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GEOMETRIC TRANSFORMATIONS BETWEEN GENERAL CONCURRENT CHARTS AND TANGENTIAL CONTACT CHARTS

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1. INTRODUCTION

The general geometric transformations between the general concurrent charts and the tangential contact charts both with three variables are researched in this paper.

Hitherto, the theories of the general concurrent charts and the general tangetial contact charts [1]–[4] were treated independently of each other, except the works of J. Záhora [6], [7]; and, moreover, it was rather difficult to construct practically the tangential contact charts by an envelope method, developed by one of the authors and O. Satō [4].

In this paper the authors discuss the general transformations [5]:

$$u(x, y)\,\bar{x} + v(x, y)\,\bar{y} + w(x, y) = 0\,, \tag{*}$$

by which a general concurrent chart of three variables in (x, y)-plane and a general tangential contact chart of three variables in $(\bar{x}, \bar{y})$-plane are transformed to each other, by an envelope method or a contact transformation method.

These transformation expressions (*) include J. Záhora's interesting noncorrelative transformation [6], [7], the pedal transformation [8], and the well-known dual transformation [9] in the projective geometry, as their special cases.

2. THE TRANSFORMATIONS THAT TRANSFORM THE GENERAL CONCURRENT CHARTS INTO THE GENERAL TANGENTIAL CONTACT CHARTS BY ENVELOPE METHOD

We now consider a relation (*), where $u(x, y)$, $v(x, y)$ and $w(x, y)$ are real-valued functions of real variables x and y and of class C^1 with respect to x, y; and we call them the characteristic functions of our transformation (*). If a point (x, y) in xy-plane is given, then by the relation (*), a straight line in $\bar{x}\bar{y}$-plane is determined. Let E and $\bar{E}$ denote the xy-plane and the $\bar{x}\bar{y}$-plane, respectively; assume that a set of points P_i $(i = 1, 2, 3, \ldots, n)$ on a given continuous curve c in E is transformed into a family of straight lines p_i $(i = 1, 2, 3, \ldots, n)$ in $\bar{E}$ by (*), and let the envelope of these straight lines be $\bar{c}$.

If an equation of the curve c is

$$f(x, y) = 0\,, \tag{2.1}$$

then the equation of the corresponding curve $\bar{c}$ can be found as follows. Assuming that $\partial f/\partial y \neq 0$ in (2.1), we have an explicit form of (2.1) as $y = y(x)$, and then the curve c is transformed into a family of straight lines in $\bar{E}$ expressed by the equation

$$u(x, y(x))\,\bar{x} + v(x, y(x))\,\bar{y} + w(x, y(x)) = 0 \tag{2.2}$$

from (*). Regarding x as a parameter, we shall find an envelope of (2.2); that is, differentiating (2.2) partially with respect to x, we have

$$\left\{\frac{\partial}{\partial x}\, u(x, y(x))\right\}\bar{x} + \left\{\frac{\partial}{\partial x}\, v(x, y(x))\right\}\bar{y} + \frac{\partial}{\partial x}\, w(x, y(x)) = 0\,,$$

and eliminating x from (2.2) and the above expression, we shall get

$$\bar{f}(\bar{x}, \bar{y}) = 0\,, \tag{2.3}$$

which is the required equation of the curve $\bar{c}$.

Let a functional relation of three variables

$$F(t_1, t_2, t_3) = 0\,, \quad a_i \leqq t_i \leqq b_i \quad (i = 1, 2, 3)\,, \tag{2.4}$$

be given where F is a real function of real variables t_1, t_2 and t_3, and of class C^1. We now assume that the equations representing the general concurrent chart for (2.4) are given by

$$f_i(x, y, t_i) = 0 \quad (i = 1, 2, 3)\,, \tag{2.5}$$

and let (t_i) $(i = 1, 2, 3)$ denote the three families of curves expressed by (2.5), respectively. Then we choose, from (t_i) $(i = 1, 2, 3)$, a triplet t_i $(i = 1, 2, 3)$, of curves t_i which intersect each other at a point P. Furthermore, we assume that the curves t_i $(i = 1, 2, 3)$ and their intersection point P in E are transformed into new curves t_i $(i = 1, 2, 3)$ and a straight line p in $\bar{E}$, respectively, by the relation (*). Then, it is easily seen that three new curves t_i $(i = 1, 2, 3)$ in $\bar{E}$ are tangent to the same straight line p (Fig. 1).

