ni}^2/\cosh^2(ba_{ni}) = B_n^2(b) = B_n^2\,,\quad \text{say.}$$

From (2.11) one has

$$(2.14)\qquad P(S_n > \mu_n - 2B_n) = \prod_{i=1}^{n}\cosh(ba_{ni})\int_{y>\mu_n-2B_n} e^{-by}\,\mathrm{d}H(y) =$$
$$= \prod_{i=1}^{n}\cosh(ba_{ni})\,e^{-b\mu_n}\int_{z>-2B_n} e^{-bz}\,\mathrm{d}H^*(z)\,,$$

where $H^*(z) = H(z + \mu_n)$ is the "new" cdf of $Z_n = S_n - \mu_n$. Noting that $E(Z_n \mid H^*) = 0$, $\operatorname{Var}(Z_n \mid H^*) = B_n^2$, and that e^{-bz} is a decreasing function in z, we get

$$(2.15)\qquad e^{2bB_n} \geqq \int_{z>-2B_n} e^{-bz}\,\mathrm{d}H^*(z) \geqq e^{-2B_nb}\int_{|z|<2B_n} e^{-bz}\,\mathrm{d}H^*(z)$$
$$\geqq e^{-2B_nb}(1 - B_n^2/4B_n^2) = (3/4)\,e^{-2B_nb}\,,$$

by Chebyshev's inequality.

It follows from (2.7), (2.8) and (2.13) that

$$B_n^2/n^2 \leqq (1/n^2)\sum_{i=1}^{n} a_{ni}^2 \leqq (1/n^2)\max_{1\leqq i\leqq n}|a_{ni}|\sum_{i=1}^{n}|a_{ni}| \to 0 \quad \text{as}\quad n\to\infty\,,$$

i.e.

$$(2.16)\qquad B_n/n = o(1)\,,\quad \text{as}\quad n\to\infty\,.$$

Since $x \tanh(bx)$ and $\log\cosh(bx)$ satisfy Lipschitz's condition, it is easy to see from (2.7) that

$$(2.17) \qquad \mu_n/n = (1/n)\sum_{i=1}^{n} a_{ni} \tanh(ba_{ni}) = \int_0^1 \varphi(u) \tanh(b\,\varphi(u))\,\mathrm{d}u + o(1),$$

and

$$(2.18) \qquad (1/n)\sum_{i=1}^{n} \log\cosh(ba_{ni}) = \int_0^1 \log\cosh(b\,\varphi(u))\,\mathrm{d}u + o(1), \quad \text{as} \quad n \to \infty.$$

Clearly, by (2.3), (2.16), (2.17) and by the note at the beginning of the proof one can choose $b_n > 0$ satisfying

$$(2.19) \qquad (1/n)\,\mu_n(b_n) - (2/n)\,B_n(b_n) = r_n,$$

or equivalently, as $n \to \infty$,

$$(2.20) \qquad \int_0^1 \varphi(u) \tanh(b_n\,\varphi(u))\,\mathrm{d}u = r + o(1).$$

Evidently

$$(2.21) \qquad b_n \to b, \quad \text{as} \quad n \to \infty,$$

where $b < 0$ is a unique solution of (2.6) Finally, the theorem follows from (2.14) to (2.21).

Corollary. Klotz's result mentioned in Section 1 remains true under a weaker assumption on $G(x)$:

$$(2.22) \qquad \int_0^\infty x\,\mathrm{d}G(x) < \infty.$$

Proof. Put $\varphi(u) = G^{-1}(u)$ in the Theorem.

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Souhrn

O KLOTZOVĚ VÝSLEDKU O ASYMPTOTICKÉ EFICIENCI ZNAMÉNKOVANÝCH POŘADOVÝCH TESTŮ

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V článku se odvozuje vzorec pro Bahadurovu eficienci testů symetrie znaménkovanými pořadími. Jde o speciální případ dřívějšího autorova výsledku z [3], zde však je důkaz proveden pomocí odlišné jednodušší metody vhodné pro třídu jednoduchých pořadových statistik. Výsledek článku platí za obecnějších předpokladů než u J. Klotze [4].

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