We now suppose that the equations

$$\bar{f}_i(\bar{x}, \bar{y}, t_i) = 0 \quad (i = 1, 2, 3)\,, \tag{2.6}$$

follow from (2.5) in the same way as we have obtained (2.3) from (2.1). Furthermore, we assume that the relation (*) transforms every pair of different curves t_i^0, t_i^1 ($i = 1, 2, 3$) of (2.5) into a pair of new different curves $\bar{t}_i^0$, $\bar{t}_i^1$ ($i = 1, 2, 3$) of (2.6). Then (2.6) are the equations representing the tangential contact chart for (2.4).

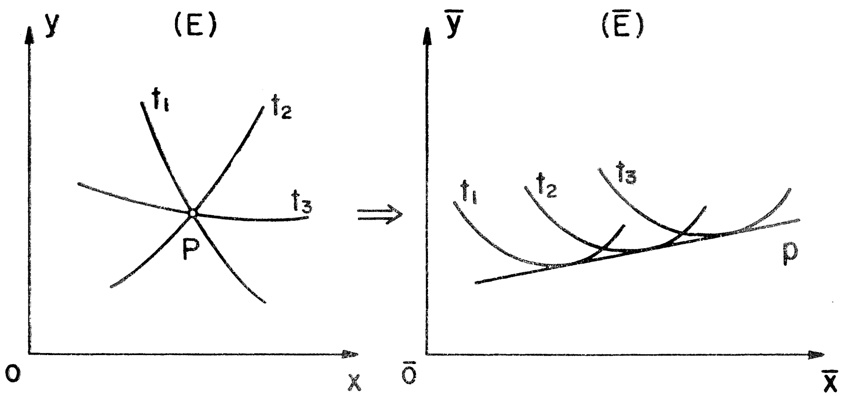

Fig. 1.

Example 1. Construct a tangential contact chart from the given concurrent chart expressed by the equations:

(2.7) $$(t_1) : y = t_1 - x\,,$$

(2.8) $$(t_2) : y = x - t_2\,,$$

(2.9) $$(t_3) : y = t_3 x\,,$$

which satisfy the functional relation

(2.10) $$t_3 = \frac{t_1 - t_2}{t_1 + t_2}\,.$$

We shall consider, as an example, the relation

(2.11) $$x\bar{x} - \bar{y} - x^2 + y = 0$$

as a special case of (*), where the characteristic functions are $u(x, y) = x$, $v(x, y) = -1$, $w(x, y) = -x^2 + y$, and try to transform (2.7), (2.8) and (2.9) by this relation.

In the first place, substituting (2.7) into (2.11), we get

$$x\bar{x} - \bar{y} - x^2 + t_1 - x = 0\,.$$

Differentiating this expression partially with respect to x, we have

$$\bar{x} - 2x - 1 = 0\,,$$

and then eliminating x from the above two expressions, we get

$$(t_1) : \bar{y} = \frac{(\bar{x} - 1)^2}{4} + t_1 . \tag{2.12}$$

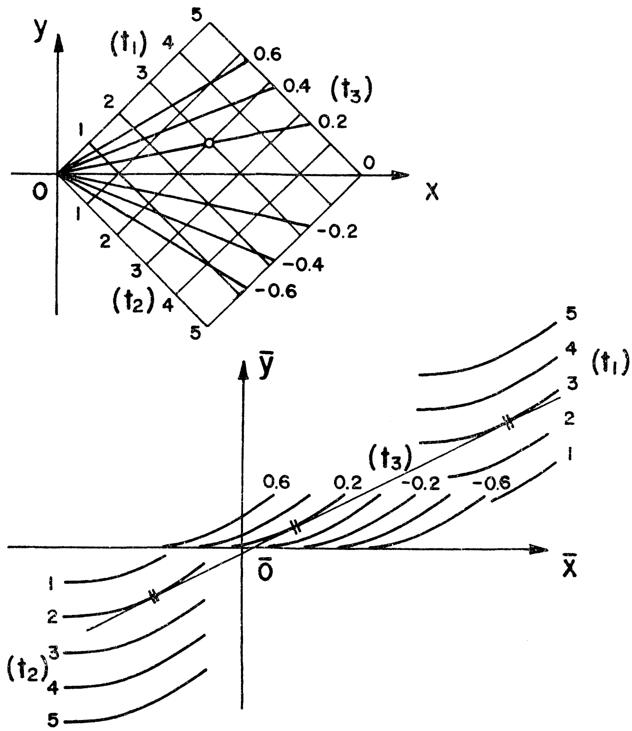

Fig. 2. Chart of $t_3 = (t_1 - t_2)/(t_1 + t_2)$. The figure shows that $t_1 = 3$, $t_2 = 2 \Rightarrow t_3 = 0{\cdot}2$.

Next, from (2.8) and (2.9), in a similar way, we get

$$(t_2) : \bar{y} = \frac{(\bar{x} + 1)^2}{4} - t_2 , \tag{2.13}$$

$$(t_3) : \bar{y} = \frac{(\bar{x} + t_3)^2}{4} , \tag{2.14}$$

respectively.

The equations (2.12), (2.13) and (2.14) are the equations representing the transformed tangential contact chart for the original concurrent chart of the given functional relation (2.10) (Fig. 2).

3. THE TRANSFORMATIONS THAT TRANSFORM GENERAL TANGENTIAL CONTACT CHARTS INTO GENERAL CONCURRENT CHARTS BY ENVELOPE METHOD

We shall again consider the relation (*) from a different point of view. It is easily seen that this relation (*) transforms a point $(\bar{x}, \bar{y})$ in $\bar{E}$ into a curve in E.

If we assume that a set of points C_i $(i = 1, 2, 3, \ldots, n)$ on a given continuous curve $\bar{g}$ in $\bar{E}$ is transformed into a family of curves c_i $(i = 1, 2, 3, \ldots, n)$ in E by (*), then we have, generally, an envelope g of a family of these curves (Fig. 3).

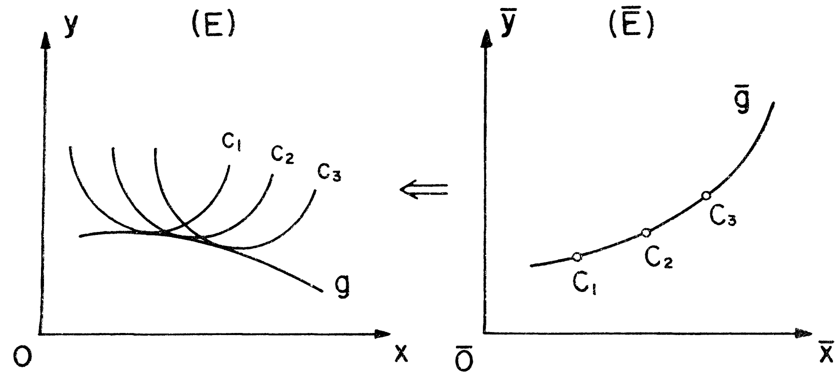

Fig. 3.

If an equation of the given curve $\bar{g}$ is

$$(3.1) \qquad \bar{g}(\bar{x}, \bar{y}) = 0\,,$$

then the equation of the corresponding curve g can be found as follows. Let an explicit form of (3.1) be $\bar{y} = \bar{y}(\bar{x})$, assuming that $\partial\bar{g}/\partial\bar{y} \neq 0$ and $\bar{y}(\bar{x})$ is a function of C^1 class; then a point $(\bar{x}, \bar{y})$ on the curve $\bar{g}$ is transformed into a curve in E expressed by the equation

$$(3.2) \qquad u(x, y)\,\bar{x} + v(x, y)\,\bar{y}(\bar{x}) + w(x, y) = 0\,,$$

by (*). Regarding $\bar{x}$ as a parameter, we shall find the envelope of the family of curves expressed by (3.2); that is, differentiating (3.2) partially with respect to $\bar{x}$, we have

$$u(x, y) + v(x, y)\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} = 0\,,$$

and eliminating $\bar{x}$ from (3.2) and the above expression, we get

$$(3.3) \qquad g(x, y) = 0\,,$$

which is the required equation of the transformed curve g.

We shall next consider a figure in E which results by transforming the straight line $\bar{y} = a\bar{x} + b$ in $\bar{E}$ by (*). Substituting $\bar{y}(\bar{x}) = a\bar{x} + b$ into (3.2), we have

$$\{u(x, y) + av(x, y)\}\, \bar{x} + bv(x, y) + w(x, y) = 0\,. \tag{3.4}$$

Putting

$$u(x, y) + a\, v(x, y) = 0\,, \quad b\, v(x, y) + w(x, y) = 0$$

and assuming the above two curves to have $s(\geqq 1)$ real intersection points, say P_i $(i = 1, 2, \ldots, s)$. Regarding $\bar{x}$ as a parameter, (3.4) represents a family of curves passing through these intersection points.

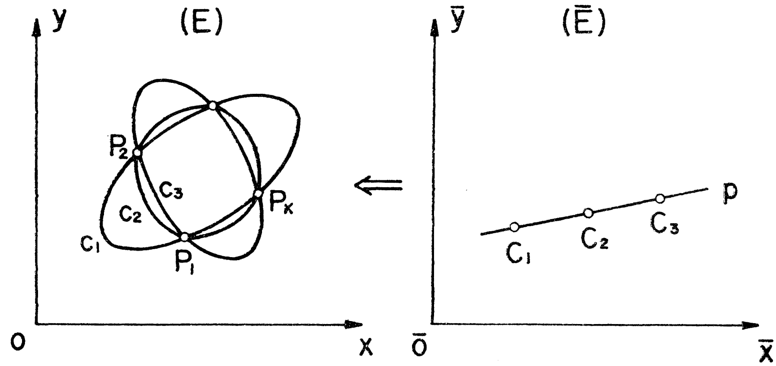

Fig. 4.

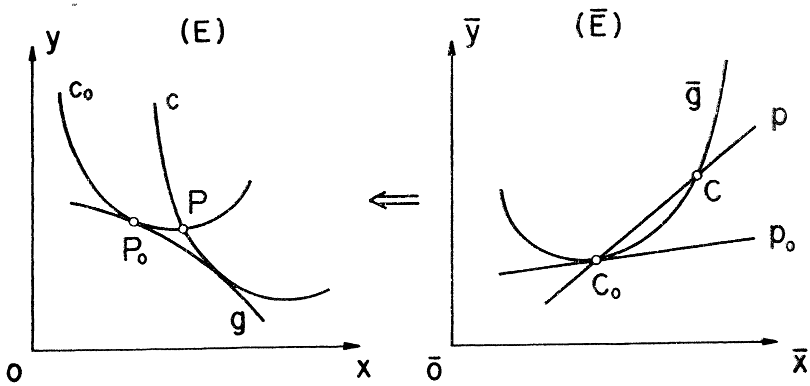

Fig. 5.

In this sense, we may say that a straight line p in $\bar{E}$ is transformed into the points P_i $(i = 1, 2, \ldots, s)$ in E by (*). We shall now pay attention to one of these points, say P_k, and discuss locally only there; therefore, we may admit that a straight line p in $\bar{E}$ is transformed into one point P_k in E by our transformation (Fig. 4). Furthermore, we assume that the transformation by (*) from the straight line p in $\bar{E}$ into the point P_k in E is univalent.

In Fig. 5 let the straight line passing through the two consecutive points C_0 and C on a curve $\bar{g}$ in $\bar{E}$ be p. Further we assume that the curve $\bar{g}$, two points C_0, C and the straight line p in $\bar{E}$ are transformed, by the relation (*), into a curve g, two consecutive curves c_0, c and a point P in E, respectively. Then we see that two curves c_0 and c pass through the point P and are tangent to the curve g. Let the point of contact of the curves c_0 and g be P_0. In $\bar{E}$, if the point C approaches the point C_0, then the straight line p also approaches p_0, which is tangent to the curve $\bar{g}$ at the point C_0, and then in E the curve c approaches the curve c_0, hence the point P approaches the point P_0. Consequently, a tangent p_0 at the point C_0 on the curve $\bar{g}$ is transformed into a point P_0, which is a point of contact of curves c_0 and g.

We shall again consider the functional relation (2.4) and we assume that the equations representing the tangential contact chart for (2.4) are given by

$$\bar{g}_i(\bar{x}, \bar{y}, t_i) = 0 \quad (i = 1, 2, 3)\,; \tag{3.5}$$

let (t_i) $(i = 1, 2, 3)$ denote the three families of curves expressed by (3.5), respectively. Then we choose a triplet t_i, of curves t_i $(i = 1, 2, 3)$ which are tangent to a straight line p. We also assume that the curves t_i $(i = 1, 2, 3)$ and the tangent p in $\bar{E}$ are transformed into new curves t_i $(i = 1, 2, 3)$ and a point P in E, respectively, by (*). Since p is a tangent of the curve t_1, the point P is on the new curve t_1 in E; similarly the point P is on both new curves t_2 and t_3. Therefore the three new curves t_i $(i = = 1, 2, 3)$ in E intersect at the same point P. (See Fig. 1, but the arrow line is inverse in this case.)

We now suppose that (3.5) yields the equations

$$g_i(x, y, t_i) = 0 \quad (i = 1, 2, 3) \tag{3.6}$$

in the same way as we have obtained (3.3) from (3.1). Furthermore, we assume that the relation (*) transforms every pair of different curves $t_{i'}^0$, t_i^1 $(i = 1, 2, 3)$ of (3.5) into a pair of new different curves $t_{i'}^0$, t_i^1 $(i = 1, 2, 3)$ of (3.6). Then (3.6) are the equations representing the general concurrent chart for (2.4).

Example 2. Construct a general concurrent chart from a given tangential contact chart expressed by the equations:

$$(t_1) : \bar{x}^2 = -8t_1\bar{y}\,, \tag{3.7}$$

$$(t_2) : \bar{y}^2 = -8t_2\bar{x}\,, \tag{3.8}$$

$$(t_3) : \bar{x}\bar{y} = t_3\,, \tag{3.9}$$

which satisfy the functional relation

$$t_1 t_2 = t_3 . \tag{3.10}$$

Assuming that the characteristic functions are $u(x, y) = y$, $v(x, y) = x$, $w(x, y) = -2xy$, we have the relation

$$y\bar{x} + x\bar{y} - 2xy = 0 , \tag{3.11}$$

as a special case of (*), and try to transform (3.7), (3.8) and (3.9) by (3.11).

In the first place, we have $\bar{y} = -\bar{x}^2/8t_1$ from (3.7), and then substituting this expression into (3.11) we get

$$y\bar{x} - \frac{x\bar{x}^2}{8t_1} - 2xy = 0 . \tag{3.12}$$

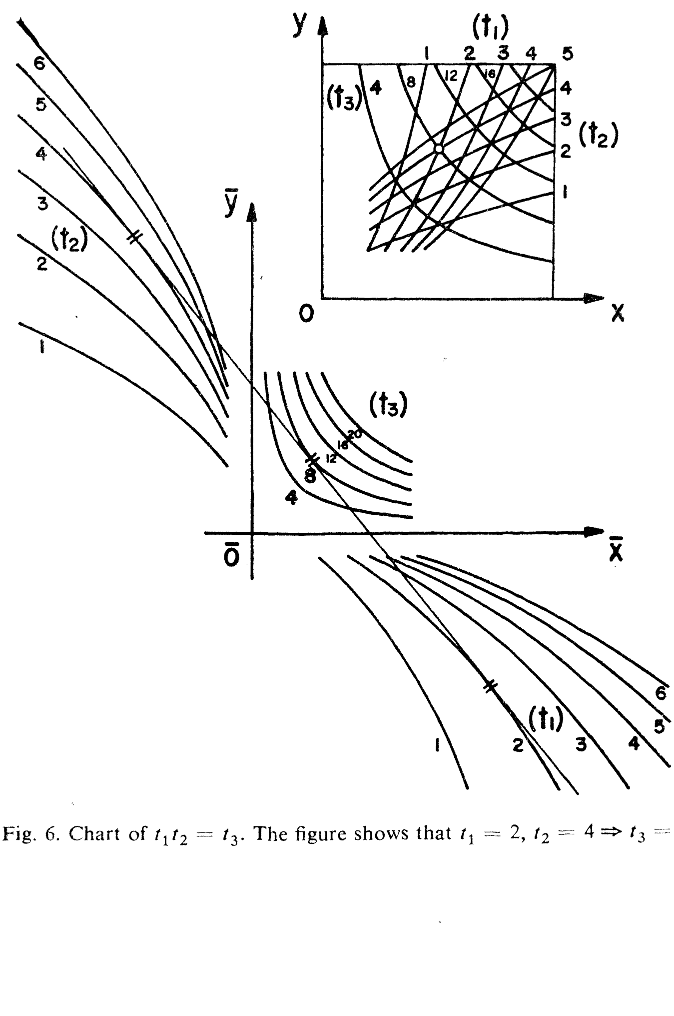

Fig. 6. Chart of $t_1 t_2 = t_3$. The figure shows that $t_1 = 2$, $t_2 = 4 \Rightarrow t_3 = 8$.

Differentiating (3.12) partially with respect to $\bar{x}$, we have

$$y - \frac{x\bar{x}}{4t_1} = 0\,,$$

and then eliminating $\bar{x}$ from (3.12) and the above expression, we get

$$(t_1) : x^2 = t_1 y\,. \tag{3.13}$$

Next, from (3.8) and (3.9), similarly as above, we get

$$(t_2) : y^2 = t_2 x\,, \tag{3.14}$$

$$(t_3) : xy = t_3\,, \tag{3.15}$$

respectively.

The equations (3.13), (3.14) and (3.15) are those of the required general concurrent chart representing (3.10) (Fig. 6).

4. THE CONTACT TRANSFORMATIONS BETWEEN GENERAL CONCURRENT CHARTS AND TANGENTIAL CONTACT CHARTS

We shall now consider a contact transformation by the relation (*). The contact transformation is a transformation that carries two curves which are tangent to each other into two new curves which are also tangent to each other. On the other hand the transformation by the relation (*), which is a special case of the general contact transformations, carries a point P into a straight line p and inversely p into P, as we have discussed in §§ 2 and 3. Therefore, we see that the contact transformation by the relation (*) is a transformation which carries a general concurrent chart into a tangential contact chart and vice versa.

We next try to find the formula by which the figures in E are transformed into new figures in $\bar{E}$ by the contact transformation expression (*). From the theory of contact transformations [10] we have

$$\begin{aligned}
&F \equiv u(x, y)\,\bar{x} + v(x, y)\,\bar{y} + w(x, y) = 0\,,\\
&F_x + pF_y = \{u_x(x, y) + p\,u_y(x, y)\}\,\bar{x} + \{v_x(x, y) + p\,v_y(x, y)\}\,\bar{y} + \\
&\qquad + w_x(x, y) + p\,w_y(x, y) = 0\,,\\
&F_{\bar{x}} + \bar{p}F_{\bar{y}} = u(x, y) + \bar{p}\,v(x, y) = 0\,,
\end{aligned} \tag{4.1}$$

where $p = \mathrm{d}y/\mathrm{d}x$, $\bar{p} = \mathrm{d}\bar{y}/\mathrm{d}\bar{x}$.

Solving (4.1) for $\bar{x}$, $\bar{y}$ and $\bar{p}$, we get

$$\bar{x} = -\frac{\begin{vmatrix} w & v \\ w_x + pw_y & v_x + pv_y \end{vmatrix}}{\begin{vmatrix} u & v \\ u_x + pu_y & v_x + pv_y \end{vmatrix}}, \quad \bar{y} = -\frac{\begin{vmatrix} u & w \\ u_x + pu_y & w_x + pw_y \end{vmatrix}}{\begin{vmatrix} u & v \\ u_x + pu_y & v_x + pv_y \end{vmatrix}},$$

(4.2)

$$\bar{p} = -\frac{u(x, y)}{v(x, y)}.$$

Expressions thus obtained are the required formulas of the transformation, which transform the line element (x, y, p) into a new line element $(\bar{x}, \bar{y}, \bar{p})$.

Given an equation of a curve c in E expressed by

(2.1) $$f(x, y) = 0,$$

then we first find $p = dy/dx$ from this expression, and putting

(4.3) $$p = p(x, y),$$

then eliminating x, y and p from the first expressions of (4.2), (2.1) and (4.3), we obtain

(2.3) $$\bar{f}(\bar{x}, \bar{y}) = 0,$$

which is an equation of the transformed curve $\bar{c}$ in $\bar{E}$.

We shall next consider a method for obtaining the formulas of transforming the figures in $\bar{E}$ into those on E.

Again solving (4.1) for x, y, p, we have a set of solutions of the form

(4.4) $$x = x(\bar{x}, \bar{y}, \bar{p}), \quad y = y(\bar{x}, \bar{y}, \bar{p}), \quad p = p(\bar{x}, \bar{y}, \bar{p}),$$

which are the formulas of transformation of the line element $(\bar{x}, \bar{y}, \bar{p})$ into that of (x, y, p).

Therefore, we can also find the equation of the curve c in E, by contact transformation method, when the equation of the curve $\bar{c}$ in $\bar{E}$ is known in advance.

Example 3. Construct a tangential contact chart from the given concurrent chart expressed by the equations

(4.5) $$(t_1) : y^2 = -\frac{2}{t_1} x + \frac{1}{t_1^2},$$

(4.6) $$(t_2) : y^2 = \frac{2}{t_2} x + \frac{1}{t_2^2},$$

$$(4.7) \qquad (t_3): y = \frac{1}{t_3},$$

which satisfy the functional relation

$$(4.8) \qquad t_3 = \sqrt{(t_1 t_2)} \quad (t_1 > 0, t_2 > 0).$$

We shall consider, as an example, the relation

$$(4.9) \qquad x\bar{x} + y\bar{y} - 1 = 0,$$

as a special case of (*), and try to transform (4.5), (4.6) and (4.7) by this relation. We use the formulas (4.2), and noting that $u = x$, $v = y$, $w = -1$, we get the following equations of transformation:

$$(4.10) \qquad \bar{x} = -\frac{p}{y - xp}, \quad \bar{y} = \frac{1}{y - xp}.$$

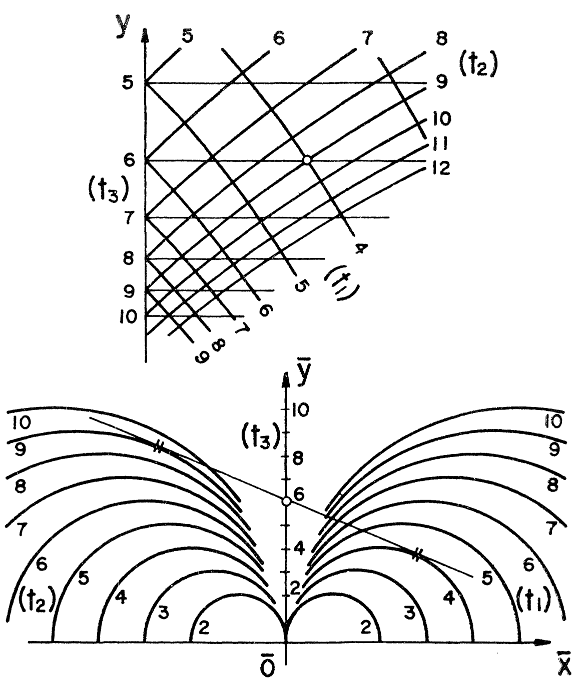

Fig. 7. Chart of $t_3 = \sqrt{(t_1 t_2)}$. The figure shows that $t_1 = 4$, $t_2 = 9 \Rightarrow t_3 = 6$.

From (4.5) we have $p = -1/(t_1 y)$, and then (4.10) becomes

$$\bar{x} = \frac{2t_1}{1 + t_1^2 y^2}, \quad \bar{y} = \frac{2t_1^2 y}{1 + t_1^2 y^2}.$$

Eliminating y from the above expressions, we get

$$(4.11) \qquad (t_1): (\bar{x} - t_1)^2 + \bar{y}^2 = t_1^2 .$$

From (4.6), similarly, we get

$$(4.12) \qquad (t_2): (\bar{x} + t_2)^2 + \bar{y}^2 = t_2^2 .$$

Lastly, from (4.7) we have $p = 0$; and again using (4.10), we get

$$(4.13) \qquad (t_3): \bar{x} = 0 , \quad \bar{y} = t_3 .$$

The equations (4.11), (4.12) and (4.13) are those required for the tangential contact chart of the given functional relation (4.8) (Fig. 7).

5. SPECIAL CASES OF OUR TRANSFORMATIONS

In this section we shall discuss some special cases of the relation (*).

5.1. J. Záhora's transformation. We assume that at least one of three families of the curves expressed by the equations (2.5), representing the general concurrent chart, is not a family of straight lines, and let its equation be

$$(5.1) \qquad f_3(x, y, t_3) = 0 .$$

The equation of the tangent at the point (x, y) to the curve (5.1) is given by

$$\frac{\partial f_3}{\partial x}(\bar{x} - x) + \frac{\partial f_3}{\partial y}(\bar{y} - y) = 0 ,$$

or

$$(5.2) \qquad \begin{gathered} \frac{\partial f_3(x, y, t_3)}{\partial x}\bar{x} + \frac{\partial f_3(x, y, t_3)}{\partial y}\bar{y} \\ - x\frac{\partial f_3(x, y, t_3)}{\partial x} - y\frac{\partial f_3(x, y, t_3)}{\partial y} = 0 . \end{gathered}$$

Assuming in (5.1) $\partial f_3/\partial t_3 \neq 0$, we may have $t_3 = t_3(x, y)$, and then substituting this expression into (5.2), we have the equation of the tangent in the following form:

$$(5.2') \qquad \Phi(x, y)\,\bar{x} + \Psi(x, y)\,\bar{y} - x\,\Phi(x, y) - y\,\Psi(x, y) = 0 .$$

We shall consider the transformation by the expression (5.2′) which is a special case of (*); and its characteristic functions are

$$(5.3) \qquad u = \Phi(x, y) , \quad v = \Psi(x, y) , \quad w = -\{x\,\Phi(x, y) + y\,\Psi(x, y)\} .$$

Assuming again in (5.1) $\partial f_3/\partial y \neq 0$, we have $y = y(x, t_3)$, and substituting this expression into (5.2′), we get

$$\Phi(x, y(x, t_3))\,\bar{x} + \Psi(x, y(x, t_3))\,\bar{y} -$$
$$- x\,\Phi(x, y(x, t_3)) - y(x, t_3)\,\Psi(x, y(x, t_3)) = 0\,,$$

which is the equation of tangent to the curve (5.1) with the parameter x. Hence the envelope of the above equation must be also the same as the curve (5.1). Therefore, by the transformation expression (5.2′), the Eq. (5.1) is transformed into the equation

$$f_3(\bar{x}, \bar{y}, t_3) = 0\,, \tag{5.4}$$

which is the same as the original curve (5.1).

Inversely, the curve (5.4), which is one of the three families of curves, representing a tangential contact chart, is again transformed into the same curve (5.1) by (5.2′).

Therefore, by keeping the third family of curves (t_3) invariant, we obtain the transformation between concurrent charts and tangential contact charts.

Example 2 in § 3 is one of these cases, and therein (3.11) is the equation of the tangent to the curve (3.15).

This transformation was precisely researched by J. Záhora [6], [7], although his method is different from ours.

5.2. Pedal transformation. Consider a curve $\bar{c}$, and let its tangent at $\bar{P}(\bar{x}, \bar{y})$ on it be $P\bar{P}$, where $P(x, y)$ is the intersection point of the tangent and a perpendicular from the origin O. Then the locus of the point $P(x, y)$, that is, the pedal curve of the original curve $\bar{c}$ with respect to the origin O is easily shown by the relation

$$x\bar{x} + y\bar{y} - (x^2 + y^2) = 0\,. \tag{5.5}$$

Hence the pedal transformation [8] expressed by (5.5) is a special case of (*), and its characteristic functions are

$$u = x\,, \quad v = y\,, \quad w = -(x^2 + y^2)\,. \tag{5.6}$$

5.3. General dual transformation. We may consider the general dual transformation in the projective geometry with the following characteristic functions:

$$u = a_{11}x + a_{12}y + a_{13}\,, \quad v = a_{21}x + a_{22}y + a_{23}\,, \tag{5.7}$$
$$w = -(a_{31}x + a_{32}y + a_{33})\,,$$

where $|a_{ik}| \neq 0\,(i, k = 1, 2, 3)$.

This dual transformation in nomography was fully discussed by H. Schwerdt [9].

5.4. Transformation by pole and polar with respect to the conic [9], [11]. As a special case of the characteristic functions (5.7), let us consider the functions

$$u = ax + hy + g\,, \quad v = hx + by + f\,, \quad w = gx + fy + c\,, \tag{5.8}$$

where

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0\,.$$

Then the relation (*) is expressed by

$$(ax + hy + g)\,\bar{x} + (hx + by + f)\,\bar{y} + gx + fy + c = 0\,, \tag{5.9}$$

which is an equation of a polar of point (x, y) with respect to the conic, whose equation is expressed by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0\,.$$

The expression, which is rearranged as a linear equation of x and y in (5.9), has the same form as (5.9). Therefore, the transformation by (5.9) is a dual and reciprocal transformation.

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Souhrn

GEOMETRICKÉ TRANSFORMACE OBECNÝCH PRŮSEČÍKOVÝCH NOMOGRAMŮ A DOTYKOVÝCH NOMOGRAMŮ

AKIRA MATSUDA, KATUHIKO MORITA

V článku se zkoumají obecné geometrické transformace obecných průsečíkových nomogramů tří proměnných a obecných dotykových nomogramů tří proměnných. Studované transformace mají tvar

$$u(x, y)\,\bar{x} + v(x, y)\,\bar{y} + w(x, y) = 0$$

a převádějí navzájem obecné průsečíkové nomogramy v rovině (x, y) a obecné dotykové nomogramy v $(\bar{x}, \bar{y})$ buď použitím metody obálek nebo metody kontaktních transformací.

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